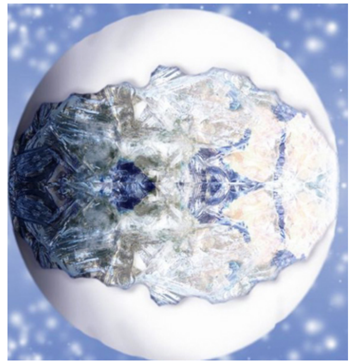


Climate models and differential equations

Jim Walsh, Oberlin College

Joint Meetings
Baltimore, MD
January 18, 2014

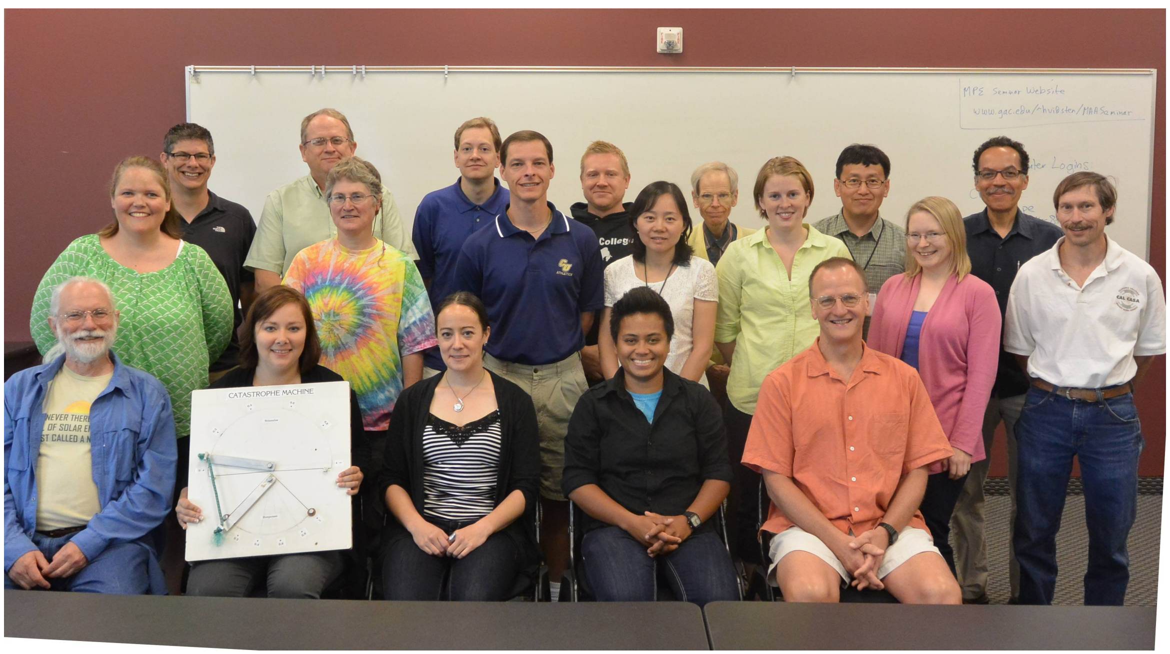
Thanks!

MCRN
Mathematics and Climate Research Network

Chris Jones, UNC, Director
Mary Lou Zeeman, Bowdoin College, Codirector
Hans Kaper, Georgetown, Codirector

<u>Co-conspirators</u>

Esther Widiasih, Samantha Oestreicher, Richard McGehee, Anna Barry



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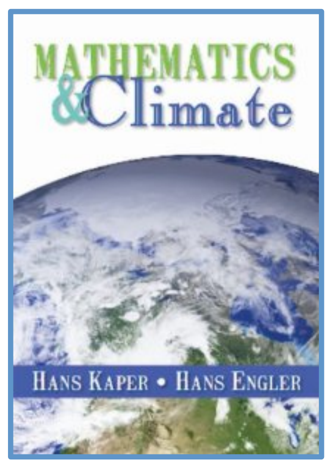

- Math 234 Differential Equations (prerequisite Multivariable Calculus)
- Fall 2013
- Text: *Differential Equations* (Blanchard, Devaney & Hall)
- QFR Curriculum Development Grant (Oberlin College HHMI grant)

