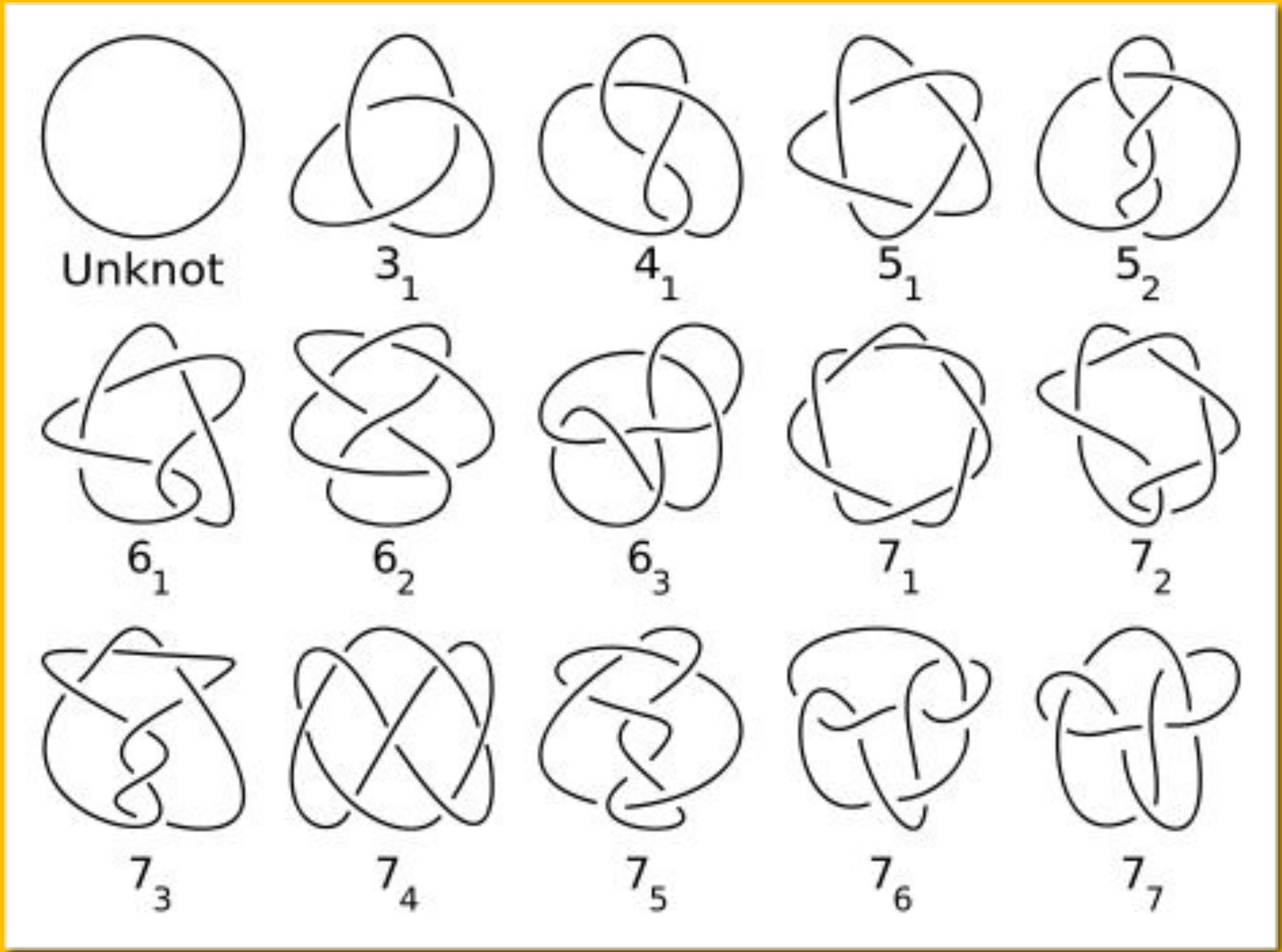
Coloring Carpets: Why They Look the Way They Do
Colin Adams and…

Michael Keyes Felix Nusbaum Tommy Clarke

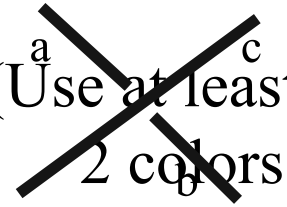
Jake Malarkey

Asher Rabinowitz


Unknot
3_1
4_1
5_1
5_2
6_1
6_2
6_3
7_1
7_2
7_3
7_4
7_5
7_6
7_7

Define a p-coloration of a picture of a knot as follows:

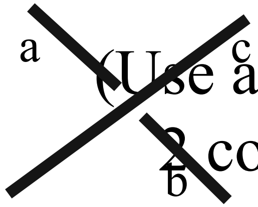
Label knot projections with p “colors”, denoted {0,1,2,…,p-1} such that at each crossing:



(Use at least 2 colors.)

a + b = 2c (mod p)

Label knot projections with p “colors”, denoted {0,1,2,…,p-1} such that at each crossing:



(Use at least 2 colors.)

$a + b = 2c \pmod{p}$ (or p divides a+b-2c)

<u>Example:</u>

A 5-coloring

5 divides
a+b-2c

Halifax **Bridges 2023 Conference Proceedings**

Oh What a Complex Rug We Weave When First We Color Then Perceive

Barry Cipra[1] and Paul Zorn[2]

[1]Northfield, Minnesota; bcipra@rconnect.com
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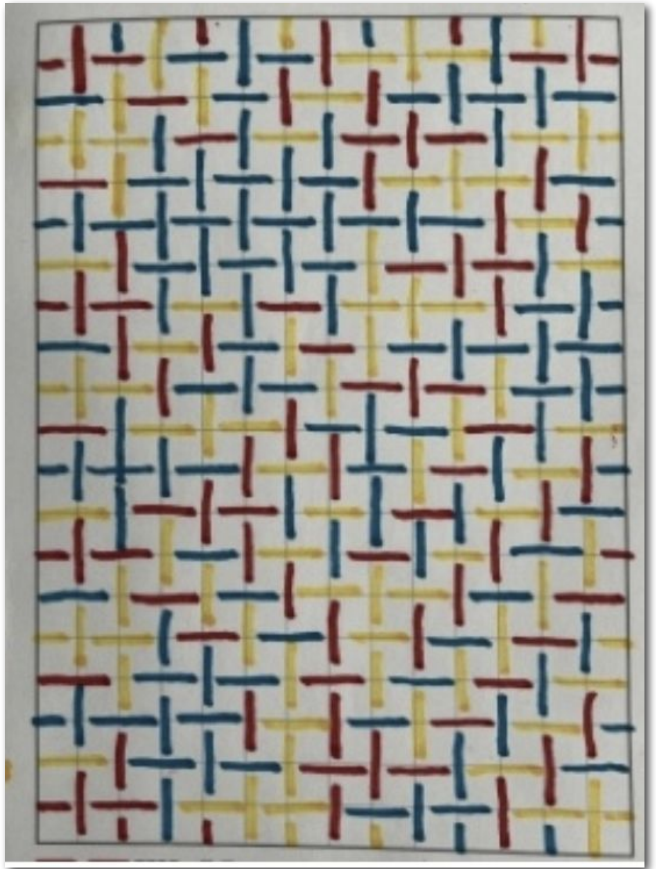
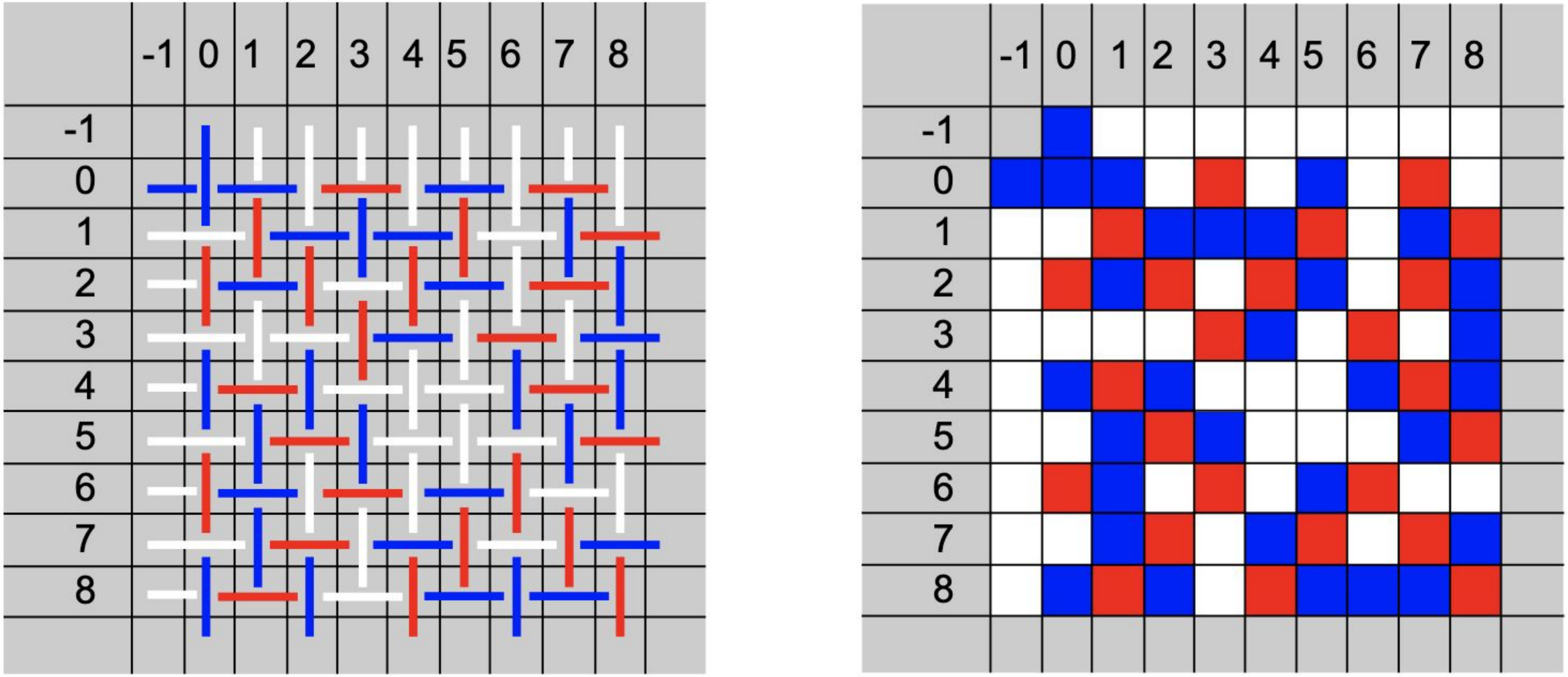
A 3-coloring

Periodic fringe

Periodic fringe



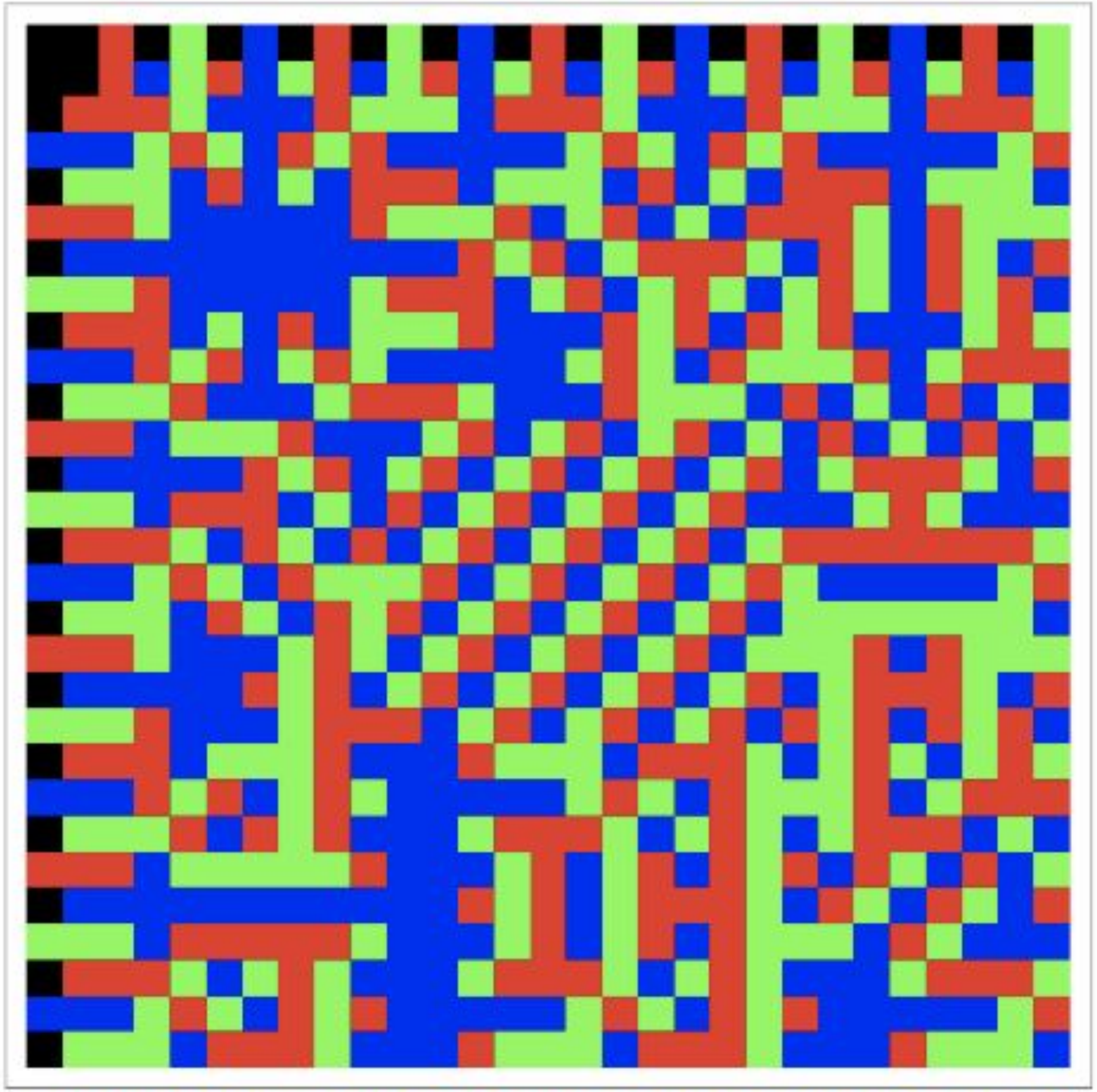


(a) Strand labelings

(b) Pixel labelings

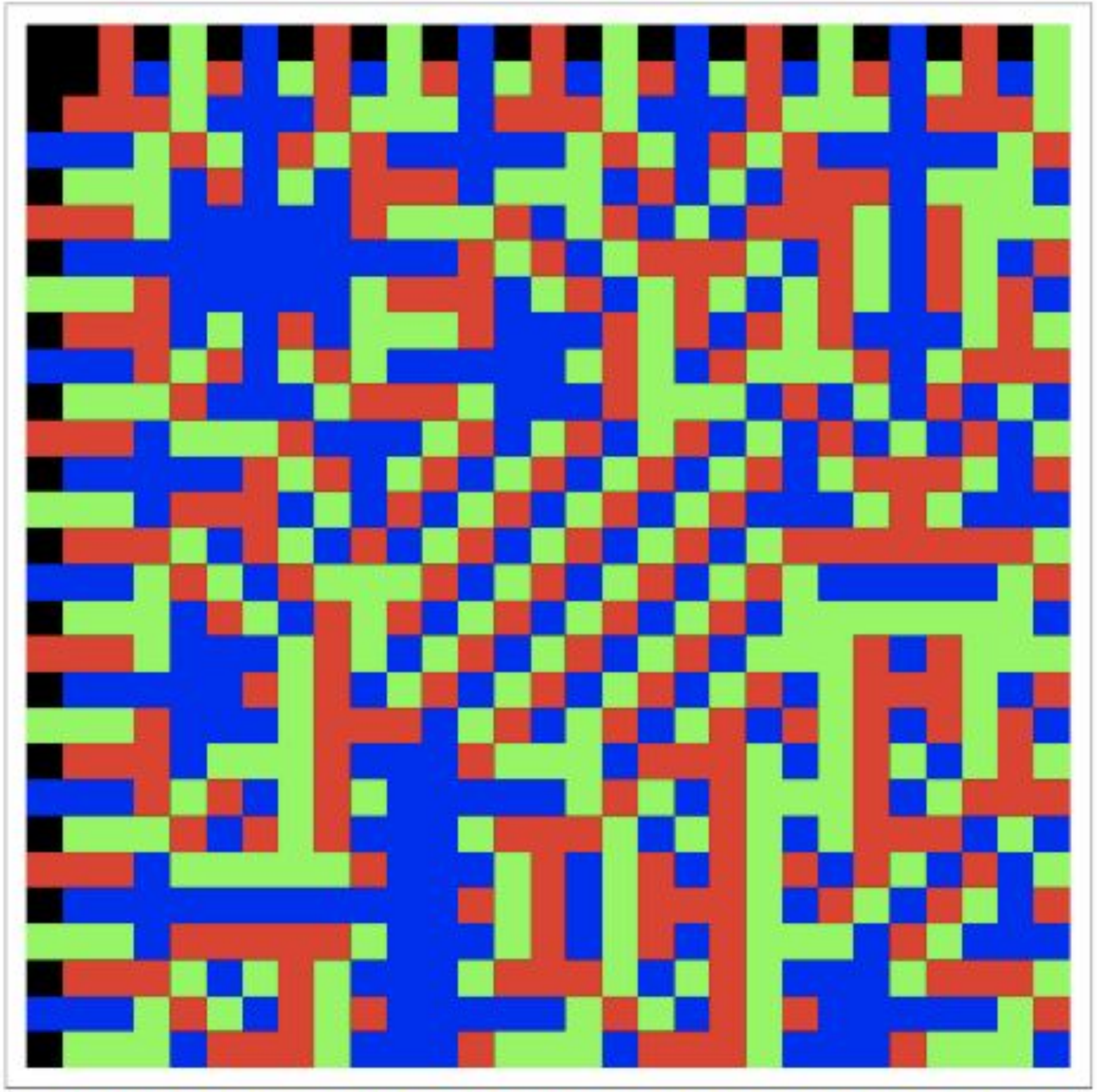
Figure 1.11: Rows and columns labeled on C_0^3

