The Four Numbers Game

Maria Nogin

California State University
Fresno, California

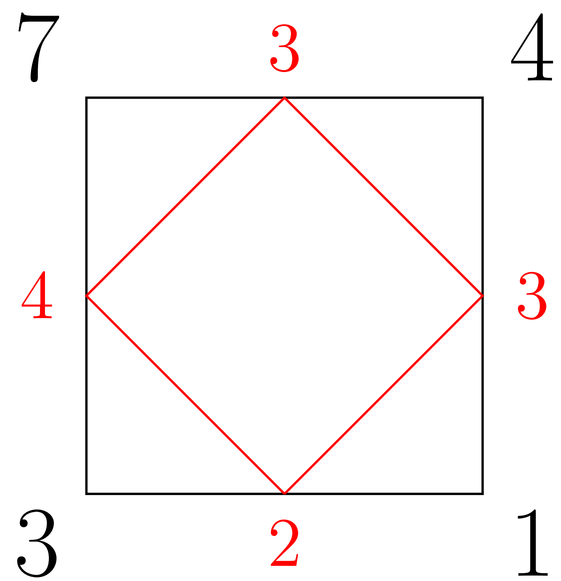
Joshua Belden

Clovis North High School
Fresno, California

Joint Mathematics Meetings
January 4, 2024

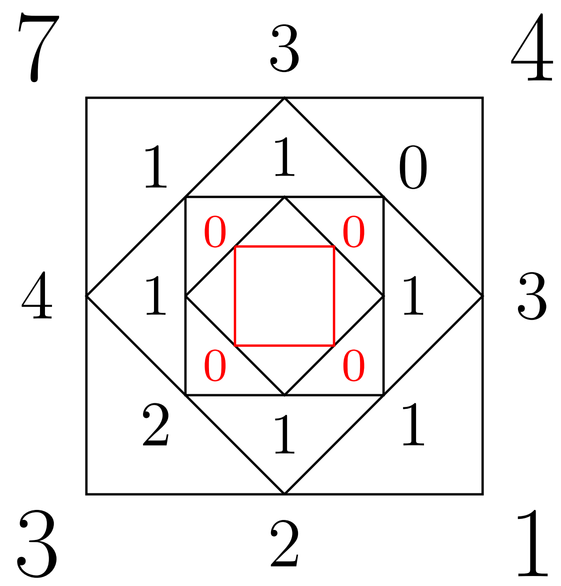
The game

- We place any four numbers at the vertices of a square.
- In each step, we connect the midpoints of the sides to form a new square and label each vertex with the difference of the endpoints of that side.



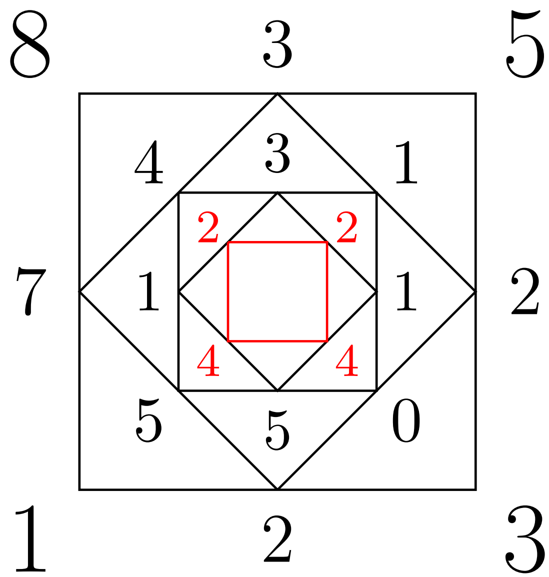
The game

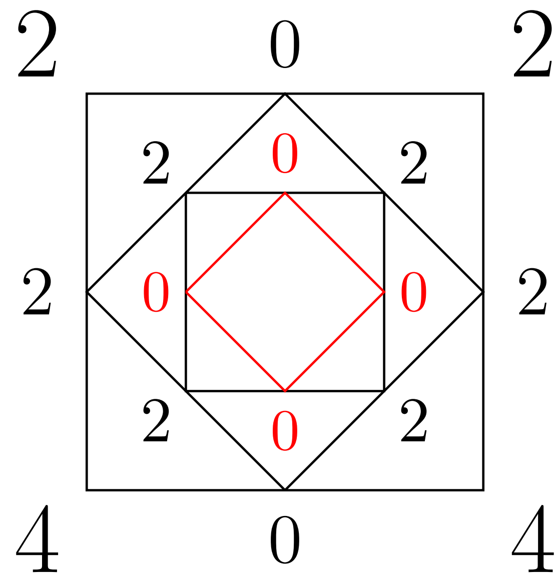
- We place any four numbers at the vertices of a square.
- In each step, we connect the midpoints of the sides to form a new square and label each vertex with the difference of the endpoints of that side.



- Keep performing steps...
    - In this example, we got four zeros after 4 steps. Any further steps will contain all zeros.

