Using Inquiry to teach Mathematics and Thinking
Slogans for the Day

• Your students are the Thermians in the movie *Galaxy Quest*.

• Focus on the question rather than the answer.

• Help your students embrace confusion.
Teacher as Amateur Cognitive Scientist

How do we get our students to think and operate as mathematicians?
Teacher as Amateur Cognitive Scientist

Getting into our own heads:
How do we operate as mathematicians?
Your students are the Thermians in the movie *Galaxy Quest*. 
Culture is, by its very nature, completely **UNCONSCIOUS**
Cultural Elements

• We hold presuppositions and assumptions that are unlikely to be shared by a student who is new to mathematical culture.

• We have skills and practices that make it easier to function in our mathematical culture.

• We know where to focus of our attention and what can be safely ignored.
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What is a definition?

To a mathematician, it is the tool that is used to make an intuitive idea subject to rigorous analysis.

To anyone else in the world, including most of your students, it is a phrase or sentence that is used to help understand what a word means.
What does it mean to say that two partially ordered sets are order isomorphic?

The student’s first instinct is not going to be to say that there is an order-preserving bijection between them.
For every $\varepsilon > 0$ there exists a $\delta > 0$ such that if...
To make matters worse, we mathematicians do some very weird things with definitions.

**Definition:** Let $\Omega$ be a collection of non-empty sets. We say that the elements of $\Omega$ are **pairwise disjoint** if given $A, B$ in $\Omega$, either $A \cap B=\emptyset$ or $A = B$.

WHY NOT....

**Definition:** Let $\Omega$ be a collection of non-empty sets. We say that the elements of $\Omega$ are **pairwise disjoint** if given any two distinct elements $A, B$ in $\Omega$, $A \cap B=\emptyset$.

???
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In beginning real analysis, we define convergence of a sequence.

\[ a_n \to L \text{ means that } \forall \varepsilon > 0 \exists N \in \mathbb{N} \ni \forall n > N, \] \[ d(a_n, L) < \varepsilon. \]
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\[ a_n \to L \text{ means that } \forall \varepsilon > 0 \ \exists N \in \mathbb{N} \ \forall n > N, \ d(a_n, L) < \varepsilon. \]
Don’t just stand there! Do something.

- $a_n \to L$ means that $\forall \varepsilon > 0 \ \exists \ n \in \mathbb{N} \ \exists \ d(a_n, L) < \varepsilon$.

- $a_n \to L$ means that $\forall \varepsilon > 0 \ \exists \ N \in \mathbb{N} \ \exists \ for \ some \ n > N, \ d(a_n, L) < \varepsilon$.

- $a_n \to L$ means that $\forall N \in \mathbb{N}, \ \exists \ \varepsilon > 0 \ \exists \ N \forall n > N, \ d(a_n, L) < \varepsilon$.

- $a_n \to L$ means that $\forall N \in \mathbb{N} \ and \ \exists \ \varepsilon > 0 \ \exists \ n > N \exists d(a_n, L) < \varepsilon$.

Think of these as “alternatives” to the definition of sequence convergence. Come up with examples of real number sequences and limits that satisfy the “alternate” definitions but for which $a_n \to L$ is false.
Focus on the question rather than the answer
ORID
Steps in effective discussion
and in effective problem-solving!

• **Observation**---just make factual observations about what the context of the problem
• **Reflection**---reflect on what the observations mean and how they are relevant to the problem
• **Interpretation**---interpret the ideas from O and R bringing specificity to the conversation
• **Decisions and conclusions**---use the interpretive discussion to draw conclusions.
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Where to focus our attention
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1. For each \( n \in \mathbb{N} \), let \( A_n = \left( \frac{1}{2}, \frac{1}{2} + \frac{1}{n} \right) \).

   (i) Compute \( \bigcup_{n \in \mathbb{N}} A_n \).  
   (ii) Compute \( \bigcap_{n \in \mathbb{N}} A_n \).

   How would your answer change if the intervals were closed instead of open?

2. Let \( \mathbb{Q}^+ \) denote the positive rational numbers. For each \( r \in \mathbb{Q}^+ \), let \( D_r = \left( \frac{1}{2}, \frac{1}{2} + r \right) \).

   (i) Compute \( \bigcup_{r \in \mathbb{Q}^+} D_r \).  
   (ii) Compute \( \bigcap_{r \in \mathbb{Q}^+} D_r \).

3. For each \( r \in \mathbb{Q}^+ \), let \( D_r = \left( \frac{1}{2} - r, \frac{1}{2} + r \right) \).

   (i) Compute \( \bigcup_{r \in \mathbb{Q}^+} D_r \).  
   (ii) Compute \( \bigcap_{r \in \mathbb{Q}^+} D_r \).

4. For each \( r \in \mathbb{Q} \), let \( K_r \) be the set of all real numbers except \( r \).

   (i) Compute \( \bigcup_{r \in \mathbb{Q}} K_r \).  
   (ii) Compute \( \bigcap_{r \in \mathbb{Q}} K_r \).
Scenario: You are teaching a set theory-based intro proofs class. Your students have been using element arguments to show one set is a subset of another and to show set equality.

**Definition:** Let $S$ be a set. The power set of $S$, denoted by $\mathcal{P}(S)$ is the set of all subsets of $S$.

They very skillfully prove that if $A$ and $B$ are sets, then

$$
\mathcal{P}(A \cap B) = \mathcal{P}(A) \cap \mathcal{P}(B) \quad \text{and} \quad \mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)
$$

They are getting to be great at this!!
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Then they are asked to prove that $A \subseteq B$ if and only if $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

And things don’t go so well.

What do you think happens? Stumbling blocks do the students encounter? Why?
Observe

A ⊆ B implies ℙ(A) ⊆ ℙ(B)

and

ℙ(A) ⊆ ℙ(B) implies A ⊆ B

Subsets! These will require element arguments

Reflect?
Reminder: they skillfully proved that if $A$ and $B$ are sets, then

$$\mathcal{P}(A \cup B) \subseteq \mathcal{P}(A) \cup \mathcal{P}(B)$$

Why is it that $A \subseteq B$ implies $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ is actually very different?
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**Element Arguments have a standard Form**

**THEOREM**: If handy-hypothesis holds, then $X \subseteq Y$.

**Proof**: Let $s \in X$.

In the middle here we use handy-hypothesis to help us show that because $s$ is an element of $X$, it also satisfies any conditions necessary to put it in $Y$.

Therefore $s \in Y$. It follows that $X \subseteq Y$. \[\square\]
Observational Reflective Interpretive Decisional

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Therefore $s \in Y$. It follows that $X \subseteq Y$.

$(\implies)$ Proving If $A \subseteq B$, then $\mathcal{P}(A) \subseteq \mathcal{P}(B)$.

a. What is the “handy hypothesis”? What is $X$ and what is $Y$?

b. What is the first line of this element argument? What is the last line?

$(\iff)$ Proving If $\mathcal{P}(A) \subseteq \mathcal{P}(B)$, then $A \subseteq B$.

a. What is the “handy hypothesis”? What is $X$ and what is $Y$?

b. What is the first line of this element argument? What is the last line?
Help your students embrace confusion.
Some class exercises that help students embrace confusion

• Calculus I---geometry and derivatives

• Intro to proofs---power set workout
explorers are we, intrepid and bold
out in the wild, equipped with our wits,
a map, and a snack
we’re searching for fun

and we’re on the right track!