3-color carpet with periodic fringe

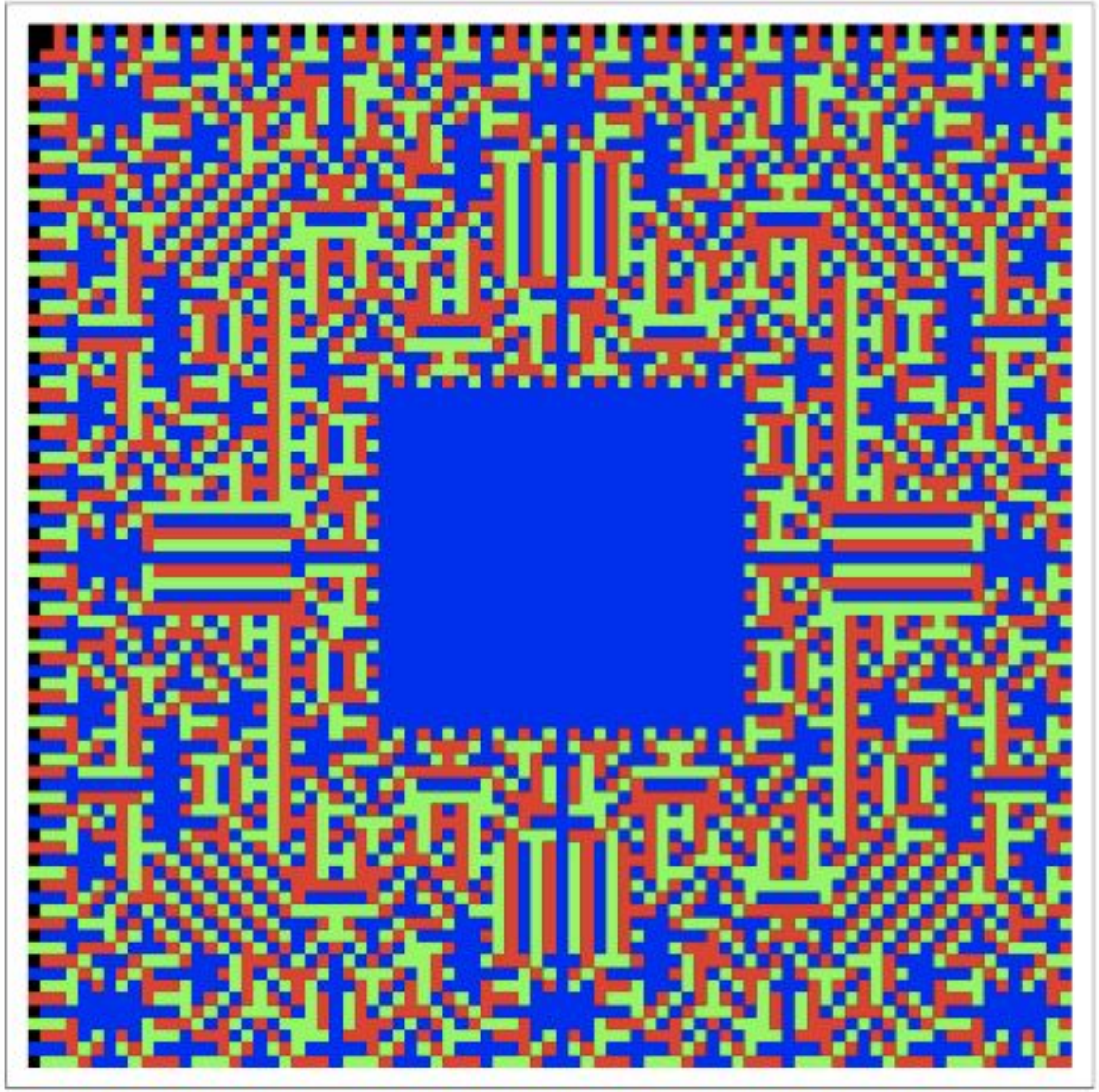


27 x 27

3-color carpet with periodic fringe

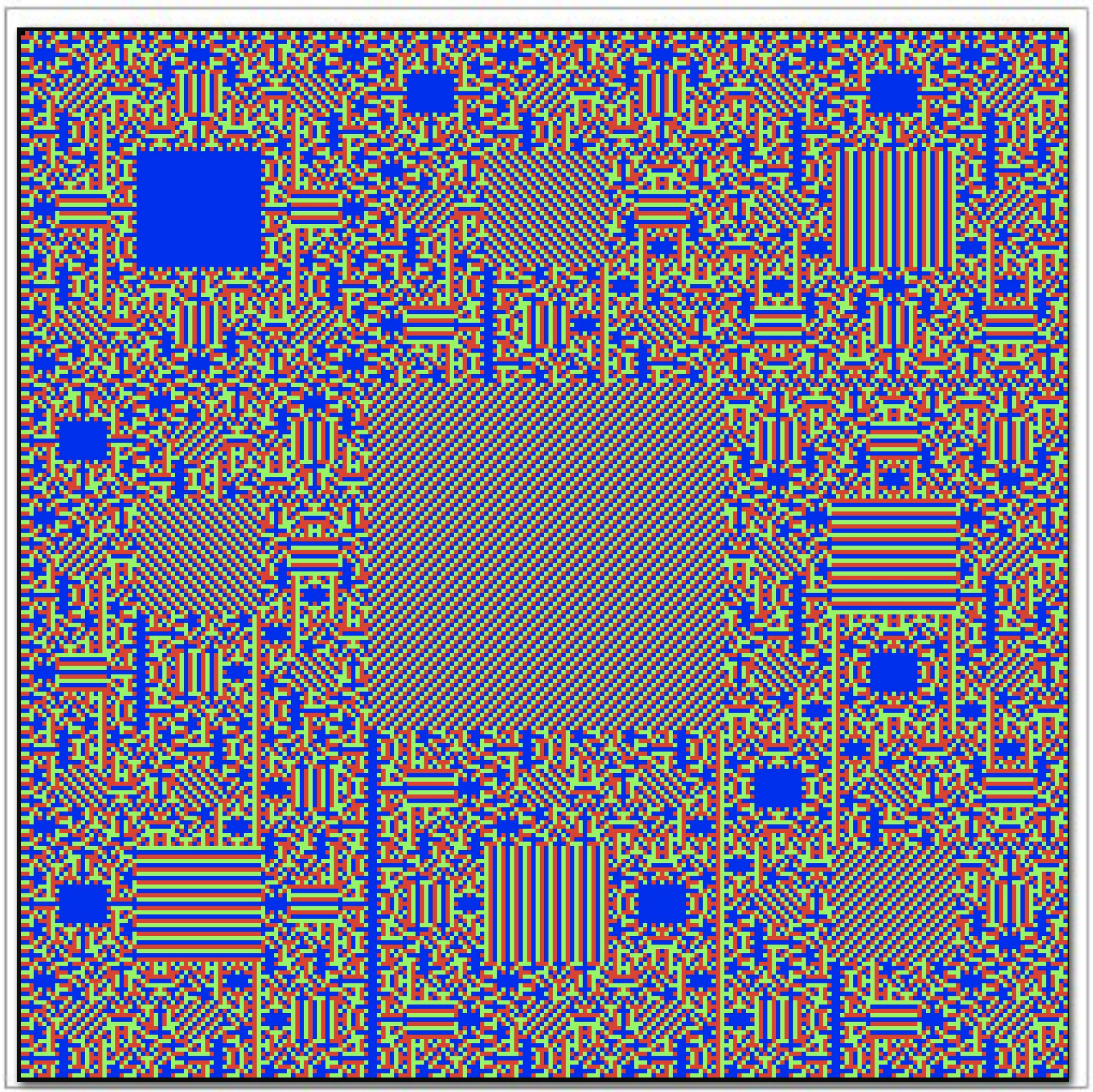


27 x 27



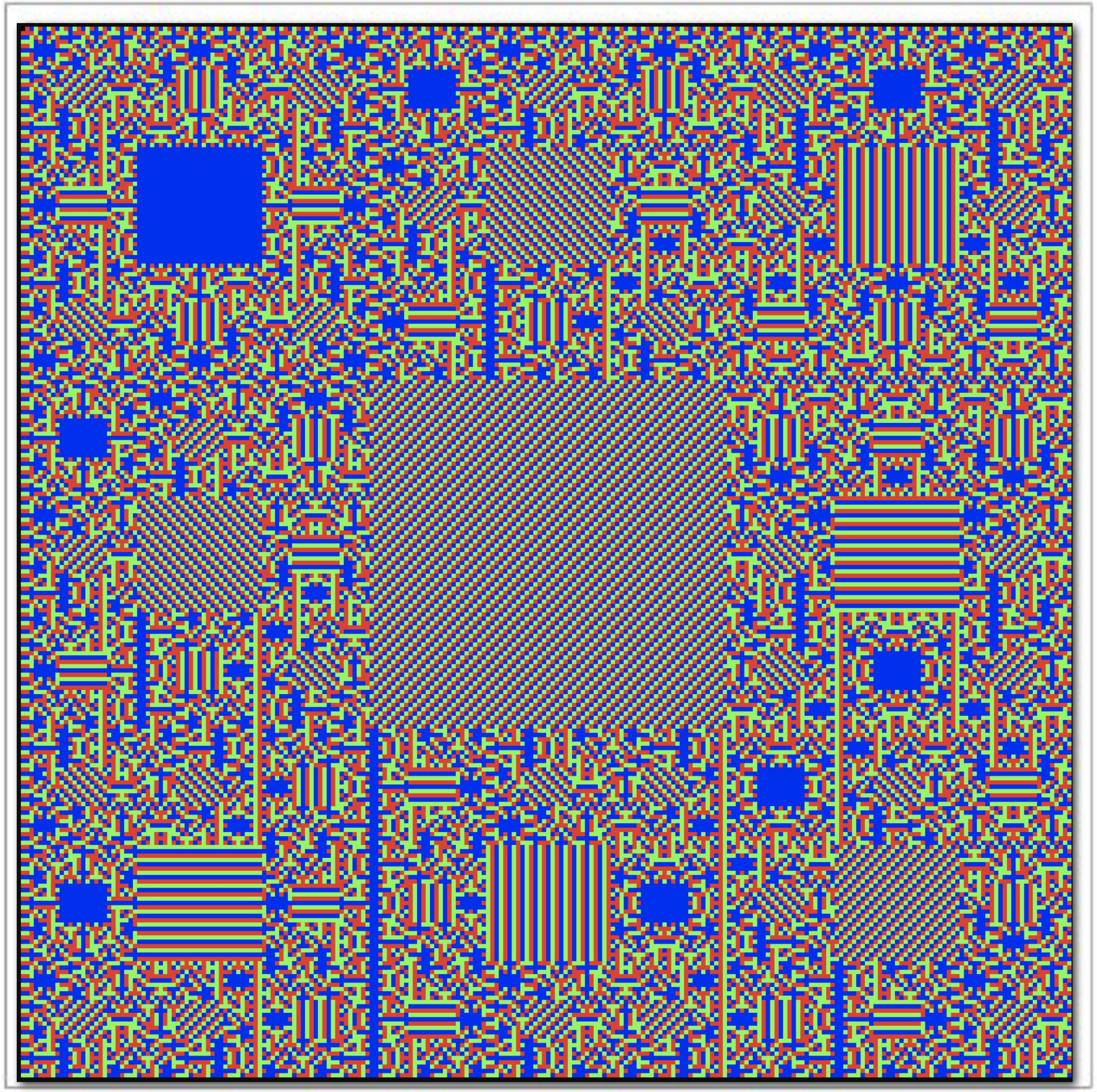
81 x 81

3-color carpet with periodic fringe

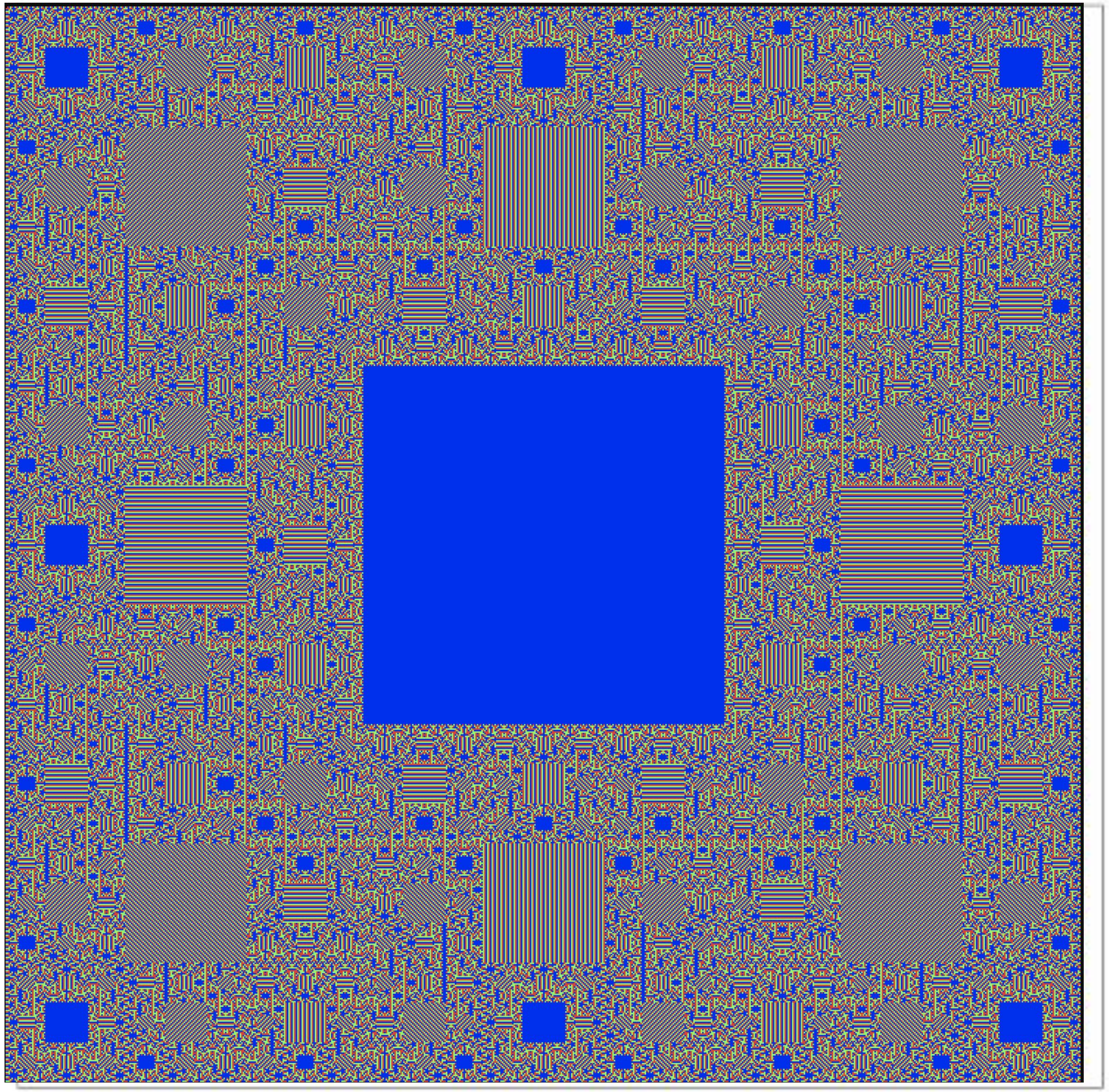


243 x 243