Second class meeting: An Energy Balance Climate Model

$$T = T(t) \quad \text{global annual average surface temperature} \quad (\text{K})$$

$$R\frac{dT}{dt} = E_{\text{in}} - E_{\text{out}} \qquad (\text{W/m}^2,\ R \ \text{ heat capacity of Earth's surface})$$

$$= Q(1-\alpha) - \sigma T^4$$

insolation
$(Q = 342 \text{ W/m}^2)$

albedo
$(\alpha = 0.3)$

outgoing radiation

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insolation ($Q = 342$ W/m^2)

albedo ($\alpha = 0.3$)

outgoing radiation

Mathematica:

```
DSolve[{T'[t] == (1 / R) * (343 * (1 - 0.3) - 5.67 * 10^(-8) * T[t]^4)}, T[t], t]
```

```
Out[1]= {{T[t] → InverseFunction[
    -(2 ArcTan[(3 #1)/(100 × 7^(3/4) 10^(1/4))] - Log[7000 - 3 × 7^(1/4) 10^(3/4) #1] + Log[7000 + 3 × 7^(1/4) 10^(3/4) #1])
      / (588 000 000 × 7^(1/4) 10^(3/4)) &][
   -(7. × 10^(-10) t)/R + C[1]]}}
```

Second class meeting: An Energy Balance Climate Model

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$$R\frac{dT}{dt} = E_{\text{in}} - E_{\text{out}} \quad (\mathrm{W/m^2},\ R \ \text{heat capacity of Earth's surface})$$

$$= \underbrace{Q}_{\text{insolation}}(1 - \underbrace{\alpha}_{\text{albedo}}) - \underbrace{\sigma T^4}_{\text{outgoing radiation}}$$

insolation
($Q = 342\ \mathrm{W/m^2}$)

albedo
($\alpha = 0.3$)

outgoing radiation

$$\boxed{\text{Equilibrium solution:} \quad T^* = \left(\frac{Q(1-\alpha)}{\sigma}\right)^{1/4} = 256\ \mathrm{K} = 1.4\ {}^\circ\mathrm{F}}$$

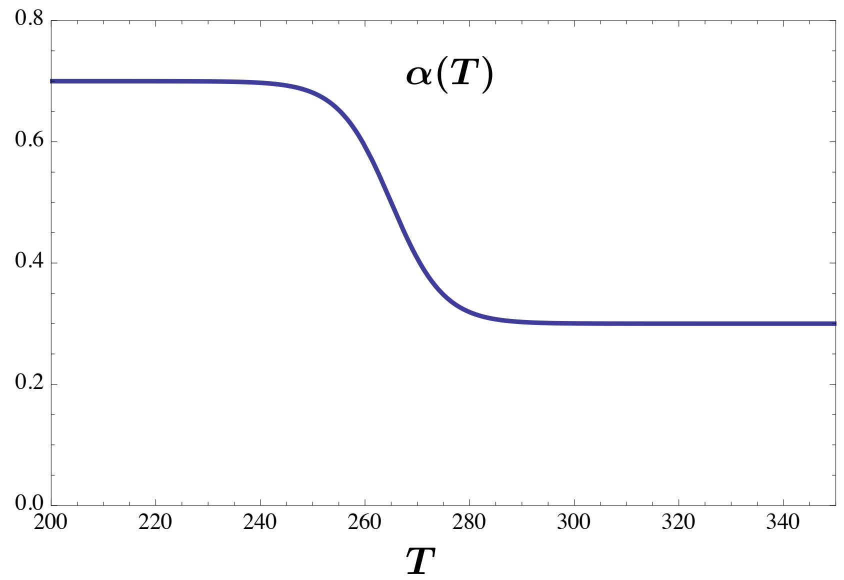
Atmosphere: replace σT^4 with $\epsilon\sigma T^4$

