
# Some Mathematics of Nonrenewable Resources: From Arithmetic to Optimal Control Theory

# Some Mathematics of Nonrenewable Resources: From Arithmetic to Optimal Control Theory.

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MAA Session on Integrating the Mathematics of Planet Earth 2013 in the College Mathematics Curriculum, II

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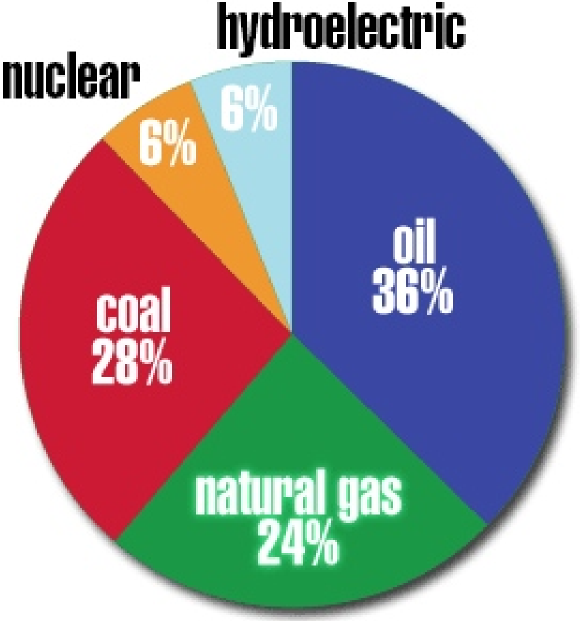
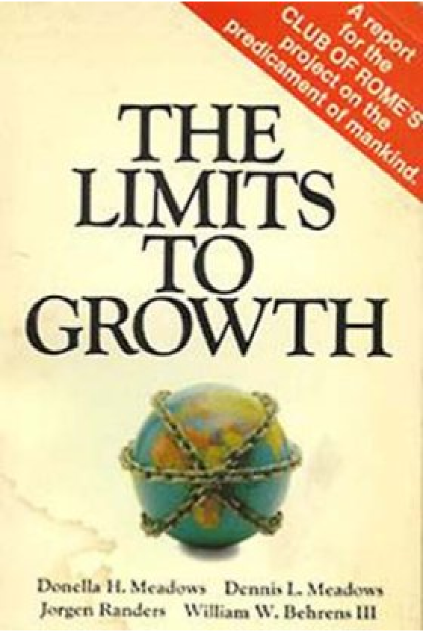
Our current civilization is heavily dependent on nonrenewable (exhaustible) resources. We use petroleum, coal, natural gas and uranium-dependent nuclear power to create electricity, heat and cool our homes, power our vehicles and manufacture our goods. Products we use every day require minerals such as copper, gold, silver, zinc and aluminum which we use up faster than the earth can replenish them.

How long will such nonrenewable resources last? Are there optimal ways to manage a dwindling supply?

We will illustrate how such questions can be approached using a variety of models that can be successfully integrated into a range of courses including college algebra, calculus of one and several variables, differential equations, discrete dynamical systems, computer simulation, and optimal control theory.

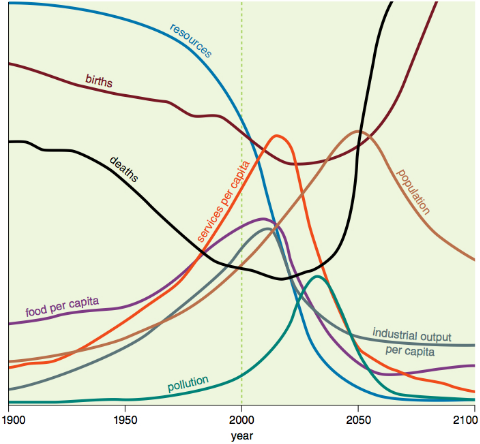

hydroelectric
nuclear
6%
6%
oil
36%
coal
28%
natural gas
24%


A report
for the
CLUB OF ROME'S
project on the
predicament of mankind.
THE
LIMITS
TO
GROWTH
Donella H. Meadows
Dennis L. Meadows
Jorgen Randers
William W. Behrens III

*Limits To Growth*, 1972

*Beyond the Limits*, 1992

*Limits To Growth, the 30 Year Update*, 2004



Graham Turner, *A Comparison of the Limits to Growth With Thirty Years of Reality*, 2008

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**The observed historical data for 1970 - 2000 most closely matches the simulated results of the Limits to Growth ”standard run” scenario for almost all the outputs reported; this scenario results in global collapse before the middle of this century.**

# What is a Nonrenewable Resource?

Nonrenewable = Exhaustible

# How To Compare Reserves

## Units of Measurement

- Tons
- Pounds
- Troy Ounces
- Flasks
- Barrels
- Cubic Feet

# Static Index

## The Static Index $s$

How long will the resource last if we keep using it at the *current* rate of consumption?