3-color carpet with periodic fringe



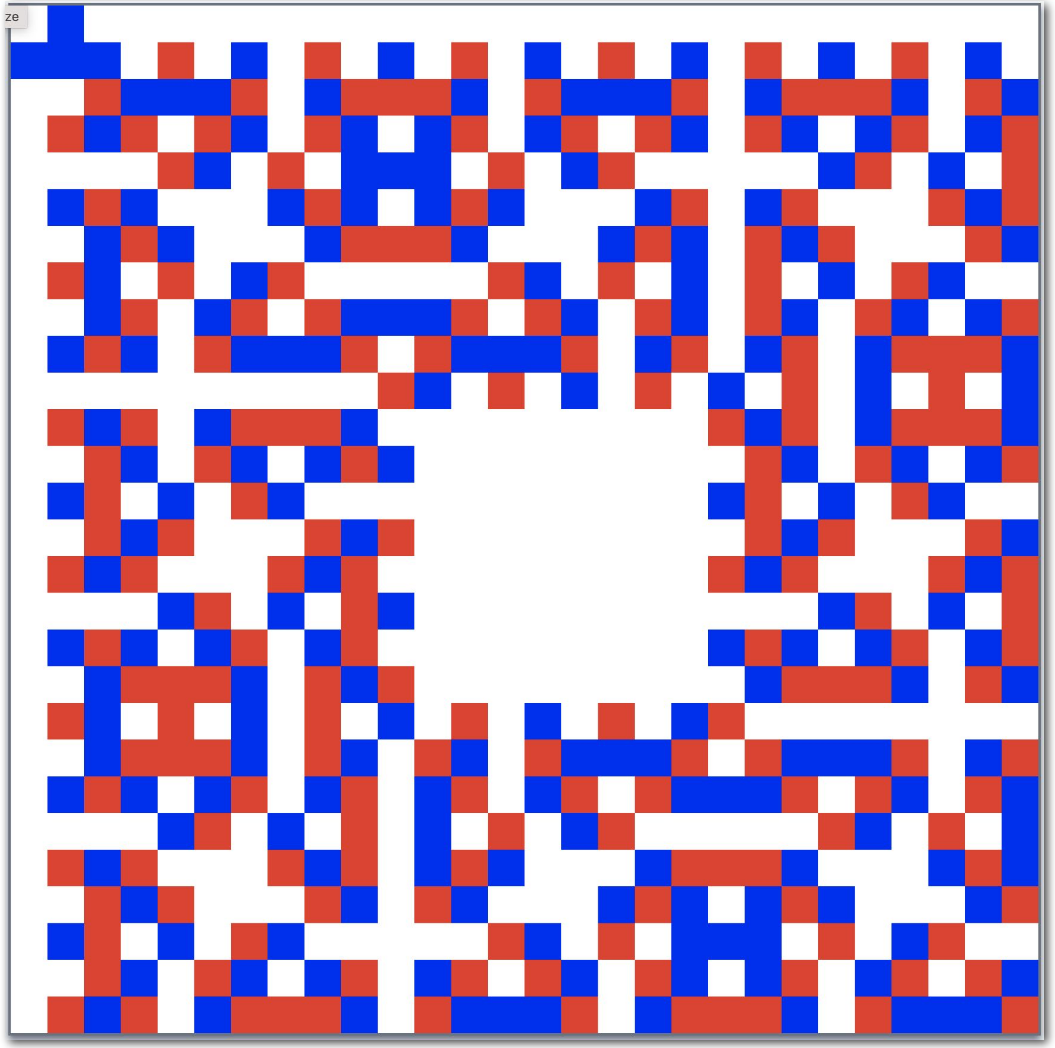
243 x 243



729 x 729

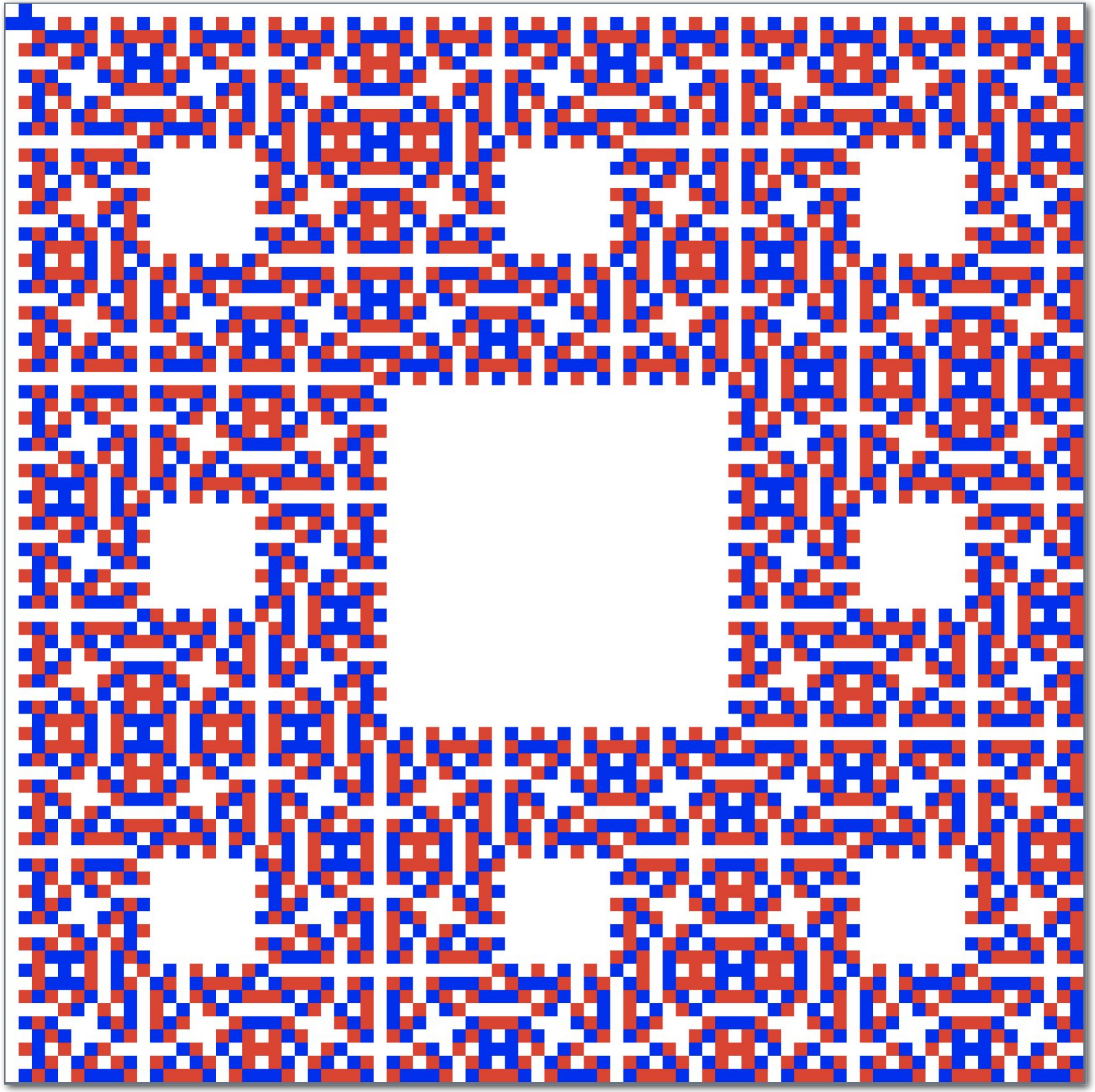
Definition 2.3

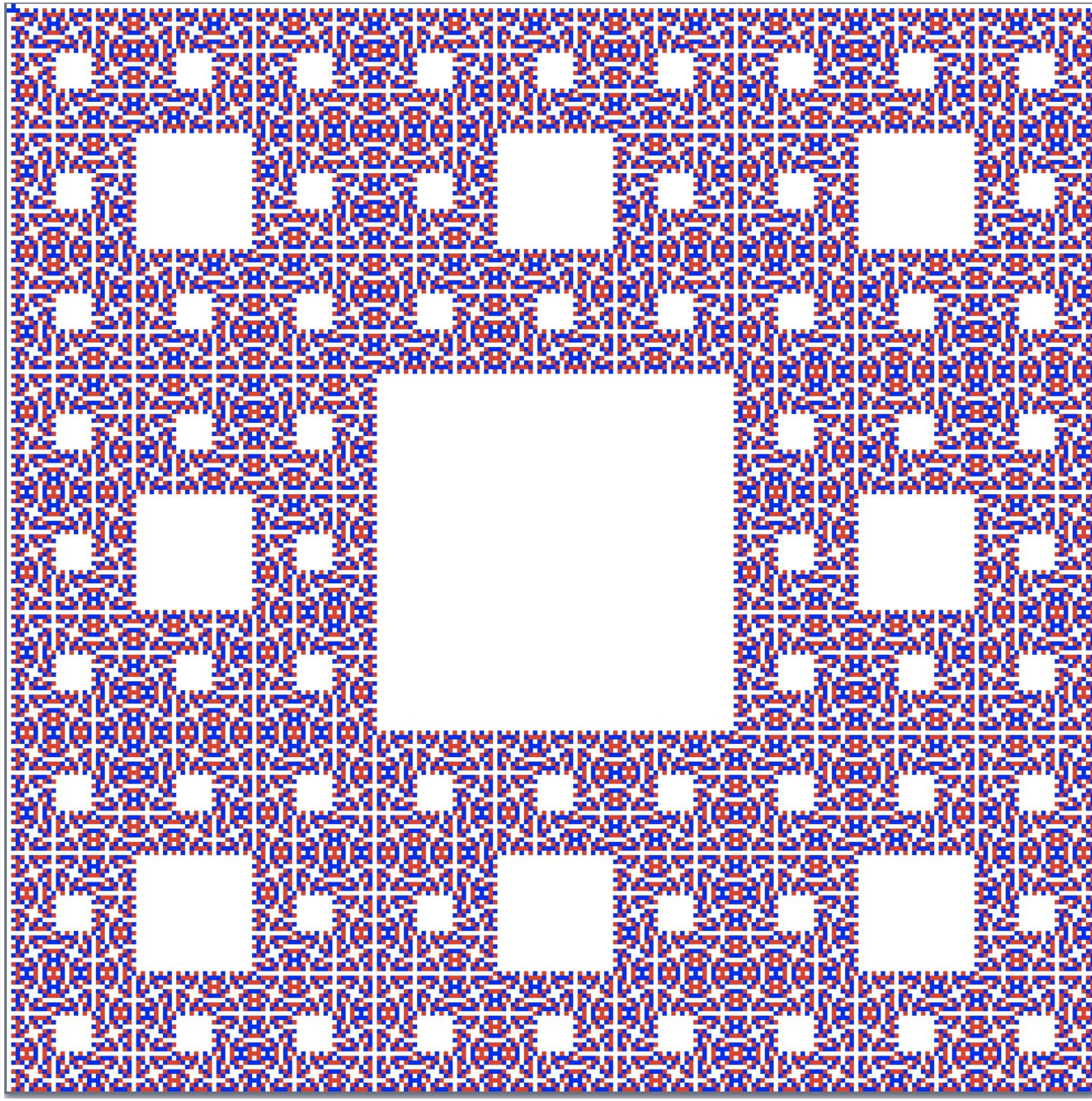
A q-colored carpet X is a **difference carpet** *if $a_0 = b_0 = 1$ and $a_i = b_i = 0$ for all $i \geq 2$. We denote it by C_0^q.*