Another example





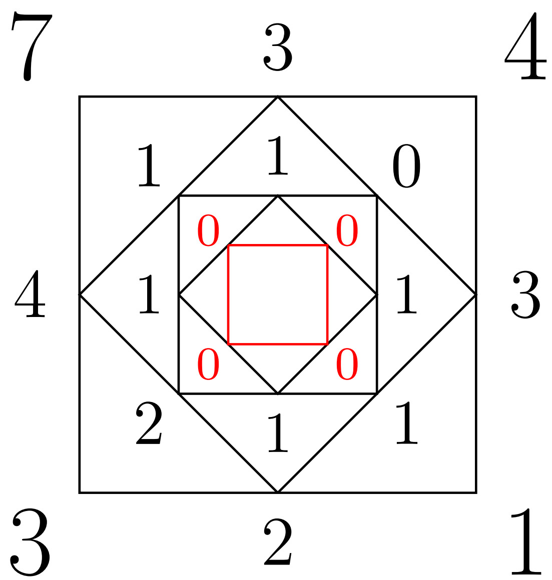
- We reached four zeros again! This time, it took 7 steps.
- Do we always reach four zeros in this game?

We experiment more. The participants choose their own four numbers to start with, perform a few steps, and see what happens... They get four zeros every time.

Length of a Game

When we reach four zeros, we say that the game is finite. The number of steps needed to reach four zeros is called the length of the game.

If four zeros are never reached, the game is called infinite.



This game has length 4.

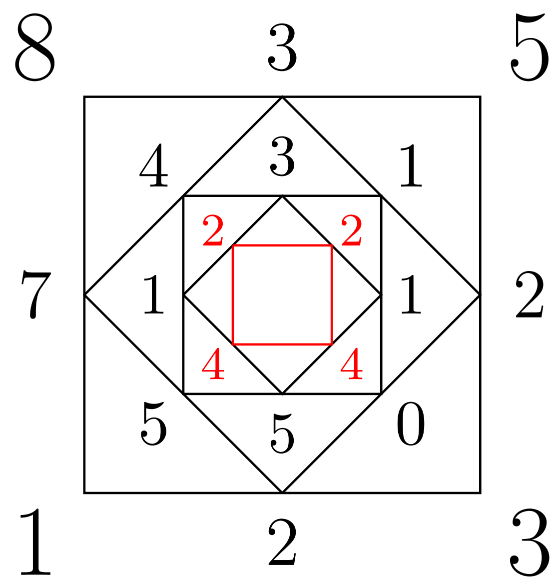
Questions and Observations

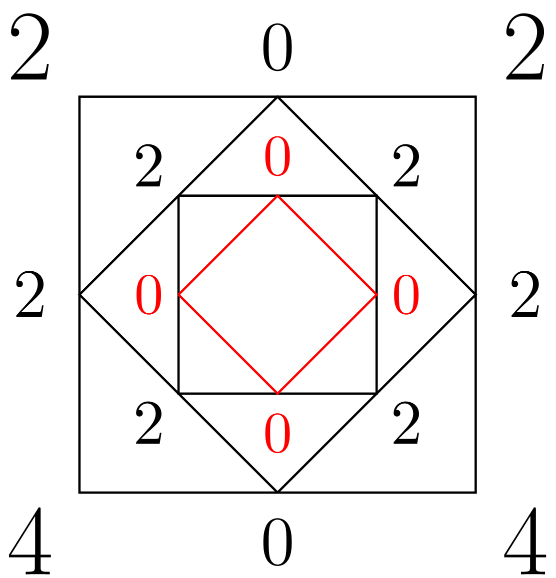
- Is every game finite?
- What could be the length of a game?
- What does the length depend on? In particular, would any of the following change the length of the game?
  - Each starting number is increased by a certain amount (no)
  - Each starting number is multiplied by a certain factor (no)
  - The four starting numbers are moved around by
    - a reflection in a line (no)
    - a rotation (no)
    - switching two adjacent numbers (sometimes)

Questions and Observations

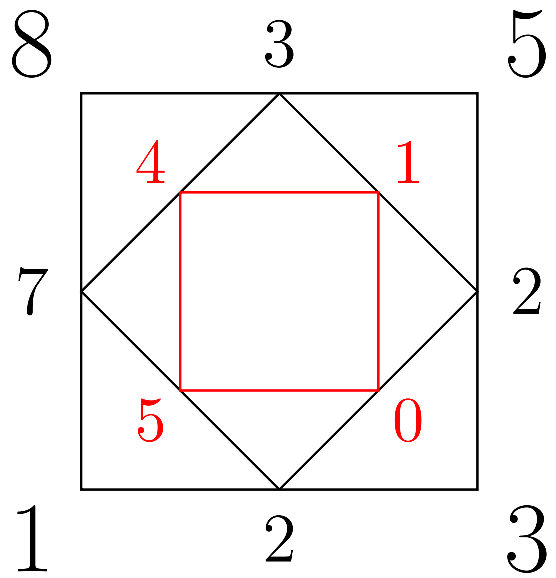
- Is every game finite?
- What could be the length of a game?