Also: phase line, linearization at T^*

Homework problem: Bifurcations (motivated by a problem in *Mathematics & Climate*)

$$\boxed{R\frac{dT}{dt} = Q(1-\alpha(T)) - 0.6\sigma T^4 = f(T)}$$

$$\boxed{\alpha(T) = 0.5 + 0.2\tanh(0.1(265-T))}$$

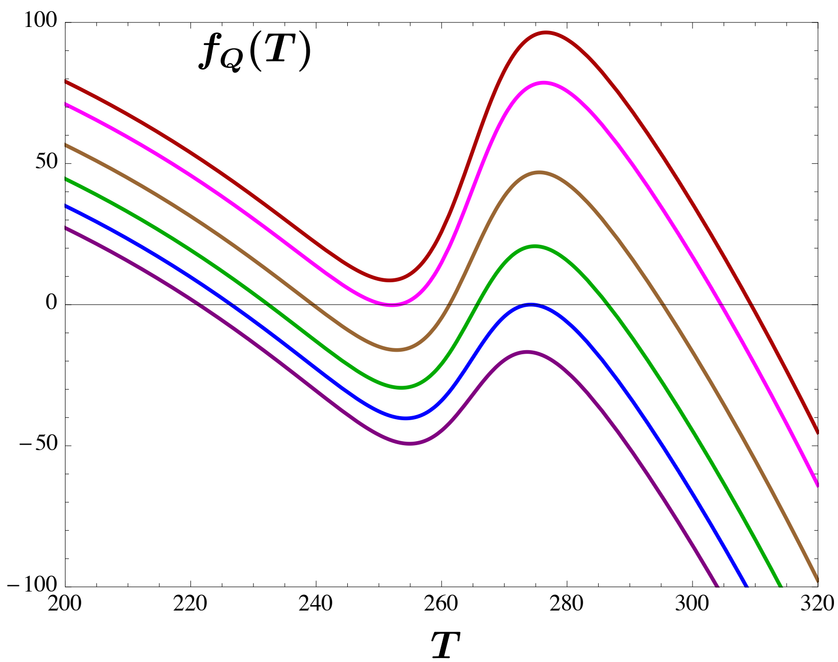


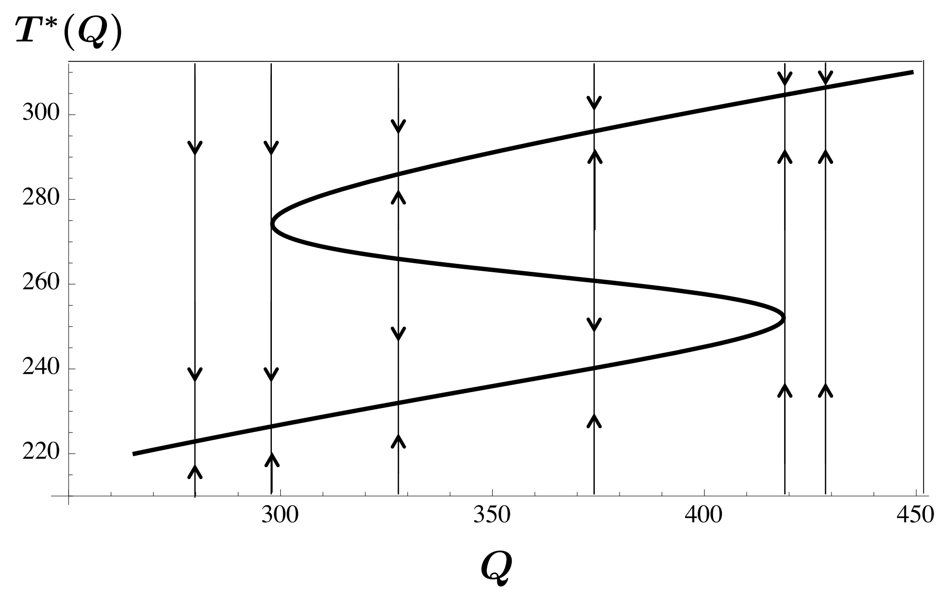
How do equilibrium solutions vary with Q?