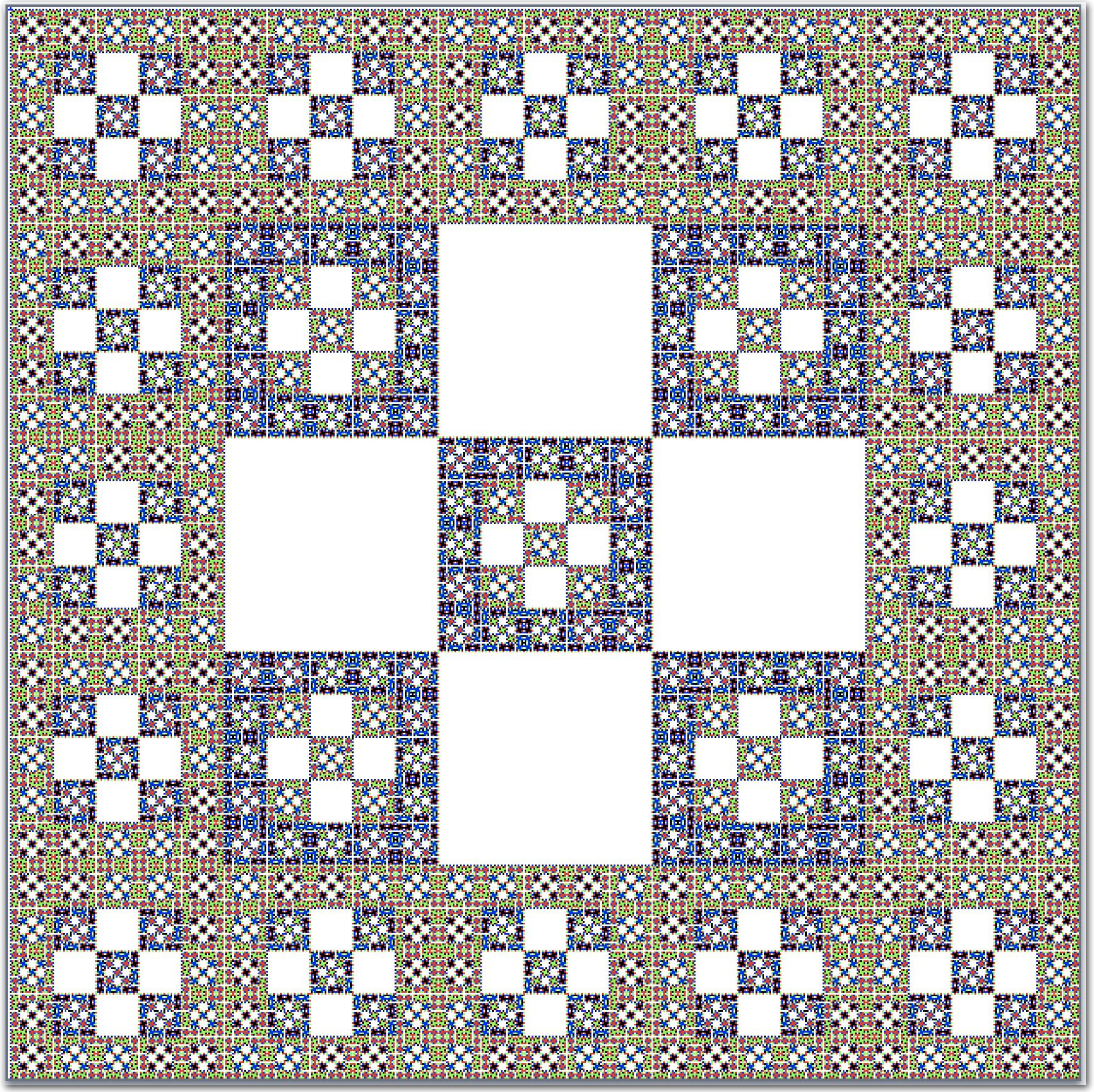


C_0^3

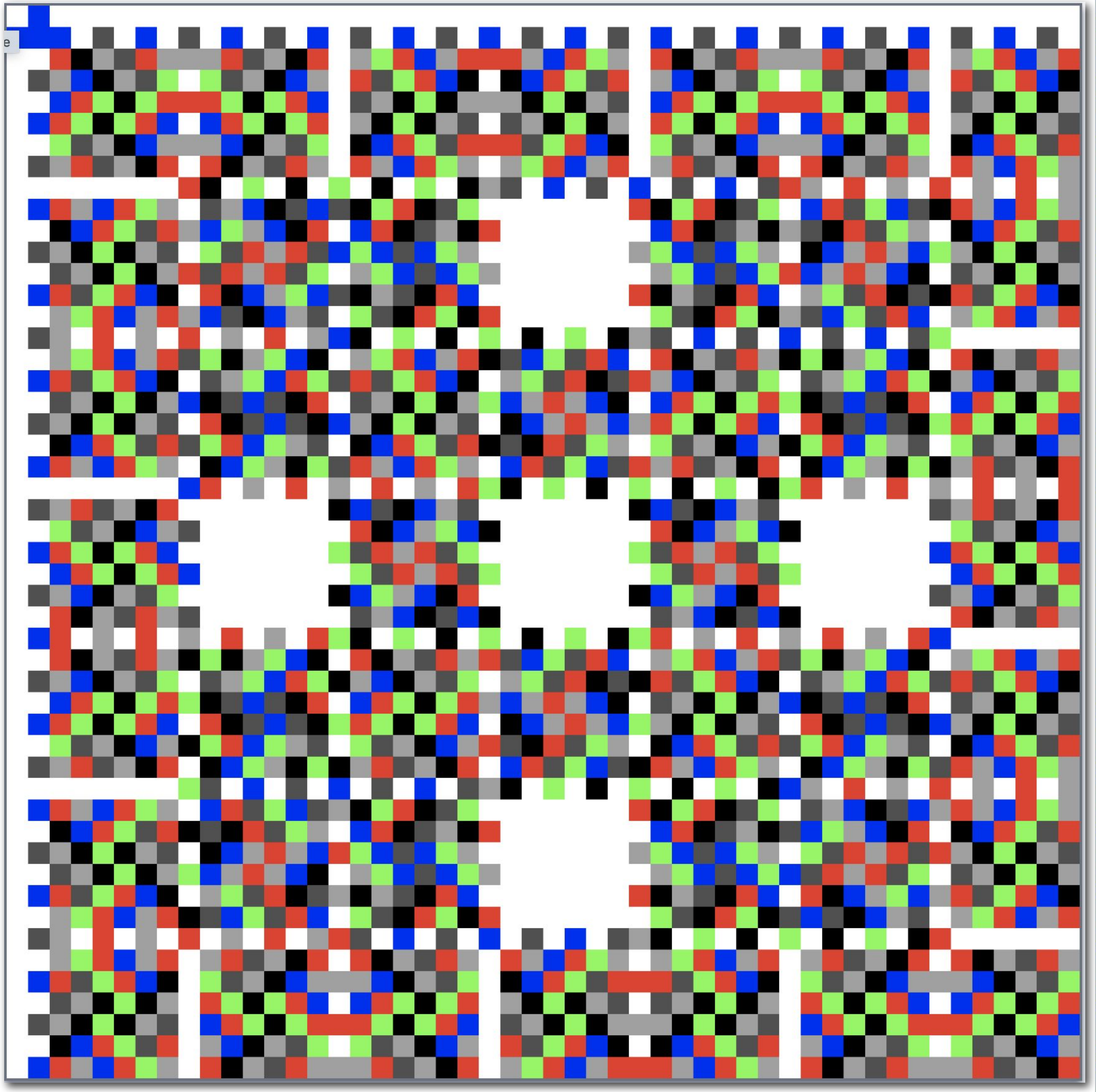
3-color difference carpet



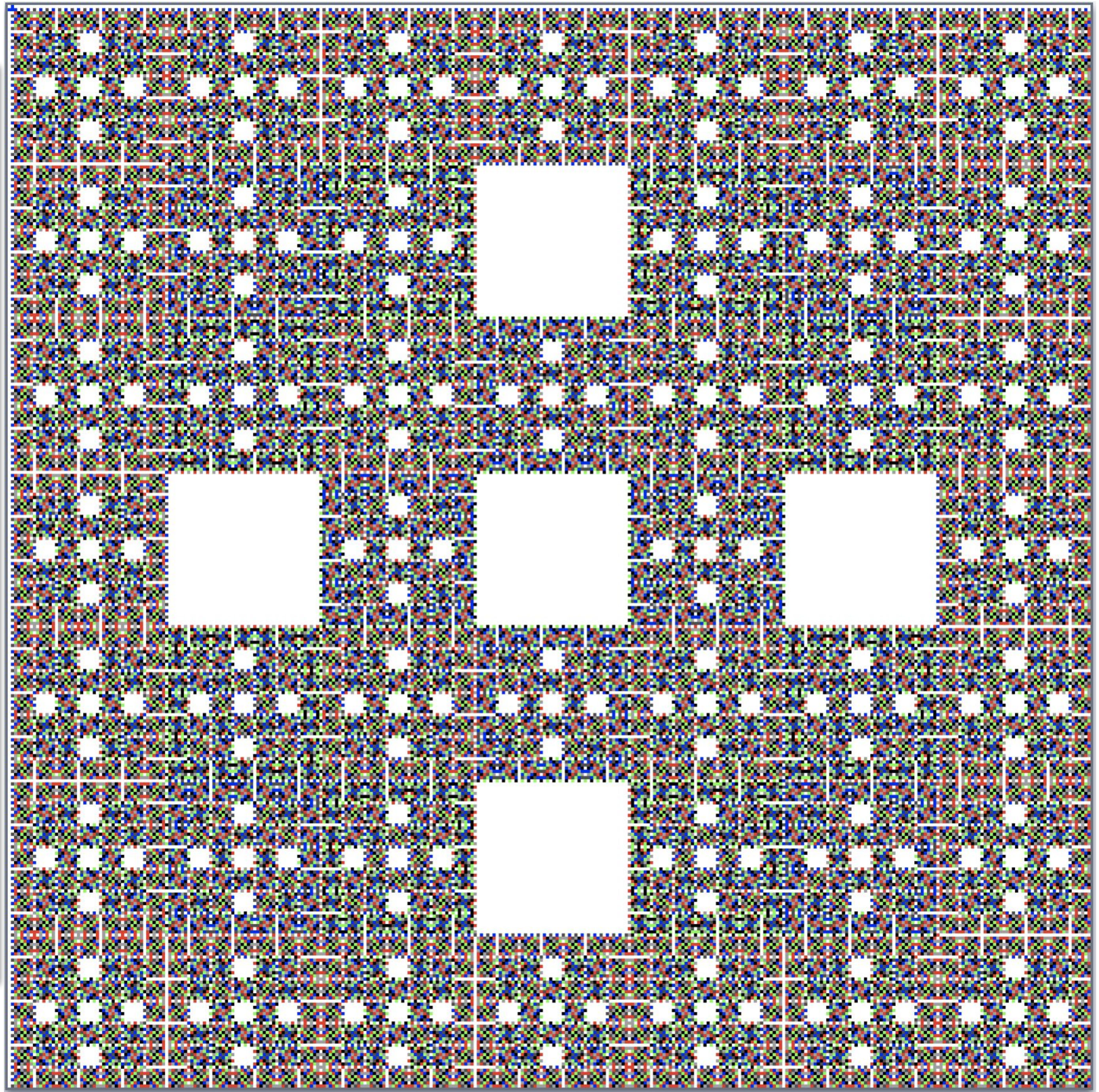


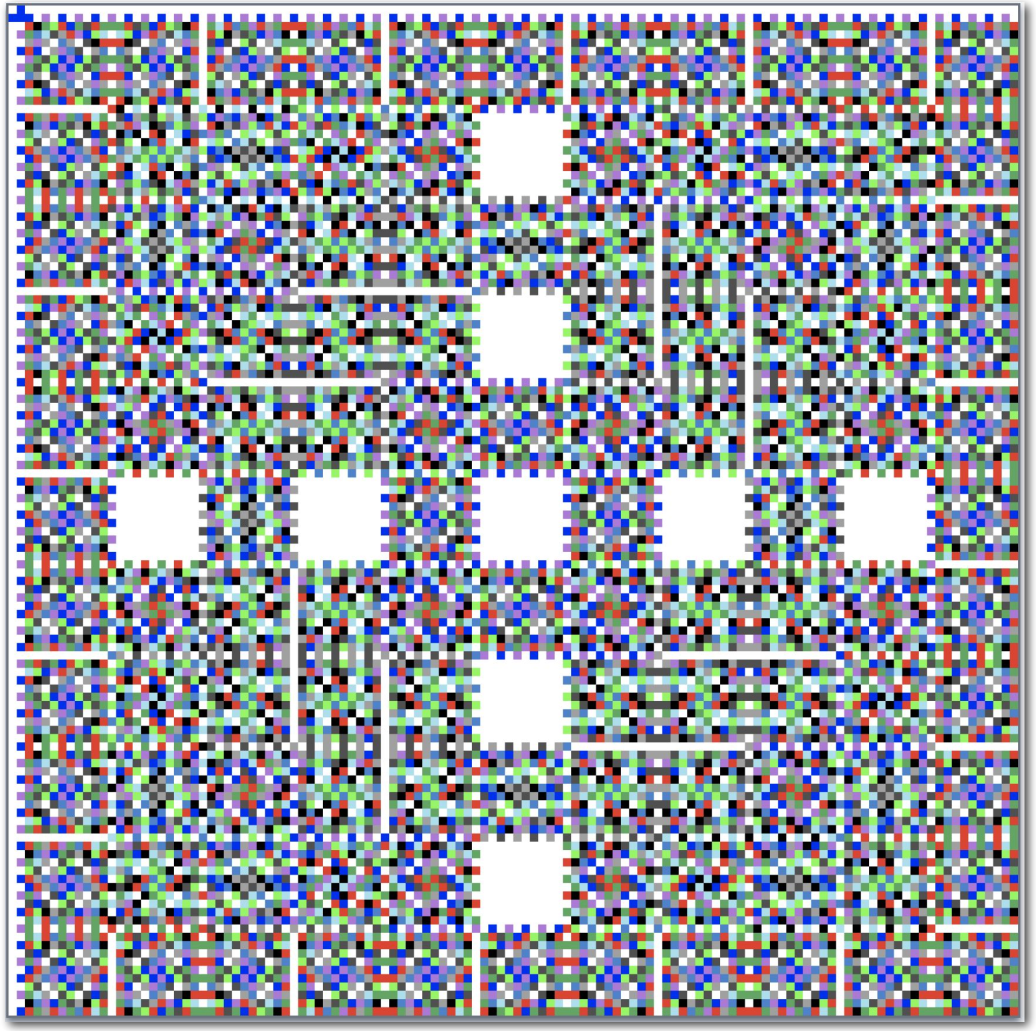


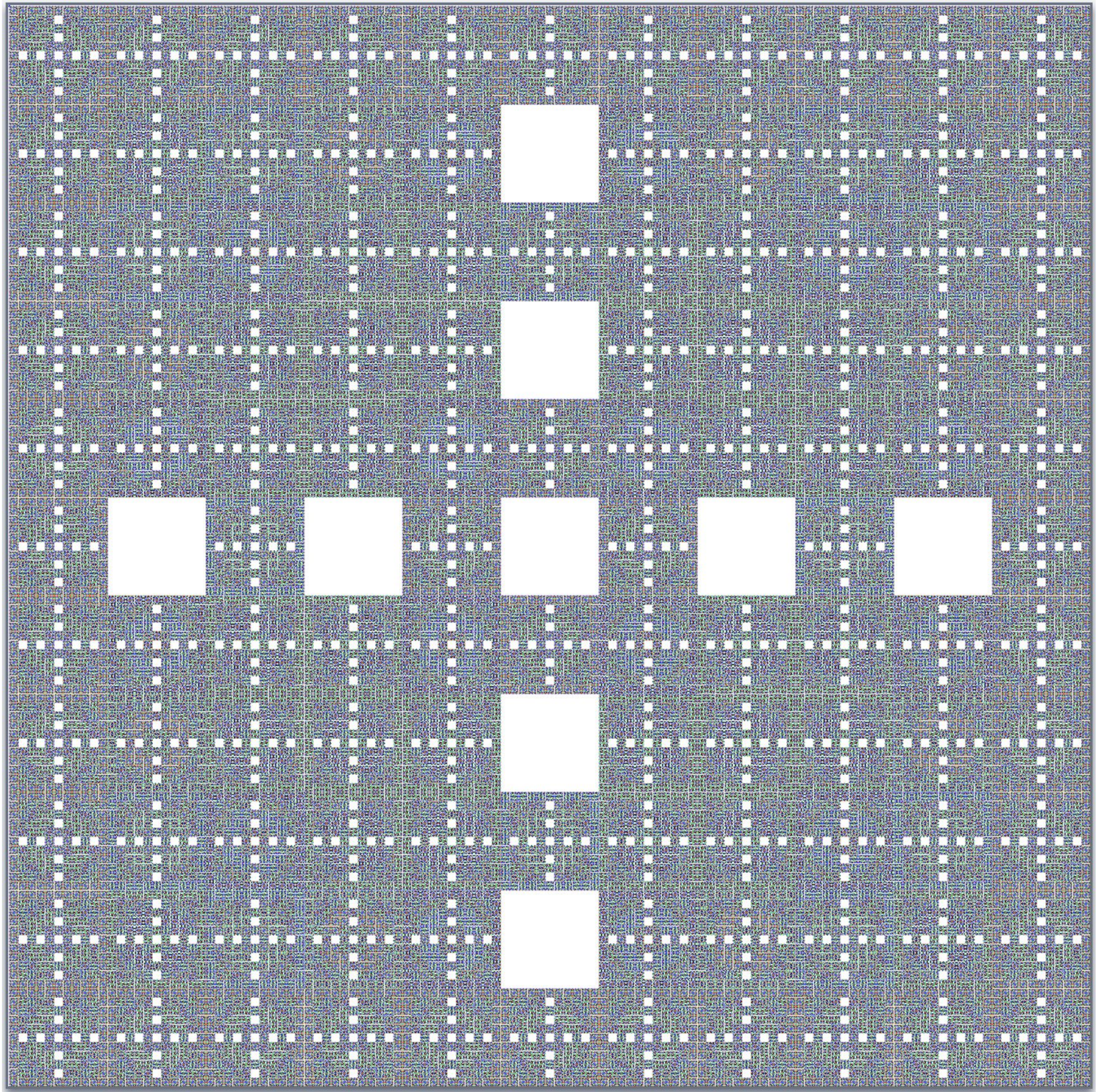
C_0^5



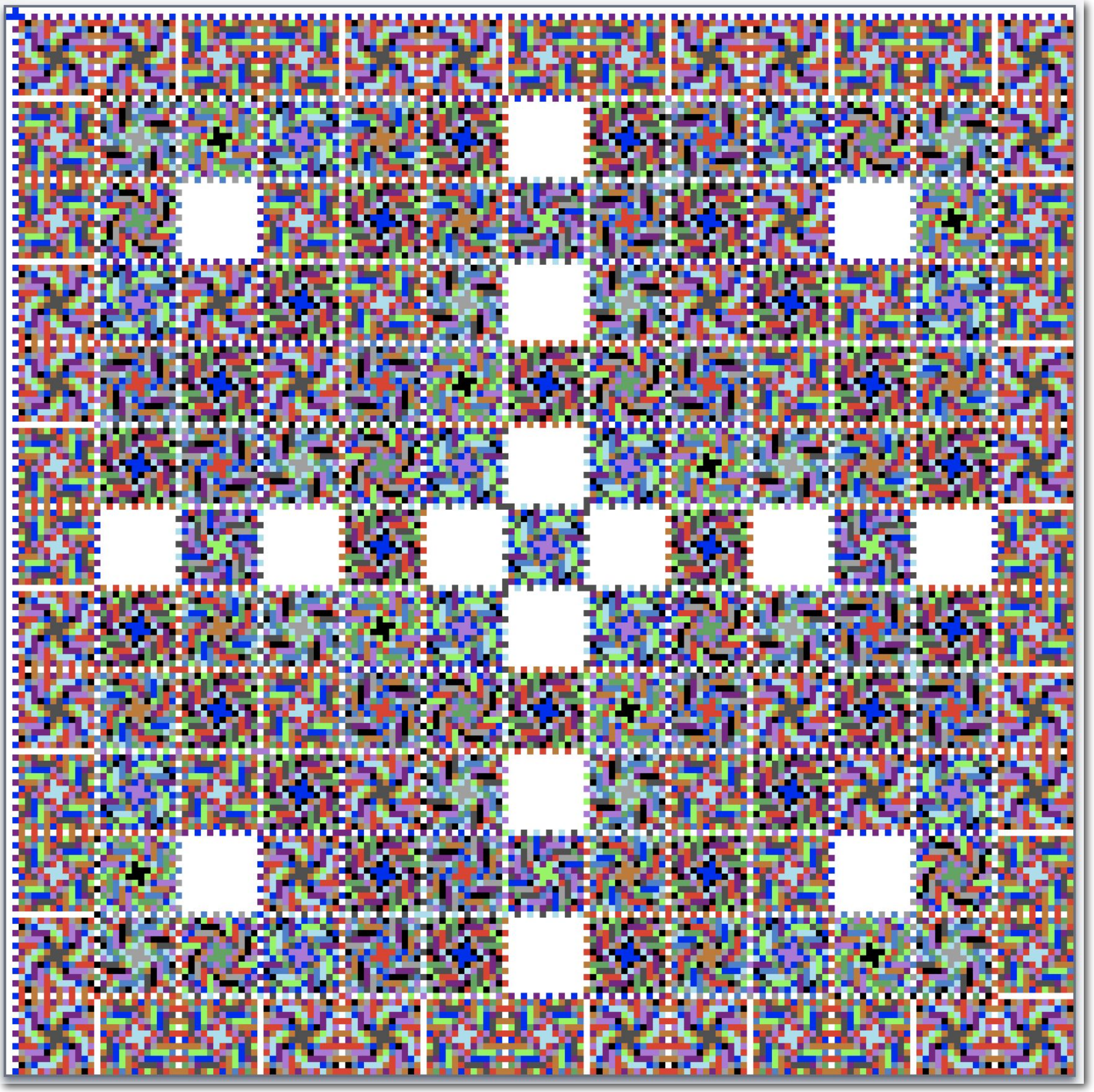
p = 7



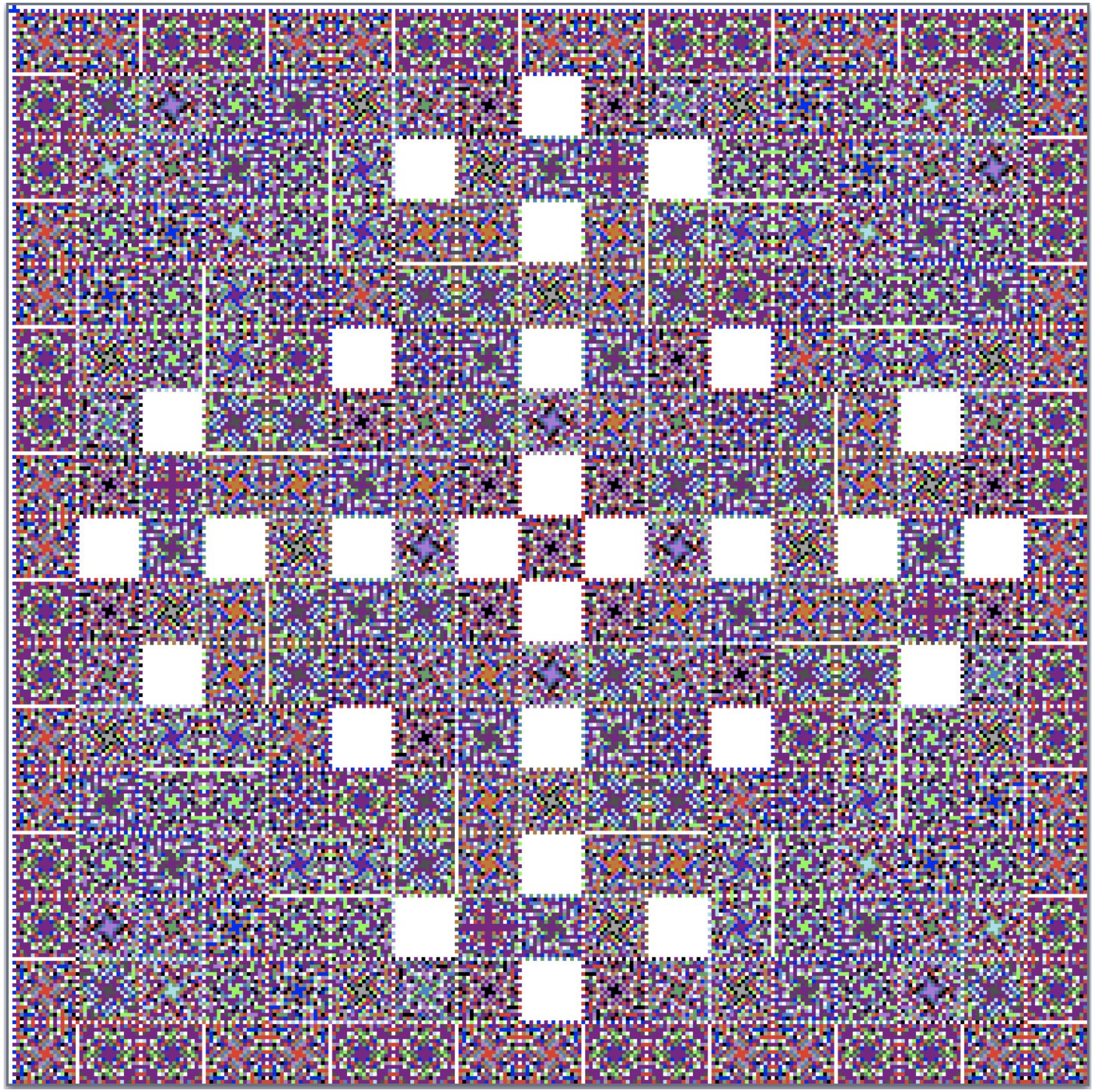




p = 11



p = 13



p = 17

Goal:

Understand why these carpets look the way they do!