Assumptions:

- Known Reserve $K$ of the Resource
- Constant rate $C$ of Consumption

$$s = \frac{K}{C}$$

# Example 1: Copper

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Known Global Reserve: 340 million tons

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## Example 2: Cobalt

Known Global Reserve: 4.8 billion pounds

## Example 1: Copper

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## Example 2: Cobalt

Known Global Reserve: 4.8 billion pounds
Current Consumption Rate: 44 million pounds per year

## Example 1: Copper

Known Global Reserve: 340 million tons
Current Consumption Rate: 9.5 million tons per year
Static Index $= \frac{340}{9.5} = 36$ years

## Example 2: Cobalt

Known Global Reserve: 4.8 billion pounds
Current Consumption Rate: 44 million pounds per year
Static Index $= \frac{4{,}800{,}000{,}000}{9{,}500{,}000} = 110$ years

# Exponential Index

# Exponential Index Measures How Long a Resource Will Last if Consumption Rate Grows Continuously at a Constant Percentage Rate

$$y(t) = \text{Consumption Rate at time } t$$

$$y(0) = C \text{ Current Consumption Rate}$$

$$\frac{dy}{dt} = ry$$

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$$y(0) = C \text{ Current Consumption Rate}$$
$$\frac{dy}{dt} = ry$$
Hence
$$\text{Consumption Rate} = Ce^{rt}$$

$A(t) =$ Total Amount Consumed over a period of $t$ years

$$A(0) = 0$$

$$\frac{dA}{dt} = Ce^{rt}$$

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$$A(0) = 0$$

$$\frac{dA}{dt} = Ce^{rt}$$

Hence

$$A(t) = \frac{C}{r}[e^{rt} - 1]$$

# System of Differential Equations

$$\frac{dy}{dt} = ry$$

$$\frac{dA}{dt} = y$$

$$y(0) = C, A(0) = 0$$

Exponential Index $T$ is time it will take to consume the total known global reserve $K$:

$$A(T) = K$$

$$\frac{C}{r}[e^{rT} - 1] = K$$

$$\frac{rK}{C} = e^{rT} - 1$$

$$e^{rT} = 1 + \frac{rK}{C} = 1 + rs$$

Hence

$$T = \frac{\ln(1+rs)}{r}$$

# Example 1: Copper

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Static Index: 36 Years

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Bureau of Mines: 2.7 percent growth rate in demand

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Exponential Index is:

$$\frac{\ln(36 * .027 + 1)}{.027} = 25 \text{ Years}$$

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**Example 2: Coal**

**Example 1: Copper**

Static Index: 36 Years

Bureau of Mines: 2.7 percent growth rate in demand

Exponential Index is:

$$\frac{\ln(36 * .027 + 1)}{.027} = 25 \text{ Years}$$

**Example 2: Coal**

Static Index: 2300 Years

**Example 1: Copper**

Static Index: 36 Years

Bureau of Mines: 2.7 percent growth rate in demand

Exponential Index is:

$$\frac{\ln(36 * .027 + 1)}{.027} = 25 \text{ Years}$$

**Example 2: Coal**

Static Index: 2300 Years

Bureau of Mines: 4.1 percent growth rate in demand

## Example 1: Copper

Static Index: 36 Years

Bureau of Mines: 2.7 percent growth rate in demand

Exponential Index is:

$$\frac{\ln(36 * .027 + 1)}{.027} = 25 \text{ Years}$$

## Example 2: Coal

Static Index: 2300 Years

Bureau of Mines: 4.1 percent growth rate in demand

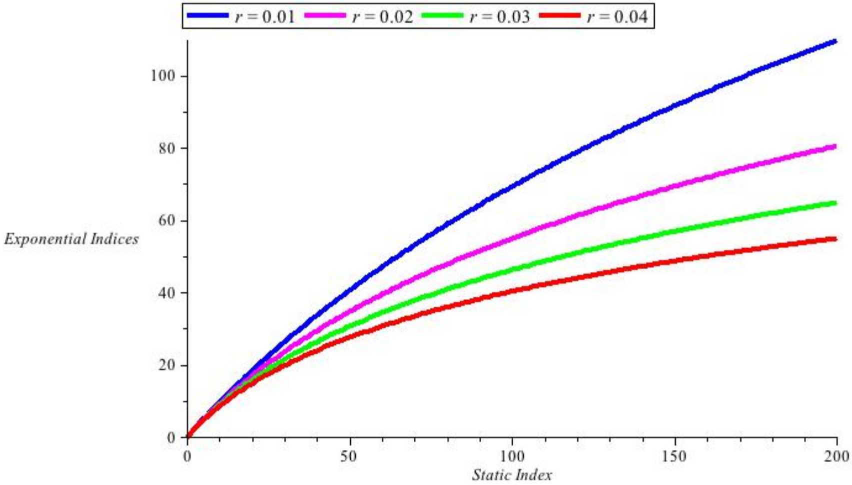
Exponential Index is:

$$\frac{\ln(2300 * .041 + 1)}{.041} = 111 \text{ Years}$$

# Comparing the Indices

| s | 0.01 | 0.02 | 0.03 | 0.04 | 0.05 | 0.06 | 0.07 | 0.08 | 0.09 | 0.10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 20 | 18 | 17 | 16 | 15 | 14 | 13 | 13 | 12 | 11 | 11 |
| 30 | 26 | 24 | 21 | 20 | 18 | 17 | 16 | 15 | 15 | 14 |
| 40 | 34 | 29 | 26 | 24 | 22 | 20 | 19 | 18 | 17 | 16 |
| 50 | 41 | 35 | 31 | 27 | 25 | 23 | 21 | 20 | 19 | 18 |
| 60 | 47 | 39 | 34 | 31 | 28 | 25 | 24 | 22 | 21 | 19 |
| 70 | 53 | 44 | 38 | 33 | 30 | 27 | 25 | 24 | 22 | 21 |
| 80 | 59 | 48 | 41 | 36 | 32 | 29 | 27 | 25 | 23 | 22 |
| 90 | 64 | 51 | 44 | 38 | 34 | 31 | 28 | 26 | 25 | 23 |
| 100 | 69 | 55 | 46 | 40 | 36 | 32 | 30 | 27 | 26 | 24 |
| 110 | 74 | 58 | 49 | 42 | 37 | 34 | 31 | 29 | 27 | 25 |
| 120 | 79 | 61 | 51 | 44 | 39 | 35 | 32 | 30 | 27 | 26 |
| 130 | 83 | 64 | 53 | 46 | 40 | 36 | 33 | 30 | 28 | 26 |
| 140 | 88 | 67 | 55 | 47 | 42 | 37 | 34 | 31 | 29 | 27 |
| 150 | 92 | 69 | 57 | 49 | 43 | 38 | 35 | 32 | 30 | 28 |

A tabulation of exponential indices (in years) for different values of $s$, the static index (also in years) and varying rates of growth $r$ of consumption.

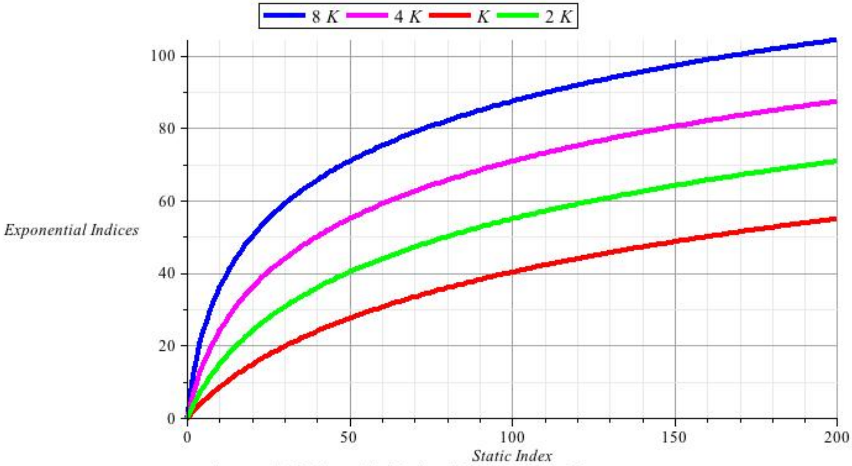


$r = 0.01$
$r = 0.02$
$r = 0.03$
$r = 0.04$
Exponential Indices
Static Index
0
20
40
60
80
100
0
50
100
150
200