Homework problem: Bifurcations

$$R\frac{dT}{dt} = Q(1-\alpha(T)) - 0.6\sigma T^4 = f_Q(T)$$

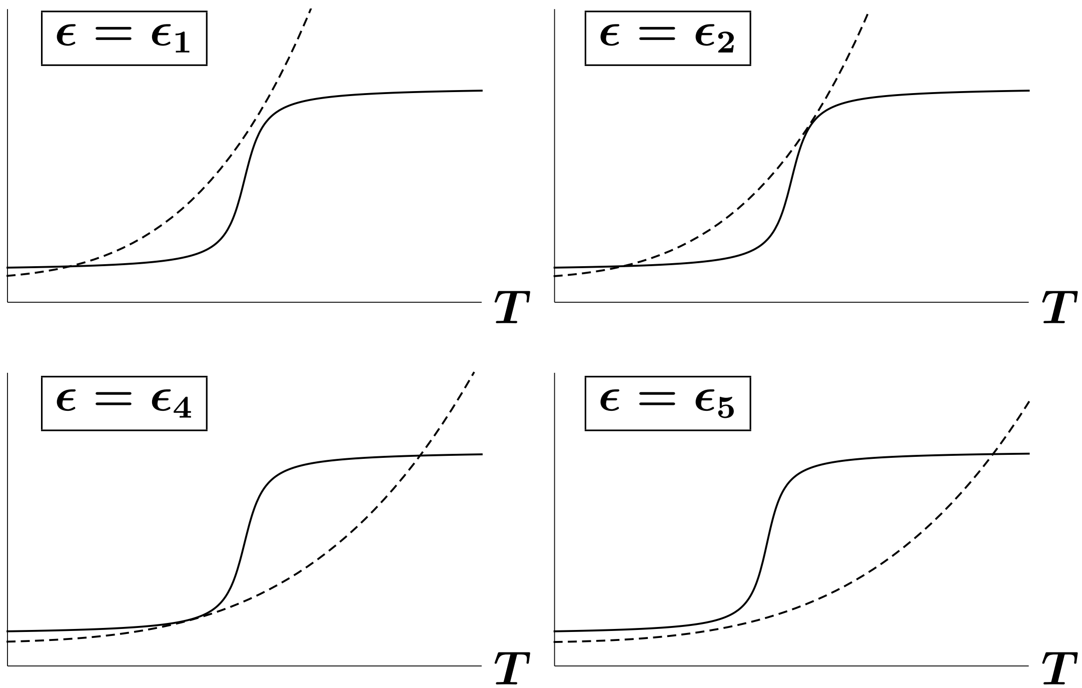
$$\alpha(T) = 0.5 + 0.2\tanh(0.1(265 - T))$$





Exam question: Bifurcations (thanks to Anna Barry)

Consider the autonomous ODE $$R\frac{dT}{dt} = E_{\text{in}}(T) - E_{\text{out}}(T), \quad E_{\text{out}}(T) = \epsilon\sigma T^4$$



Note: The dashed curve in each plot is $E_{\text{out}}(T)$. The solid curve is $E_{\text{in}}(T)$.

(a) Draw the phase line for each of the above ϵ-values.
(b) Draw a bifurcation diagram with ϵ decreasing on the horizontal axis.
(c) In a brief paragraph, discuss the bifurcation that occurs at $\epsilon = \epsilon_2$ in the context of the model and, in particular, in terms of the concentration of greenhouse gases such as CO_2.

Homework problem:

Suppose we model the OLR in the global surface temperature model via a linear term of the form $A + BT$:

$$\boxed{R\frac{dT}{dt} = Q(1-\alpha) - (A + BT), \qquad (T\ ^\circ\mathrm{C})}$$

(a) Explain, in terms of the model, the requirement that $B > 0$.

(b) Find the general solution of this equation. What is the behavior of solutions over time?

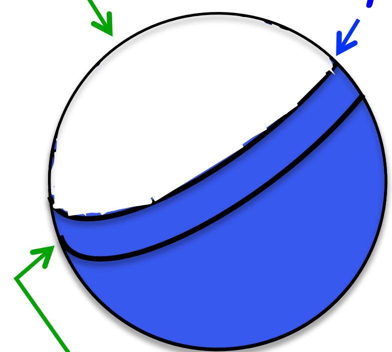
(c) The parameters $A = 202$ W m^{-2} and $B = 1.9$ W m^{-2} $(^\circ\mathrm{C})^{-1}$ have been estimated via satellite measurements.

   (i) Using $Q = 342$ W m^{-2} and $\alpha = 0.3$, compute the Earth's average surface temperature T^* at equilibrium. Why might you expect this value to be fairly close to 15.4°C, the Earth's current annual global average surface temperature?