Definition 2.1

Let $q \geq 0$. *Define* $\mathbb{V}_q$ *to be the set of all* q*-colored carpets*

Theorem 2.1

When p *is a prime,* $\mathbb{V}_p$ *forms a vector space over* $\mathbb{Z}_p$.

When q *is composite,* $\mathbb{V}_q$ *forms a* $\mathbb{Z}_q$*-module.*

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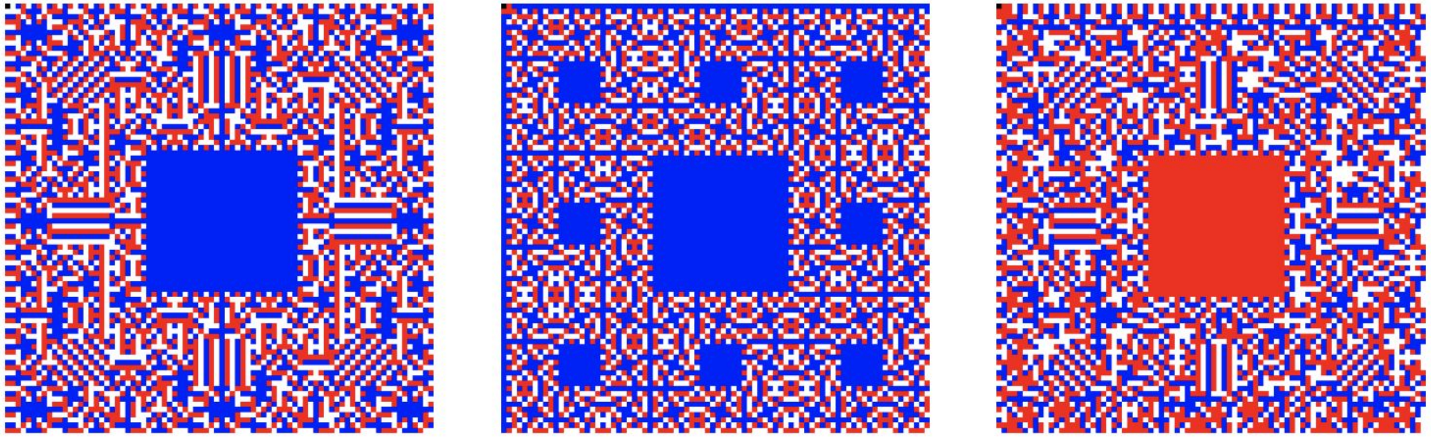

(a) X (b) Y (c) $X + Y$

Figure 2.1: Adding two 3-colored carpets.

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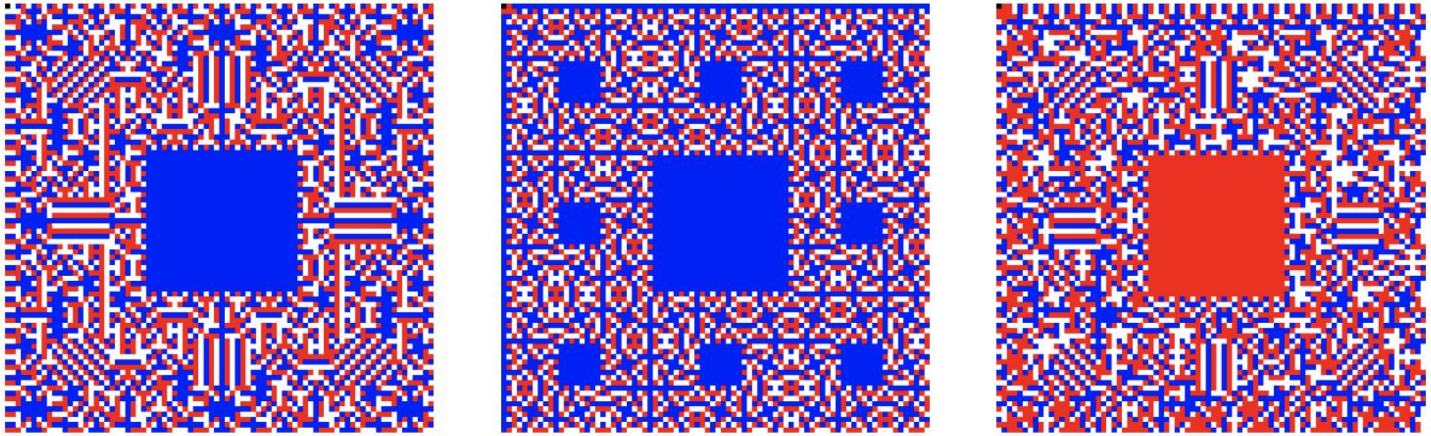

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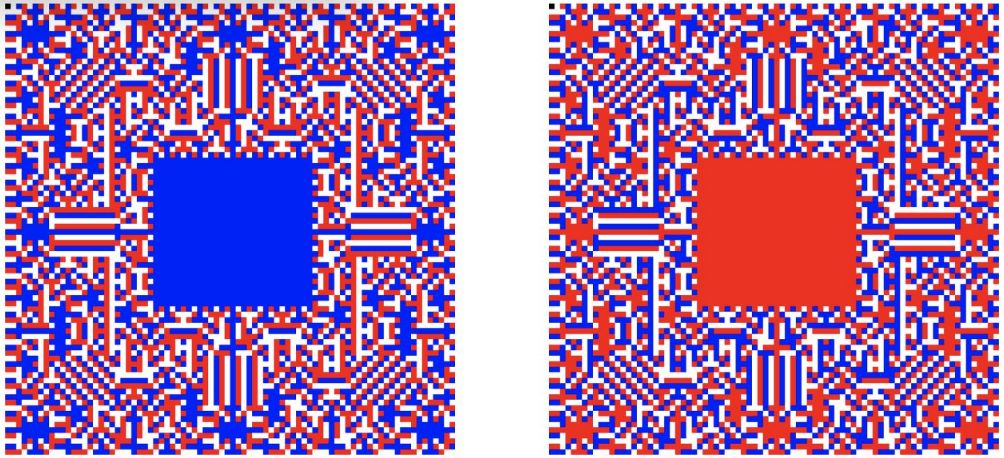
Figure 2.1: Adding two 3-colored carpets.

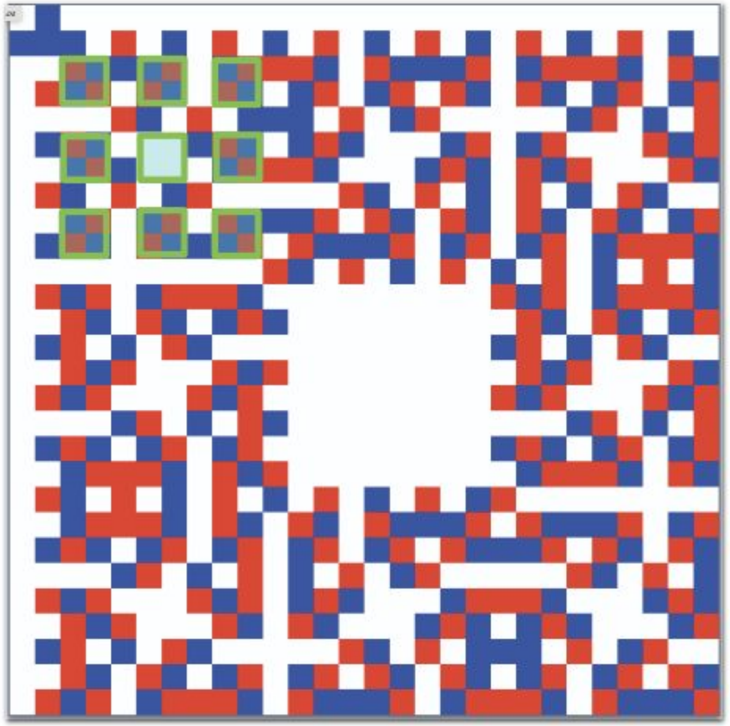


(a) X (b) 2X

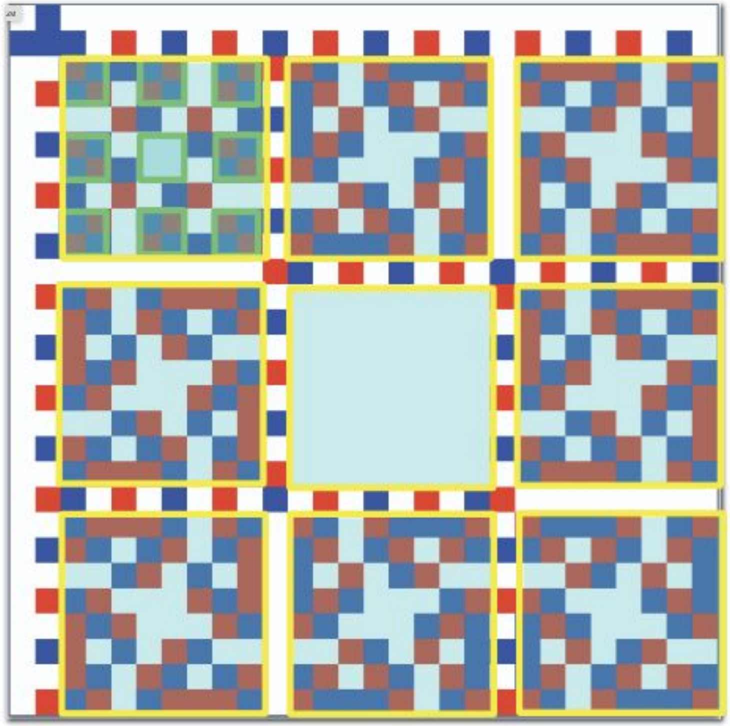
Figure 2.2: Multiplying a p-colored carpet by 2 permutes colors.

Definition 4.1

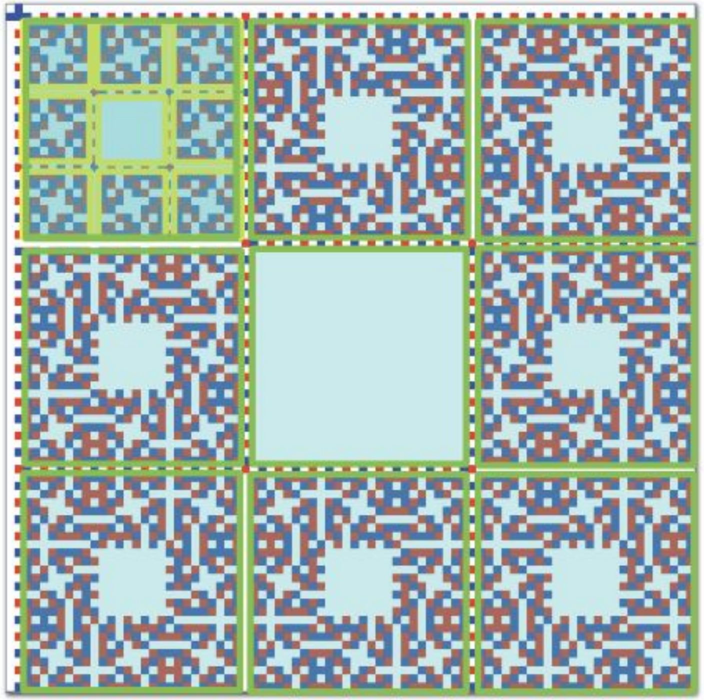
In C_0^q, for $n \geq 1$, an n**-brick** *is a $(q^n - 1) \times (q^n - 1)$ square of strands with the top left strand located at $(aq^n + 1, bq^n + 1)$ for some $a, b \in \mathbb{Z}_{\geq 0}$. Let an n-brick in C_0^q with top left strand located at $(aq^n + 1, bq^n + 1)$ be denoted $\boldsymbol{Br_n(a, b)}$. We say an n-brick is* **vertical** *if $a + b$ is even and* **horizontal** *if $a + b$ is odd, since this will determine if the strand in the upper left corner is vertical or horizontal. We denote the color of the strand in the upper left corner of $Br_n(a, b)$ by $\mathbf{UL_n(a, b)} = C_0^q(aq^n + 1, bq^n + 1)$.*