Look at ALL numbers in the process. What observations can we make?





A more compact notation



$$(8, 5, 3, 1)$$
$$(3, 2, 2, 7)$$
$$(1, 0, 5, 4)$$
$$(1, 5, 1, 3)$$
$$(4, 4, 2, 2)$$
$$(0, 2, 0, 2)$$
$$(2, 2, 2, 2)$$
$$(0, 0, 0, 0)$$

Each step: $(a, b, c, d) \quad \rightarrow \quad (|a-b|, |b-c|, |c-d|, |d-a|).$

Examples

$(7,4,1,3)$
$(3,3,2,4)$
$(0,1,2,1)$
$(1,1,1,1)$
$(0,0,0,0)$

$(8,5,3,1)$
$(3,2,2,7)$
$(1,0,5,4)$
$(1,5,1,3)$
$(4,4,2,2)$
$(0,2,0,2)$
$(2,2,2,2)$
$(0,0,0,0)$

$(75,34,12,0)$
$(41,22,12,75)$
$(19,10,63,34)$
$(9,53,29,15)$
$(44,24,14,6)$
$(20,10,8,38)$
$(10,2,30,18)$
$(8,28,12,8)$
$(20,16,4,0)$
$(4,12,4,20)$
$(8,8,16,16)$
$(0,8,0,8)$
$(8,8,8,8)$
$(0,0,0,0)$

Each step: $(a,b,c,d) \quad \rightarrow \quad (|a-b|,|b-c|,|c-d|,|d-a|).$

Order of numbers

Consider all possible orders of the numbers in the 4-tuple (a, b, c, d).

- $a \geq b \geq d \geq c$

  $(a, b, c, d) \to (a - b, b - c, d - c, a - d) \to$
  $(|a + c - 2b|, b - d, |a + c - 2d|, b - d) = (x, y, z, y) \to$
  $(p, q, q, p) \to (|p - q|, 0, |p - q|, 0) \to (r, r, r, r) \to (0, 0, 0, 0)$

- $a \geq c \geq b \geq d$

  $(a, b, c, d) \to (a - b, c - b, c - d, a - d) \to$
  $(a - c, b - d, a - c, b - d) \to (x, x, x, x) \to (0, 0, 0, 0)$

- $a \geq b \geq c \geq d$

  $(a, b, c, d) \to (a - b, b - c, c - d, a - d) \to$
  $(|a + c - 2b|, |b + d - 2c|, a - c, b - d)$

Parity of numbers

Each number in the 4-tuple (a, b, c, d) is either even or odd. Consider all possible combinations.

- (e, e, e, e)
- $(e, e, e, o) \to (e, e, o, o) \to (e, o, e, o) \to (o, o, o, o) \to (e, e, e, e)$
- $(e, e, o, o) \to (e, o, e, o) \to (o, o, o, o) \to (e, e, e, e)$
- $(e, o, e, o) \to (o, o, o, o) \to (e, e, e, e)$
- $(e, o, o, o) \to (o, e, e, o) \to (o, e, o, e) \to (o, o, o, o) \to (e, e, e, e)$
- $(o, o, o, o) \to (e, e, e, e)$

More questions

- Could a game have ANY finite length? (Can we create a game as long as we would like?)
- Can any 4-tuple be obtained as a step in such a game?
- What if instead of just nonnegative integers, we allow the starting numbers to be any integers? What can we say about the length of the game?
- What about any rational numbers?
- What about any real numbers?
- What if instead of playing with 4 numbers, we play a similar game with a different number of numbers? Say, 3, or 5, or even more numbers?

Examples of three and six numbers game

3 numbers	6 numbers
$(1, 2, 3)$	$(1, 2, 1, 2, 1, 3)$
$(1, 1, 2)$	$(1, 1, 1, 1, 2, 2)$
$(0, 1, 1)$	$(0, 0, 0, 1, 0, 1)$
$(1, 0, 1)$	$(0, 0, 1, 1, 1, 1)$
	$(0, 1, 0, 0, 0, 1)$

These games have infinite lengths.

More questions

- For what numbers k, each k-number game is finite?
- Do there exist k-number games of any finite length for any k?

Games Modulo n

Given $(a_1, a_2, ..., a_k)$, define the next step by
$\big((a_1 - a_2) \bmod n,\ (a_2 - a_3) \bmod n, \ldots,\ (a_k - a_1) \bmod n\big)$

Examples of a games modulo 5:

4 numbers	5 numbers
$(1, 2, 3, 4)$	$(0, 1, 4, 2, 3)$
$(4, 4, 4, 3)$	$(4, 2, 2, 4, 3)$
$(0, 0, 1, 4)$	$(2, 0, 3, 1, 4)$
$(0, 4, 2, 4)$	$(2, 2, 2, 2, 2)$
$(1, 2, 3, 4)$	$(0, 0, 0, 0, 0)$
infinite	finite

Open question. For each $k, n \in \mathbb{N}$, what are possible game lengths and possible cycle lengths of a k-number game modulo n?

Thank you!