# Changes in Known Global Reserves

# Actual Total Reserve = n Times Known Reserve

$$T = \frac{\ln(nsr+1)}{r}$$



Exponential Indices with 1, 2, 4, and 8 Times Known Reserves

| Mineral | Years<br>Lomborg<br>0% Growth | Years<br>Benchmark<br>0% Growth | Years<br>Diederen<br>2% Growth | Years<br>Latest<br>2.57% Growth |
|---|---|---|---|---|
| Iron Ore | 215 | 78 | 46 | 42 |
| Cobalt | 320 | 91 | 57 | 46 |
| Aluminum | 230 | 137 | 63 | 58 |
| Silver | 15 | 23 | 10 | 18 |
| Gold | 18 | 21 | 13 | 16 |
| Zinc | 42 | 22 | 13 | 17 |
| Tin | 47 | 20 | 15 | 15 |
| Copper | 43 | 39 | 23 | 26 |
| Nickel | 117 | 52 | 28 | 32 |

Table : Years of Supply Left For Certain Minerals Based On Reserves and Annual Production Using Data from *USGS Mineral Commodity Summaries, 20111*

| Mineral | 1989 | 2002 | 2008 | Price Multiplier 2002 - 2008 |
|---|---|---|---|---|
| Iron Ore | $41.20 | $23.50 | $56.60 | 2.4 |
| Cobalt | $22,700 | $15,500 | $51,800 | 3.3 |
| Aluminum | $35.30 | $18.40 | $20.00 | 1.1 |
| Silver | $232,500 | $134,000 | $398,000 | 3.0 |
| Gold | $16,200,000 | $9,060,000 | $21,200,000 | 2.3 |
| Zinc | $2,380 | $772 | $1,480 | 1.9 |
| Tin | $15,100 | $5,830 | $18,900 | 3.2 |
| Copper | $3,800 | $1,510 | $5,330 | 3.5 |
| Nickel | $17,500 | $6,130 | $16,000 | 2.6 |

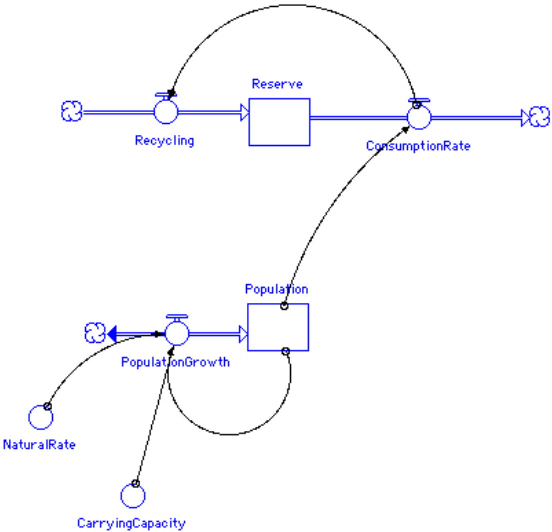
Table : Mineral Price Variation For Select Years; Prices in constant 1998 U.S. dollars/ton. From Mark Henderson, *The Depletion Wall*, 2012

# A Simulation Model

Assume:

- per capita usage remains constant
- Population grows logistically
- Recycling occurs