1-bricks



2-bricks

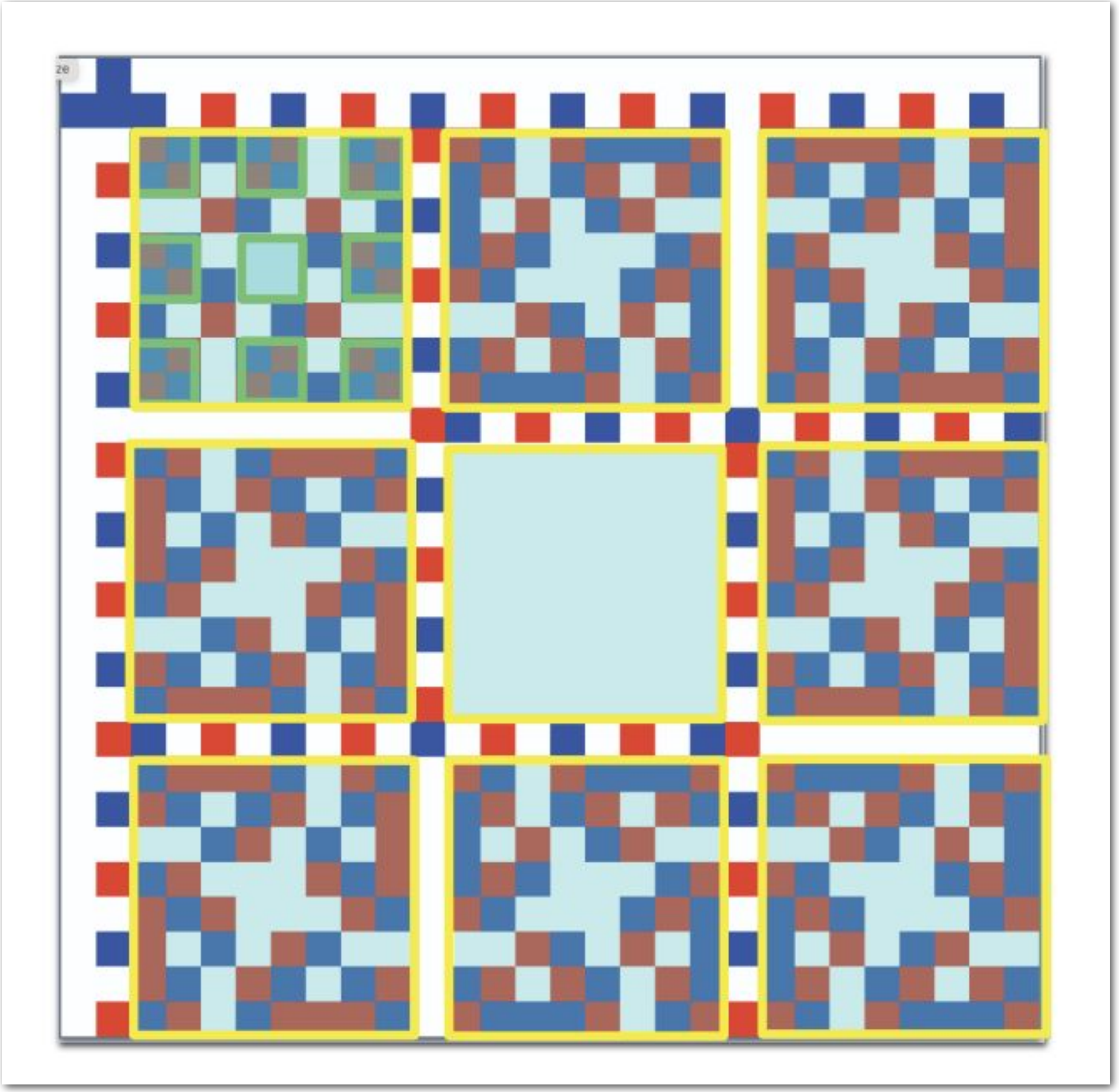


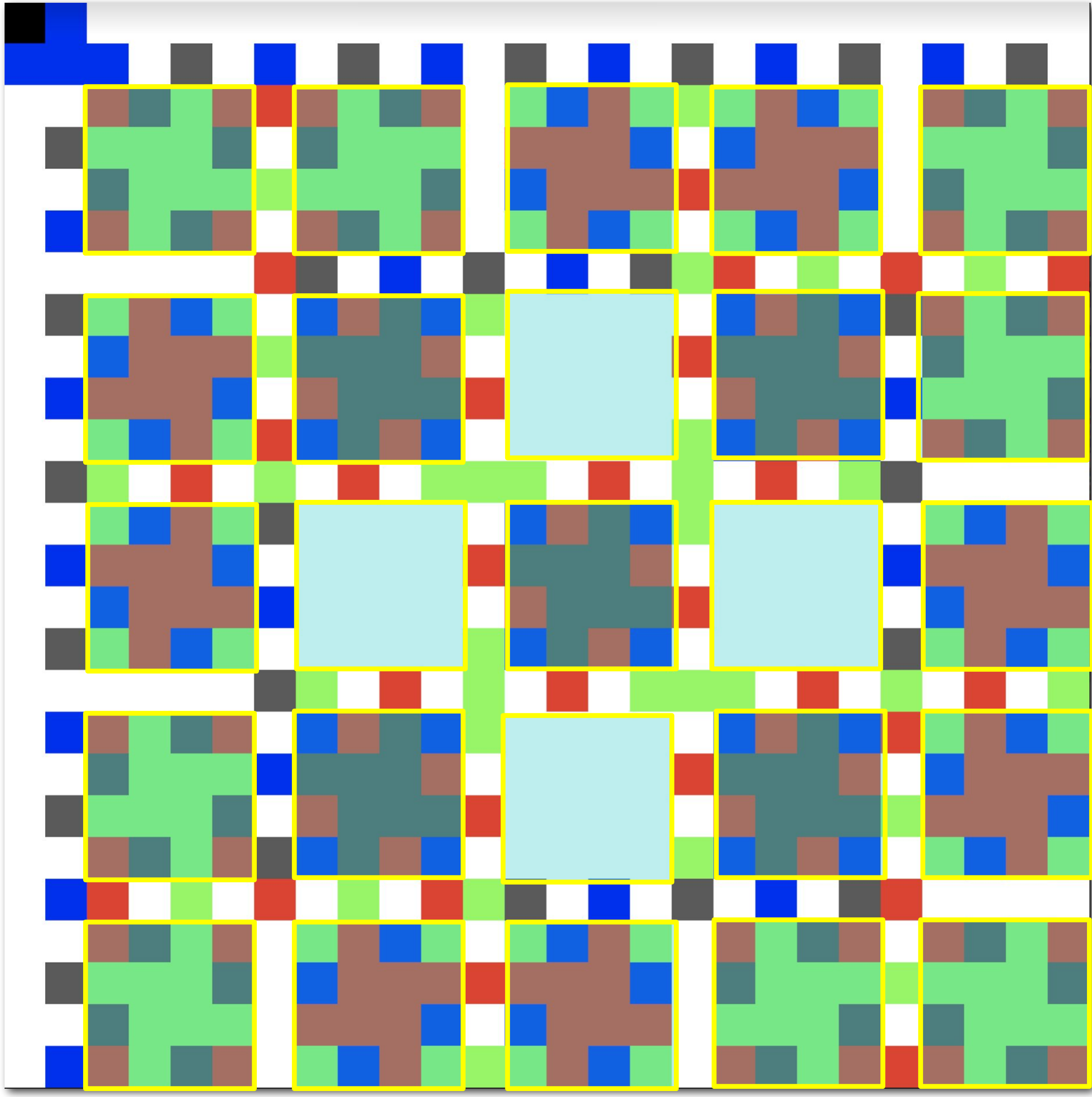
3-bricks

Definition 4.2

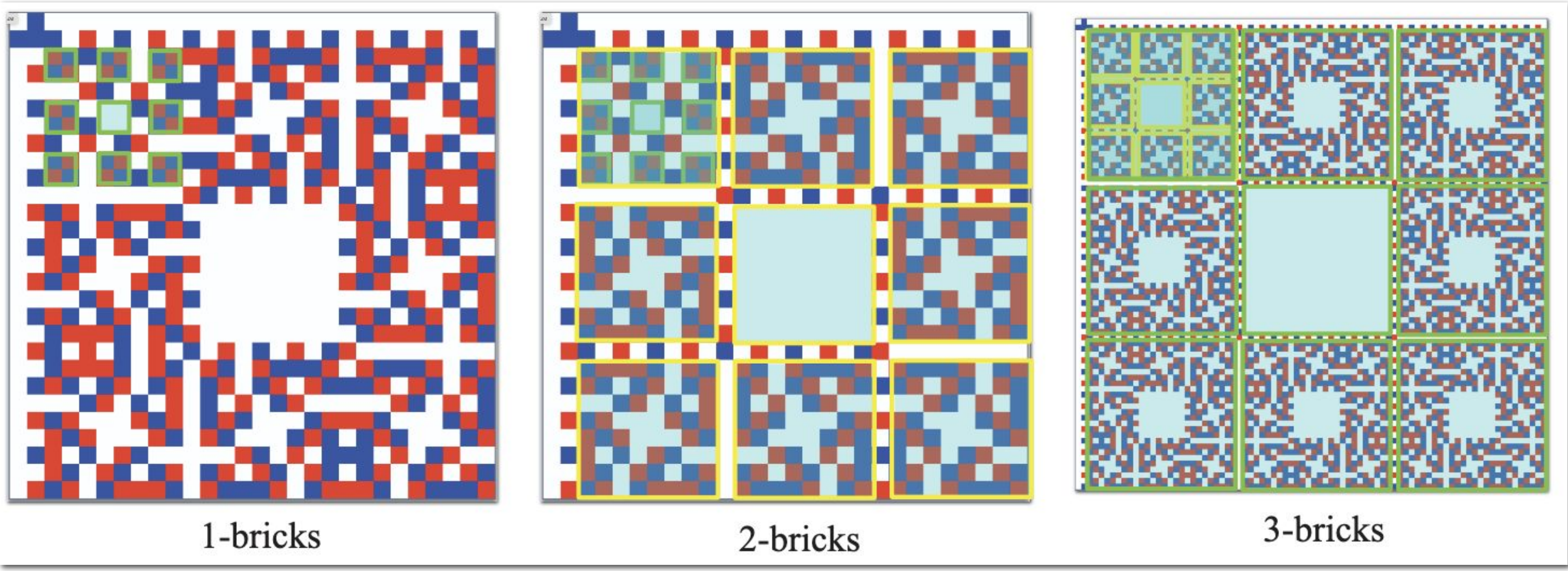
The **grout between** n**-bricks** *is the set of all strands not contained in any n-brick.*

<u>Thm.</u> Every other tile in the grout is white,
except at intersections of horizontal and vertical grout.

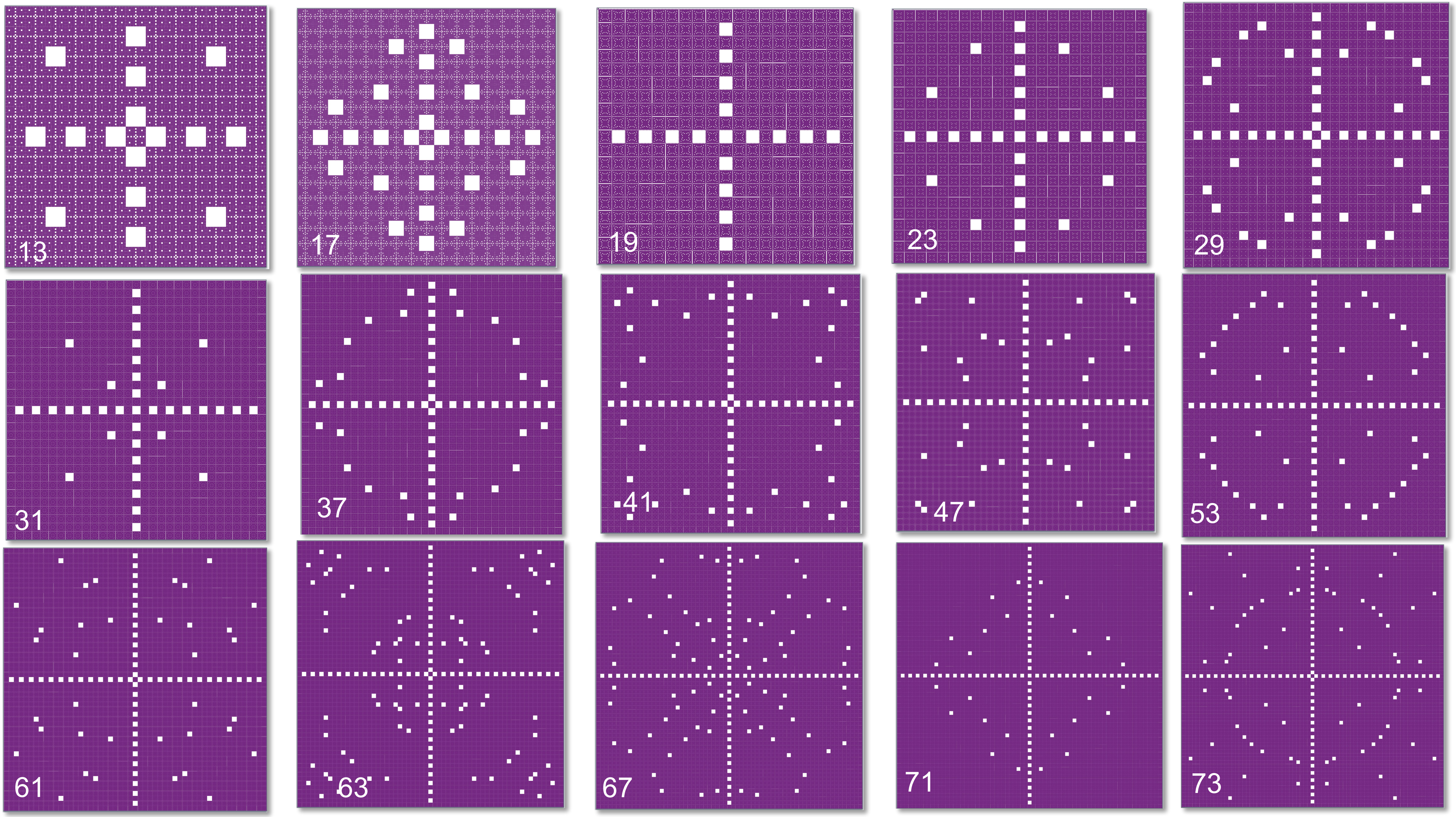




So, now we can build a difference carpet via bricks using the hierarchical structure.



Where do the white (n-1)-bricks appear in an n-brick?


13
17
19
23
29
31
37
41
47
53
61
63
67
71
73

But what about C_0^q when q is not a prime?

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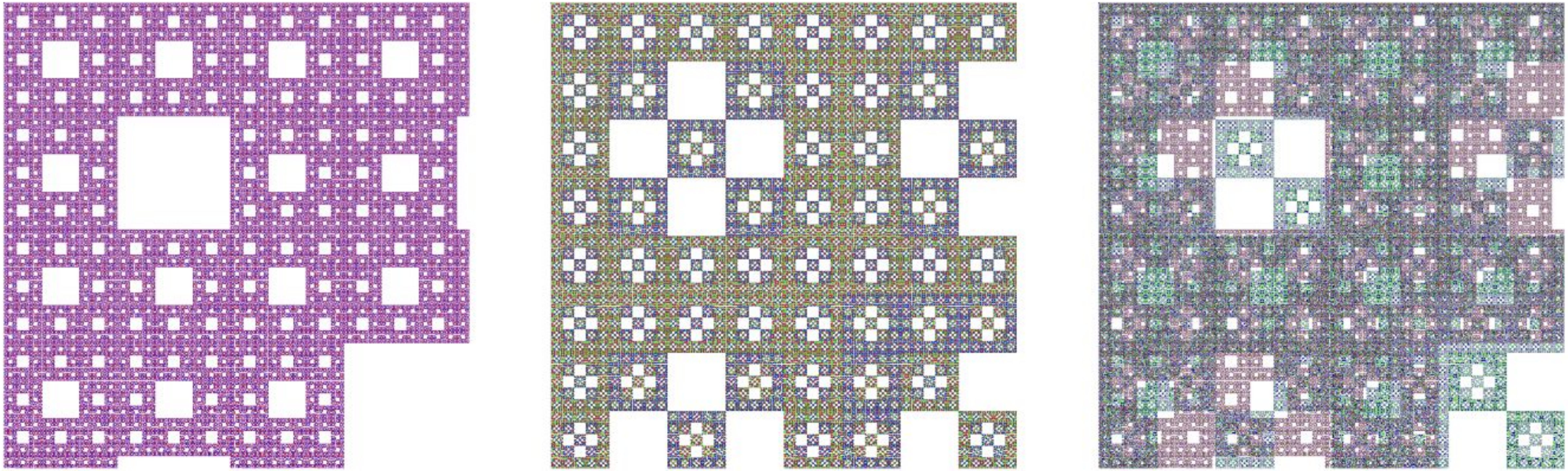


(a) C_0^3 (b) C_0^5 (c) C_0^{15}

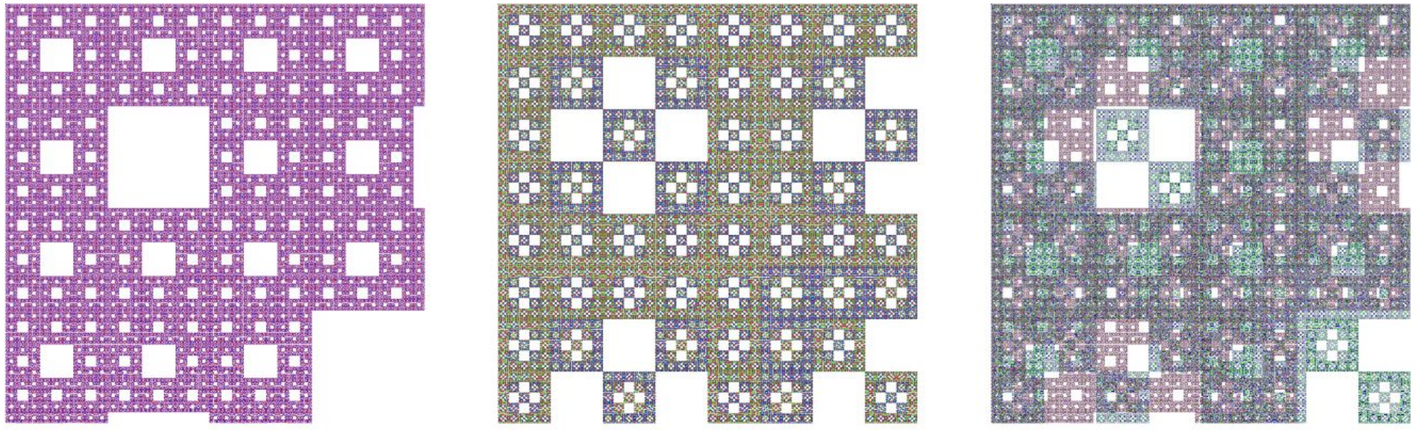
Figure 5.1: C_0^{15} appears to be formed by overlaying C_0^3 and C_0^5.

Definition 5.1

For positive integers $\ell, m, n > 1$ we say that carpets $X \in \mathbb{V}_\ell$ and $Y \in \mathbb{V}_m$ **overlay** *to form q-colored carpet Z if and only if there exist constants a, b, c such that $a(bX + cY) = Z$. We call such an action on X and Y* **an overlay**, *and this overlay* **forms** *Z.*

Example 5.1 Take carpets C_0^3, C_0^5, and C_0^{15}. Then, $5C_0^3$ and $3C_0^5$ are recolored copies of C_0^3 and C_0^5 that are well-defined as 15-colored carpets.