   (ii) How does the magnitude of T^* vary with the parameter A? Discuss in the context of the OLR term in the model.

Latitude-dependent EBM: A project (following Esther Widiasih)

- $y = \sin(\text{latitude})$
- $T(y,t)$ – temp. at latitude y (zonal ave.)
- symmetry across the equator; no land
- ice cover above *ice line* η; no ice below η



$$\begin{cases} \dfrac{d\eta}{dt} = \epsilon(T(\eta,t) - T_c) \\ R\dfrac{\partial T}{\partial t} = Qs(y)(1-\alpha_\eta(y)) - (A + BT(y,t)) - C\left(T(y,t) - \overbrace{\int_0^1 T(y,t)dy}^{\overline{T}}\right) \end{cases}$$

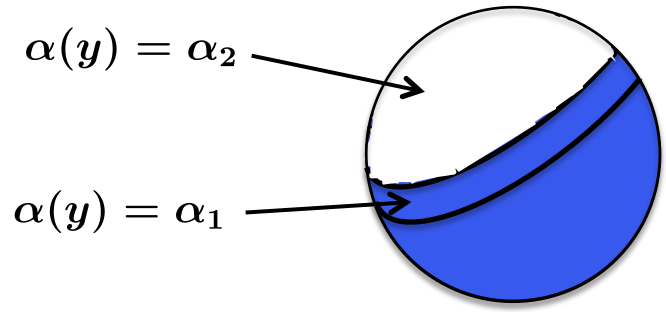
$0 < \epsilon \ll 1$

T_c critical temp.

M. I. Budyko, The effect of solar radiation variation on the climate of the Earth, *Tellus* **21** (1969), 611-619.

E. Widiasih, Instability of the ice free Earth: dynamics of a discrete time energy balance model, to appear in *SIAM J. Appl. Dyn. Syst.*

Latitude-dependent EBM: A project



$$\begin{cases} \dfrac{d\eta}{dt} = \epsilon(T(\eta,t) - T_c) \\ R\dfrac{\partial T}{\partial t} = Qs(y)(1-\alpha_\eta(y)) - (A + BT(y,t)) - C\left(T(y,t) - \int_0^1 T(y,t)dy\right) \end{cases}$$

Legendre polynomials $p_0(y), p_2(y)$

$$T(y,t) = \begin{cases} U(y,t), & y < \eta \\ V(y,t), & y > \eta \end{cases}$$

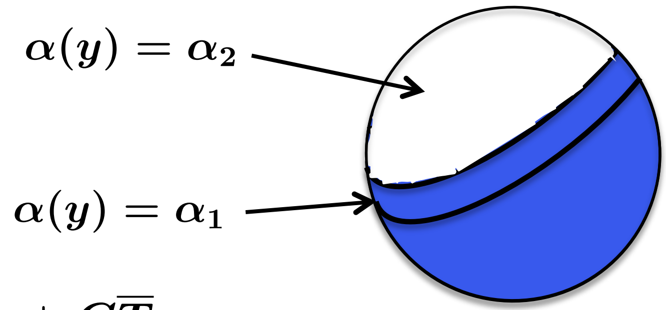
$$U(y,t) = u_0(t)p_0(y) + u_2(t)p_2(y)$$
$$V(y,t) = v_0(t)p_0(y) + v_2(t)p_2(y)$$
$$T(\eta,t) = \tfrac{1}{2}(U(\eta,t) + V(\eta,t))$$

Plug in U, V, equate coefficients of $p_0(y),\ p_2(y)$ ...

Change variables $w = \frac{1}{2}(u_0 + v_0), \quad z = u_0 - v_0$ ...

R. McGehee, E. Widiasih, A quadratic approximation to Budyko's ice albedo feedback model with ice line dynamics, preprint.