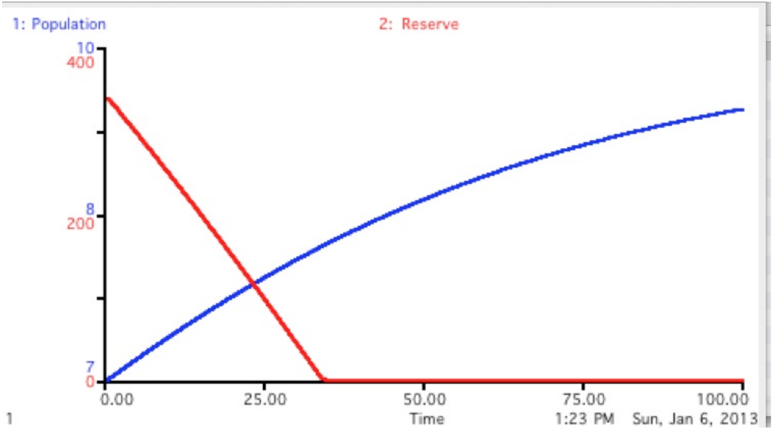
# A STELLA Simulation Model



# Some Details

- Resource: Copper
- Reseve: 340 million tons
- Current Population: 7 million
- per capita usage: 9.7 million/ 7 million
- Carrying Capacity: 10 million people
- Natural Growth Rate: 2 percent
- 2 percent of consumption is recycled

Population(t) = Population(t - dt) + (PopulationGrowth) * dt
INIT Population = 7
INFLOWS:
PopulationGrowth =
NaturalRate*Population*(1-Population/CarryingCapacity)
Reserve(t) = Reserve(t - dt) + (Recycling - ConsumptionRate) * dt
INIT Reserve = 340
INFLOWS:
Recycling = .03 *ConsumptionRate
OUTFLOWS:
ConsumptionRate = (9.5/7) *Population
CarryingCapacity = 10
NaturalRate = 0.02


1: Population
2: Reserve
10
400
8
200
7
0
0.00
25.00
50.00
75.00
100.00
Time
1
1:23 PM Sun, Jan 6, 2013

Harold Hotelling
1895-1973

THE JOURNAL OF
POLITICAL ECONOMY

*Volume 39* APRIL 1931 *Number 2*

# THE ECONOMICS OF EXHAUSTIBLE RESOURCES

## I. THE PECULIAR PROBLEMS OF MINERAL WEALTH

CONTEMPLATION of the world's disappearing supplies of minerals, forests, and other exhaustible assets has led to demands for regulation of their exploitation. The feeling that these products are now too cheap for the good of future generations, that they are being selfishly exploited at too rapid a rate, and that in consequence of their excessive cheapness they are being produced and consumed wastefully has given rise to the conservation movement. The method ordinarily proposed to stop the wholesale devastation of irreplaceable natural resources, or of natural resources replaceable only with difficulty and long delay, is to forbid production at certain times and in certain regions or to hamper production by insisting that obsolete

"The Economics of Exhaustible Resources"

*Journal of Political Economy*, 1931

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## A Simple Optimal Control Model

$X_t =$ amount of resource in period $t$
$X_0 =$ initial stock
$Y_t =$ "harvest" level
$F(X_t) =$ net growth rate (through recycling or exploration)
$\pi_t = \pi(X_t, Y_t) =$ Net benefits in period $t$
$\delta =$ discount rate
$p = \frac{1}{1+\delta} =$ discount factor

Problem: Find the harvest schedule $Y_t$ which will
Maximize $\pi = \sum_{t=0}^{T} p^t \pi(X_t, Y_t)$
Subject to $X_{t+1} - X_t = F(X_t) - Y_t$

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Form Lagrangian

$$L = \sum_{t=0}^{T} p^t \{\pi(X_t, Y_t) + p\lambda[X_t + F(X_t) - Y_t - X_{t+1}]\}$$

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Form Lagrangian

$$L = \sum_{t=0}^{T} p^t \{\pi(X_t, Y_t) + p\lambda[X_t + F(X_t) - Y_t - X_{t+1}]\}$$

Necessary Conditions For Maximum:

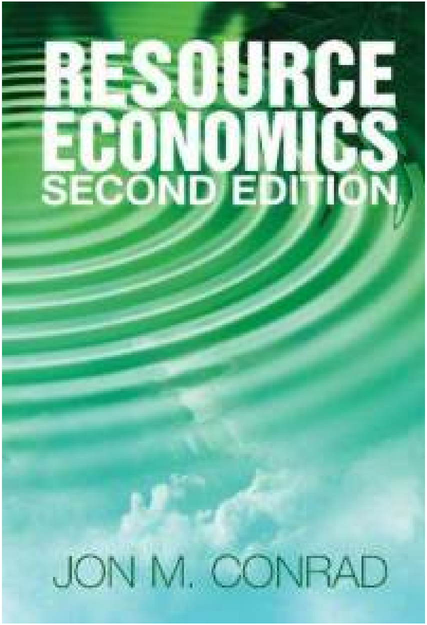
$$\frac{\partial L}{\partial Y_t} = p^t \{\frac{\partial \pi(X_t, Y_t)}{\partial Y_t} - p\lambda_{t+1}\} = 0$$