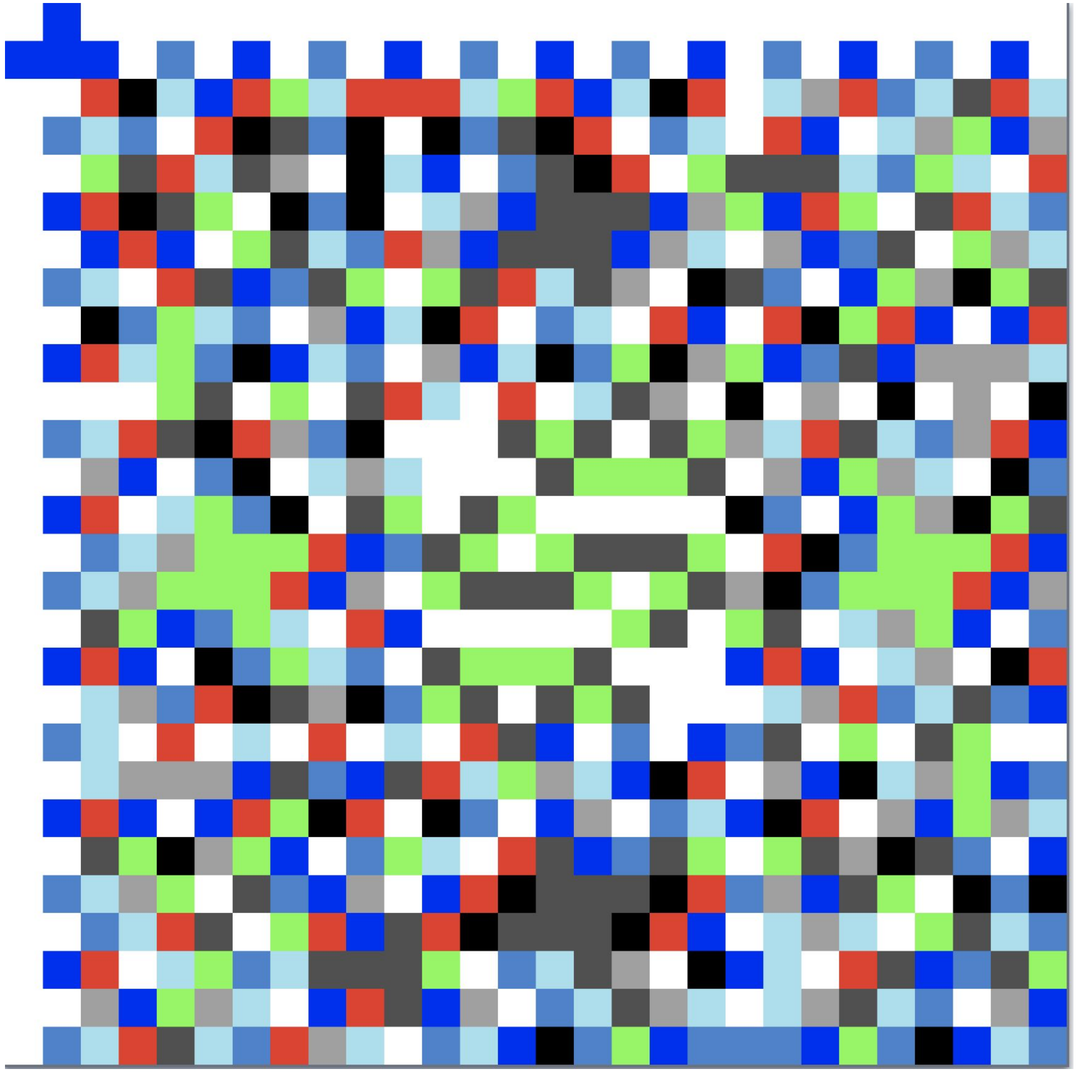
Note that the fringe of $5C_0^3 + 3C_0^5$ has zeros everywhere except for an 8 in both the top row and leftmost column. So, $2(5C_0^3 + 3C_0^5)$ has the same fringe as C_0^{15}, and thus we can say that C_0^3 and C_0^5 overlay to form C_0^{15}.



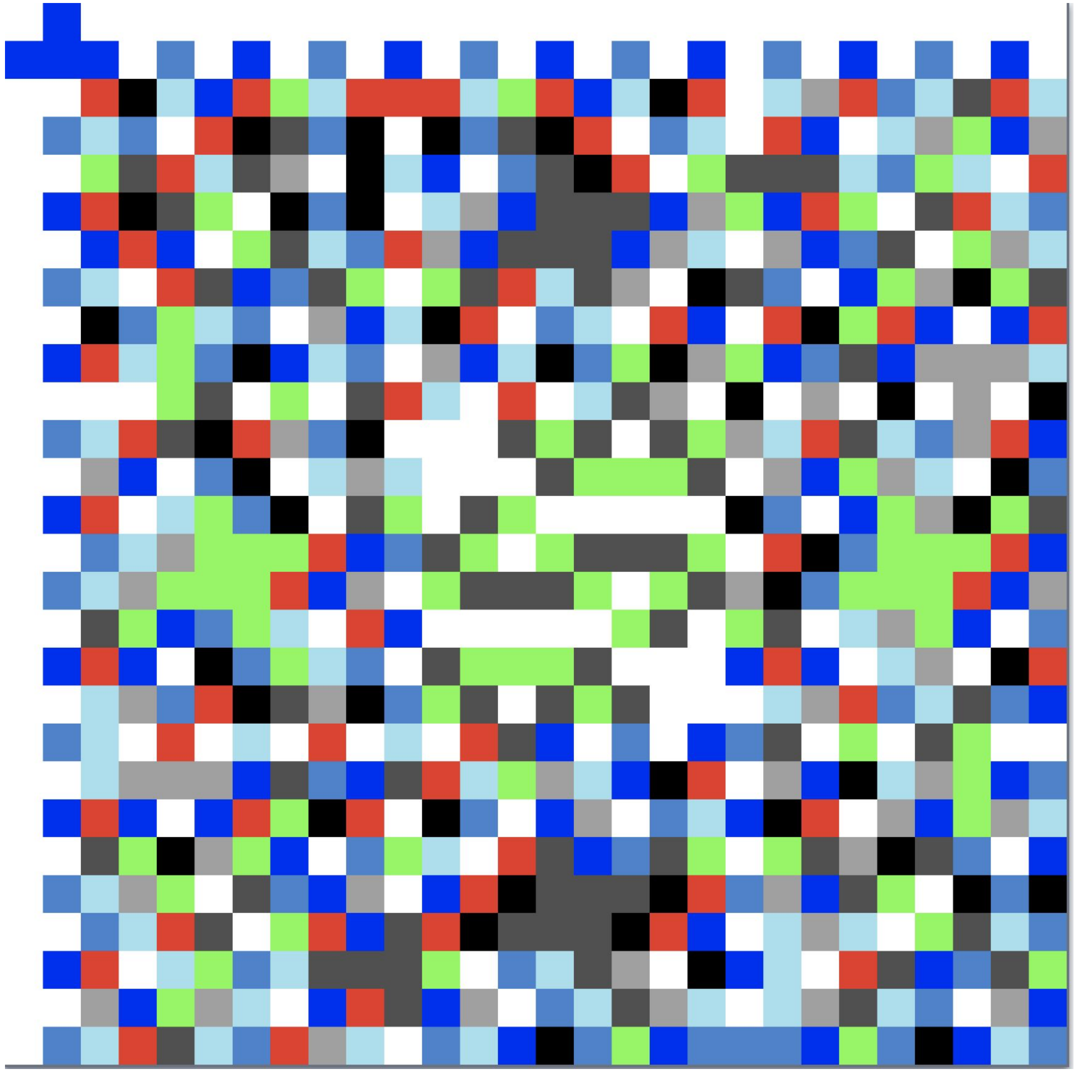
(a) C_0^3 (b) C_0^5 (c) C_0^{15}

So, we understand C_0^q when $q = p_1 p_2 \ldots p_n$ where the p_i are all distinct primes.

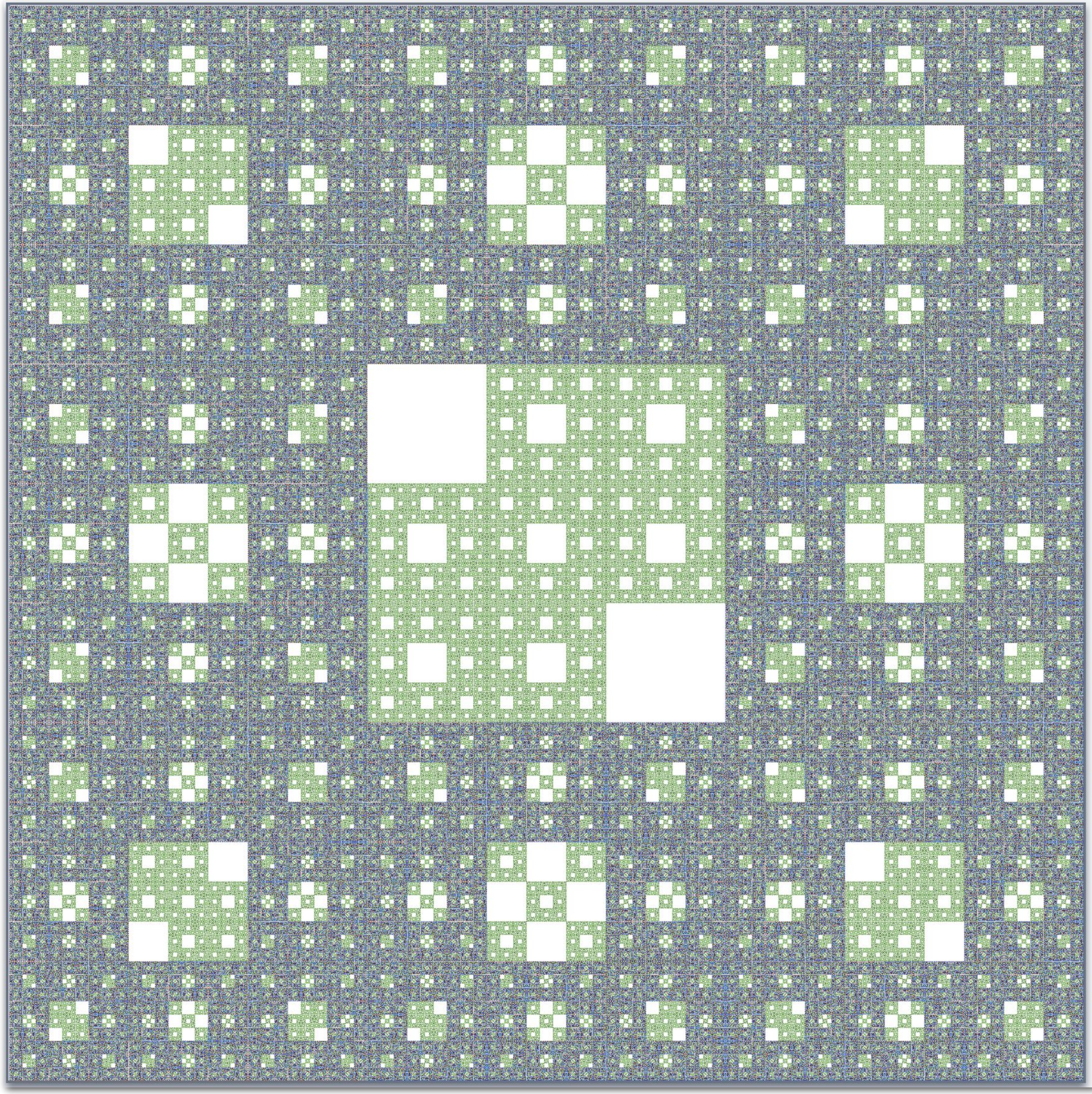
So, we understand C_0^q when $q = p_1 p_2 \ldots p_n$ where the p_i are all distinct primes. But what if it's not?