Latitude-dependent EBM: A project



$$
\left\{
\begin{aligned}
\dot{\eta} &= \epsilon(T_b - T_c) \\
R\dot{w} &= Q(1-\alpha_0) - A - (B+C)w + C\overline{T} \\
R\dot{z} &= Q(\alpha_2 - \alpha_1) - (B+C)z \\
R\dot{u}_2 &= Qs_2(1-\alpha_1) - (B+C)u_2 \\
R\dot{v}_2 &= Qs_2(1-\alpha_2) - (B+C)v_2,
\end{aligned}
\right.
$$

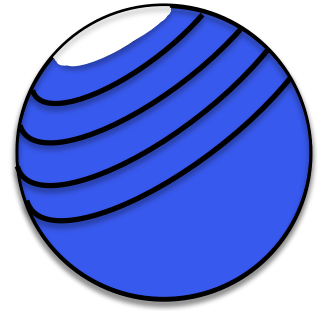
completely decouple, linear

where $\alpha_0 = \frac{1}{2}(\alpha_1 + \alpha_2)$, $\quad \overline{T} = w + \left(\eta - \frac{1}{2}\right)z + P_2(\eta)(u_2 - v_2)$,

$$T_b = w + \frac{1}{2}(u_2 + v_2)p_2(\eta).$$

Assume z, u_2, v_2 are at equilibrium ...

Latitude-dependent EBM: A project



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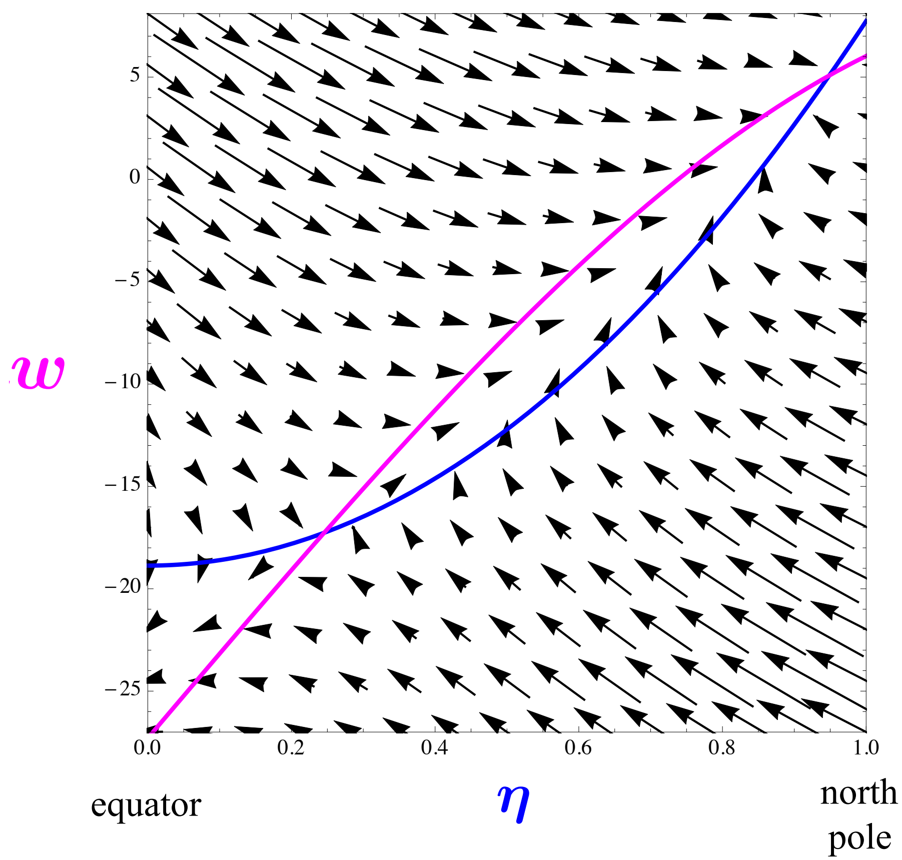

What did they think?

1	2	3	4	5
disagree	somewhat disagree	neutral	somewhat agree	agree

Question 1. **This course served to increase my desire to learn more about mathematical modeling via differential equations.**
Average response: **4.1**

Question 2. **The inclusion of material on climate modeling was a positive aspect of this course.**
Average response: **3.95**

Question 3. **The inclusion of material on climate modeling served to increase my desire to learn more about mathematical modeling.**
Average response: **3.65**

Question 4. **I would have an interest in taking a mathematics and climate course having Math 234 as a prerequisite.**
Average response: **3.15**

Question 5. **The required use of *Mathematica* was a positive aspect of this course.**
Average response: **3.5**

What did they think?

Second time will be the charm!

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