$$\frac{\partial L}{\partial X_t} = p^t \{\frac{\partial \pi(X_t, Y_t)}{\partial X_t} + p\lambda_{t+1}[1 + F'(X_t)]\} - p^t \lambda_t = 0$$

$$\frac{\partial L}{\partial [p\lambda_{t+1}]} = p^t \{X_t + F(x_t) - Y_t - X_{t+1}\} = 0$$

## More Complex Models

- Demand Functions
- Reserve-Dependent Costs
- Pollutants (Degradable and Nondegradable)
- Recycling
- Risky Development




Cambridge University Press; 2010

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- Partha S. Dasgupta and Geoffrey M. Heal, *Economic Theory and Exhaustible Resources*, Cambridge University Press, 1980.
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- Marck C. Henderson, *The Depletion Wall: Non-Renewable Resources, Population Growth, and the Economics of Poverty*, Waves of the Future, 2012.
- Michael Olinick, "Modelling Depletion of Nonrenewable Resources," *Mathematical and Computing Modelling*, **15**, 91

# THANK YOU

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A model assuming that a sufficiently high price, $p$, a substitute will become available.

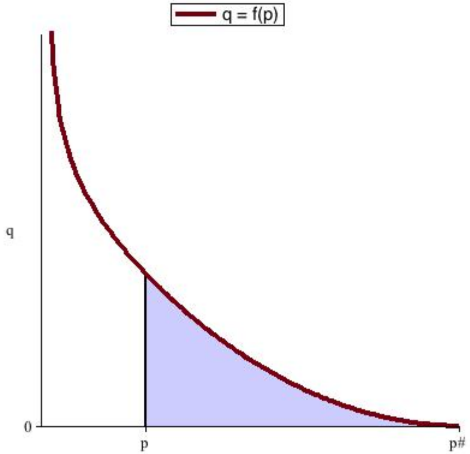
Example: Solar energy might substitute for fossil fuel

$$
\begin{aligned}
p(t) &= \text{price of the resource at time } t \\
q &= f(p) \text{ is the demand function: the quantity demanded} \\
& \quad \text{at a price } p \\
\bar{p} &= \text{price at which substitute completely replaces the resource.} \\
c &= G(q) \text{ is the cost function} \\
Q(t) &= \text{the available stock or reserve of the resource at time } t, \\
& \quad Q(0) = Q_0 > 0 \\
r &= \text{social discount rate; } r > 0 \\
T &= \text{the horizon time: the latest time at which the substitute} \\
& \quad \text{will become available regardless of the price of the natural} \\
& \quad \text{resource, } T > 0
\end{aligned}
$$

## Assumptions
Demand function:

- $f' \leq 0$
- $f(p) > 0$ for $p < \bar{p}$
- $f(p) = 0$ for $p \geq \bar{p}$

# A Typical Demand Function



Assumptions

Cost function $G(q)$ ($q$ is demand)

- $G(0) = 0$
- $G(q) > 0$ for $q > 0$
- $G' > 0$ and $G" \geq 0$
- $G'(0) < p$ (producers can make positive profit at a price $p$ below $\bar{p}$ )

Let $c = G(q) = G(f(p)) = g(p)$
Note: $g(p) > 0$ for $p < \bar{p}$ and $g(p) = 0$ for $p \geq \bar{p}$
Let $\pi(p) = pf(p) - g(p)$ denote the profit function of the producers.
$\pi$ is called *producers' surplus*
Let $\underline{p}$ be the smallest price at which $\pi$ is nonnegative.
We assume $\pi(p)$ is concave in interval $[\underline{p}, \bar{p}]$
Consumer Surplus:

$$\psi(p) = \int_p^{\bar{p}} f(y)dy$$

Maximize

$$J = \int_0^T [\psi(p) + \pi(p)]e^{-rt}dt$$

subject to

$$Q'(t) = -f(p), Q(0) = Q_0, Q(T) \geq 0$$

and

$$p \in [\underline{p}, \bar{p}]$$