p = 9



p = 9

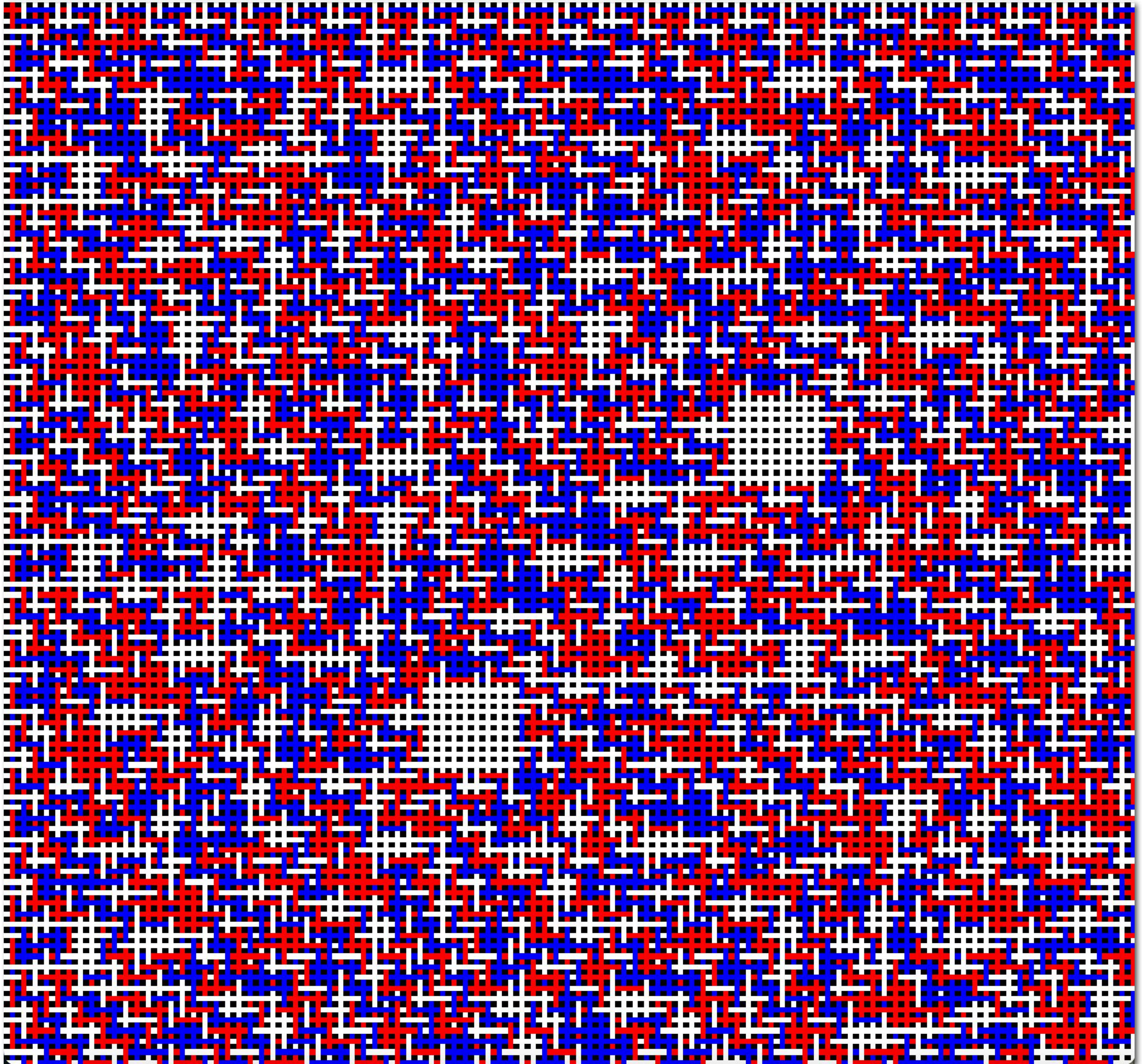


Other Weaves

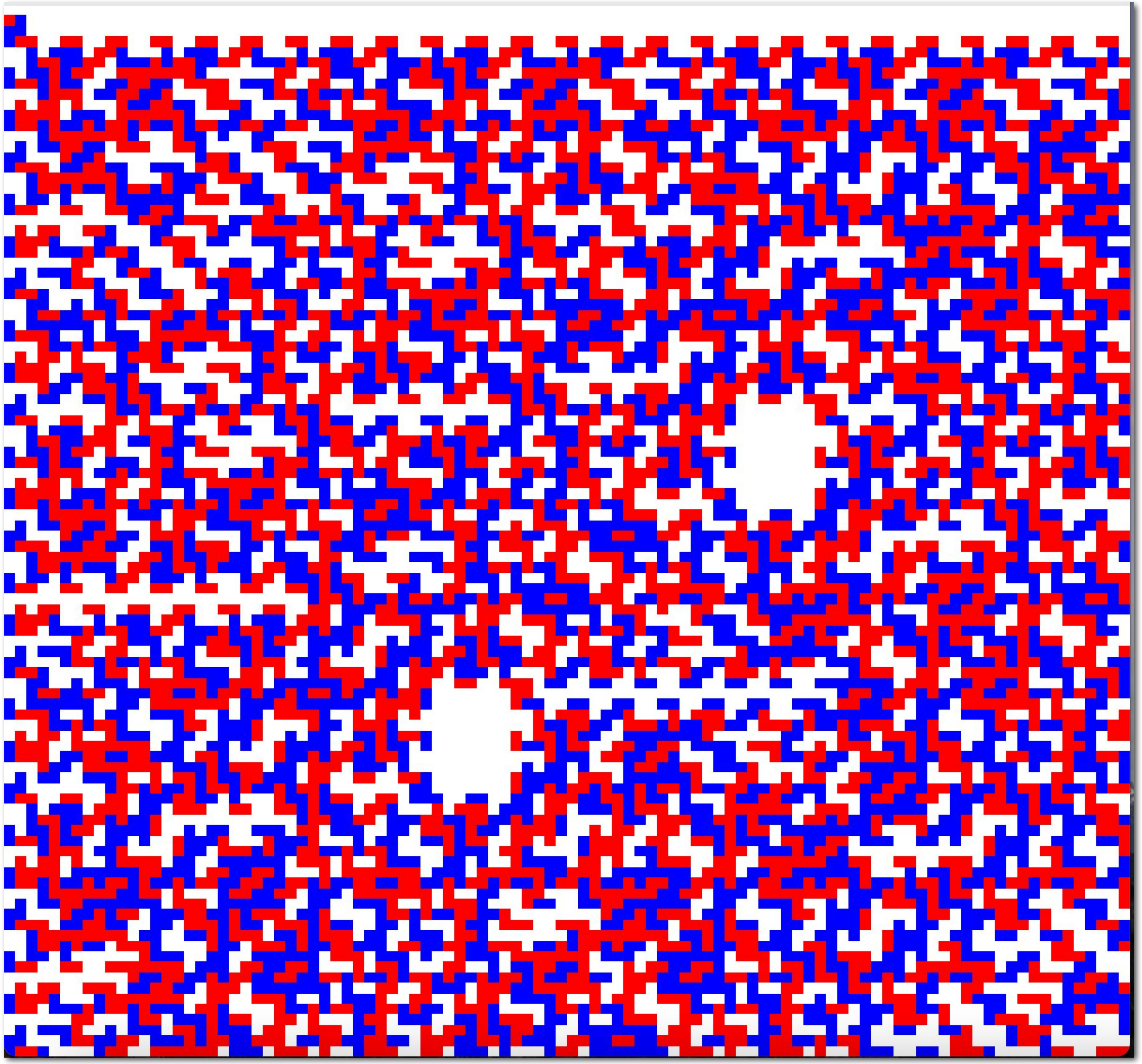
2 over 2 under

Other Weaves

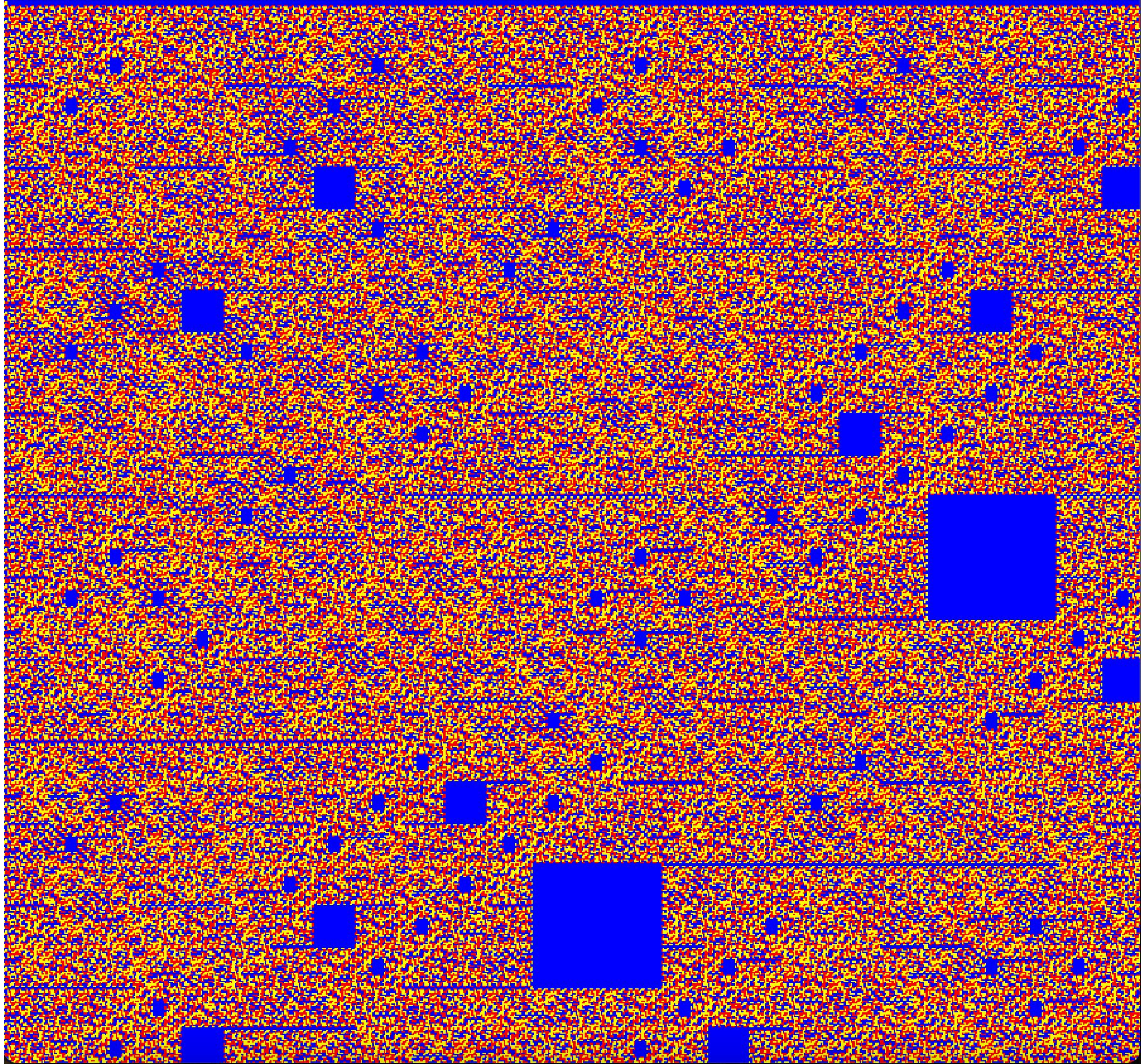
2 over 2 under weave



2 over 2 under
mod 3
difference carpet
(unthinned)

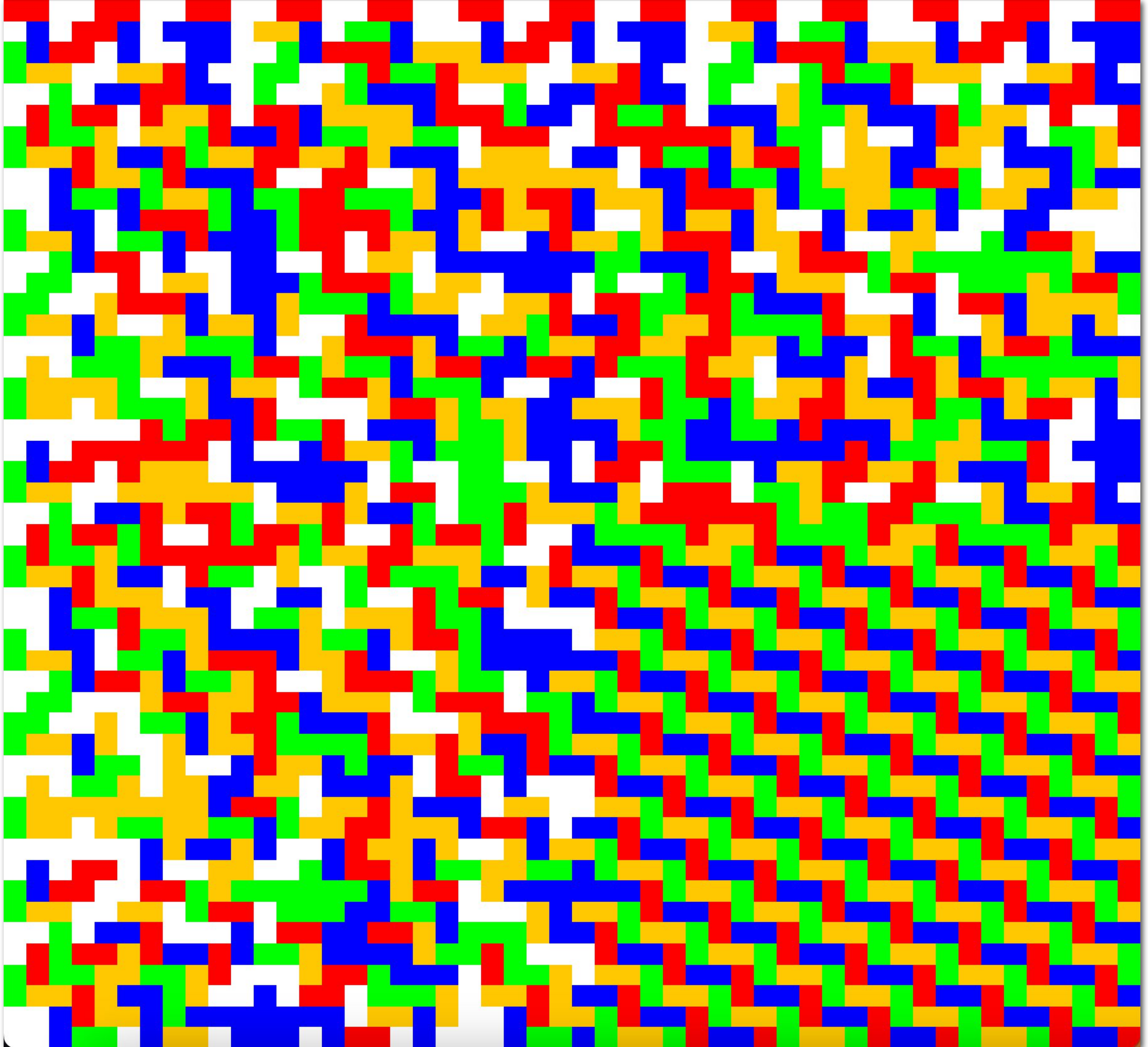


2 over 2 under
mod 3
difference carpet

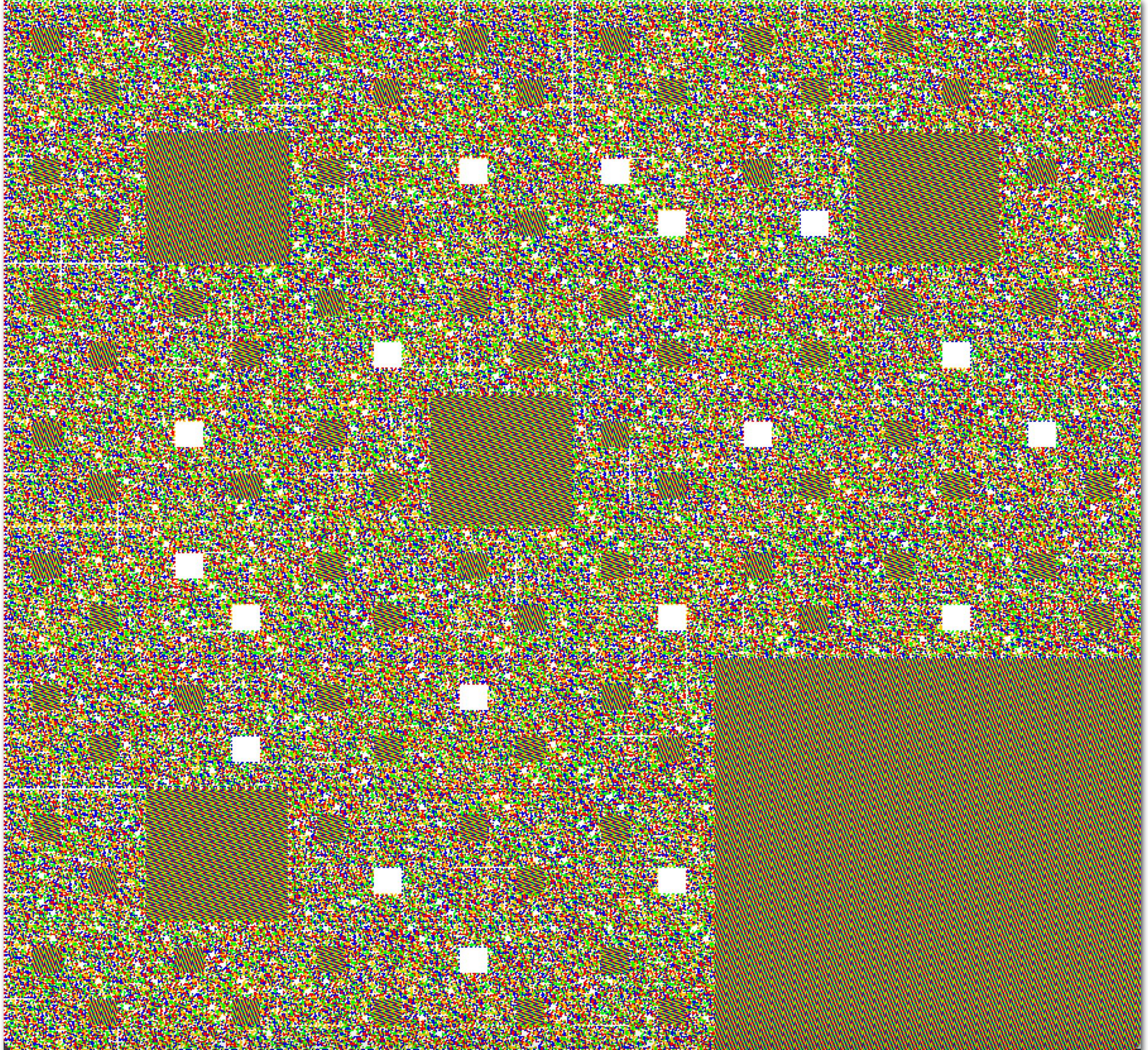


2 over 2 under
mod 3
difference carpet
(colors changed)

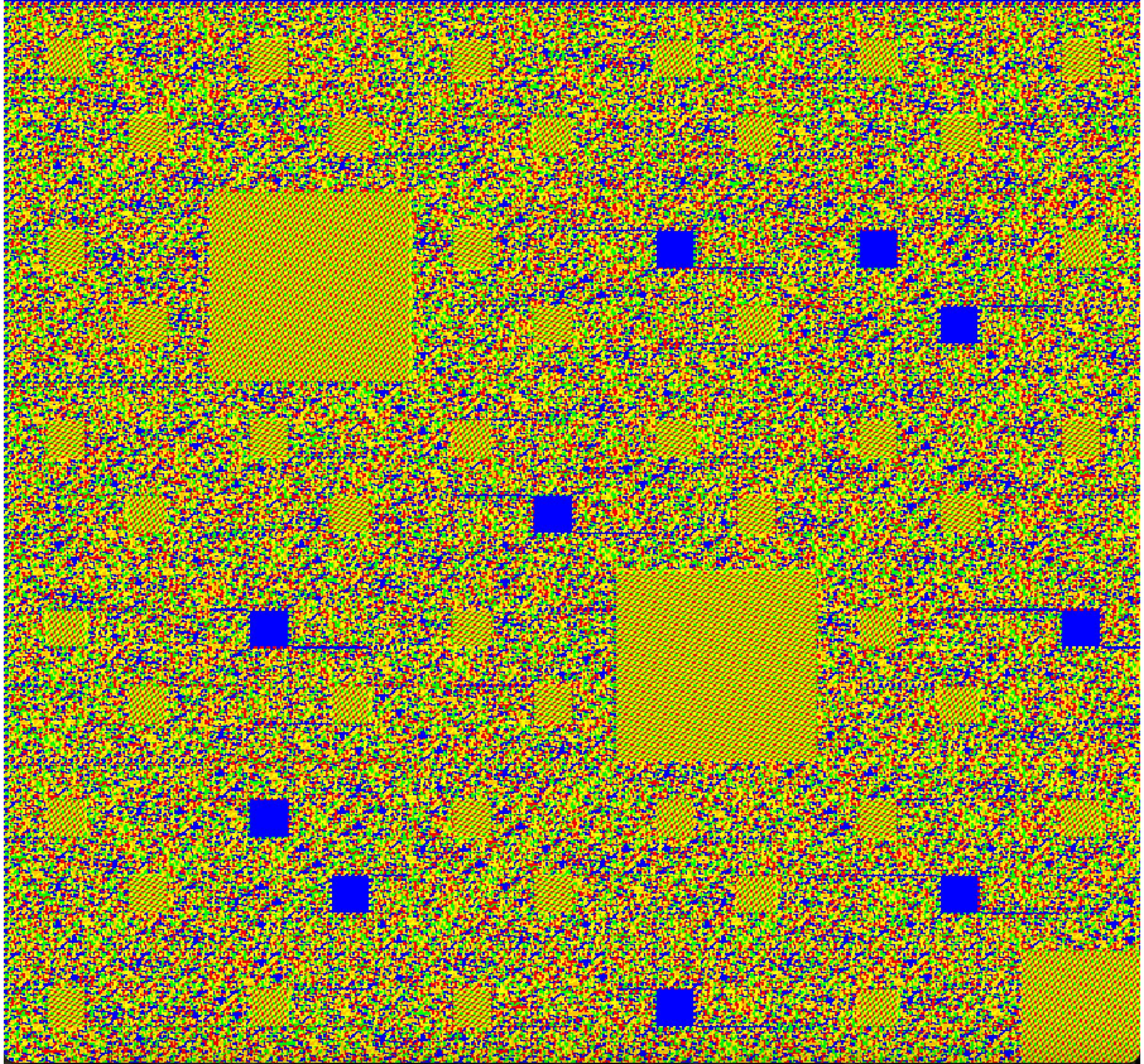
2 over 2 under
mod 5
difference carpet
(unthinned)



2 over 2 under
mod 5
difference carpet

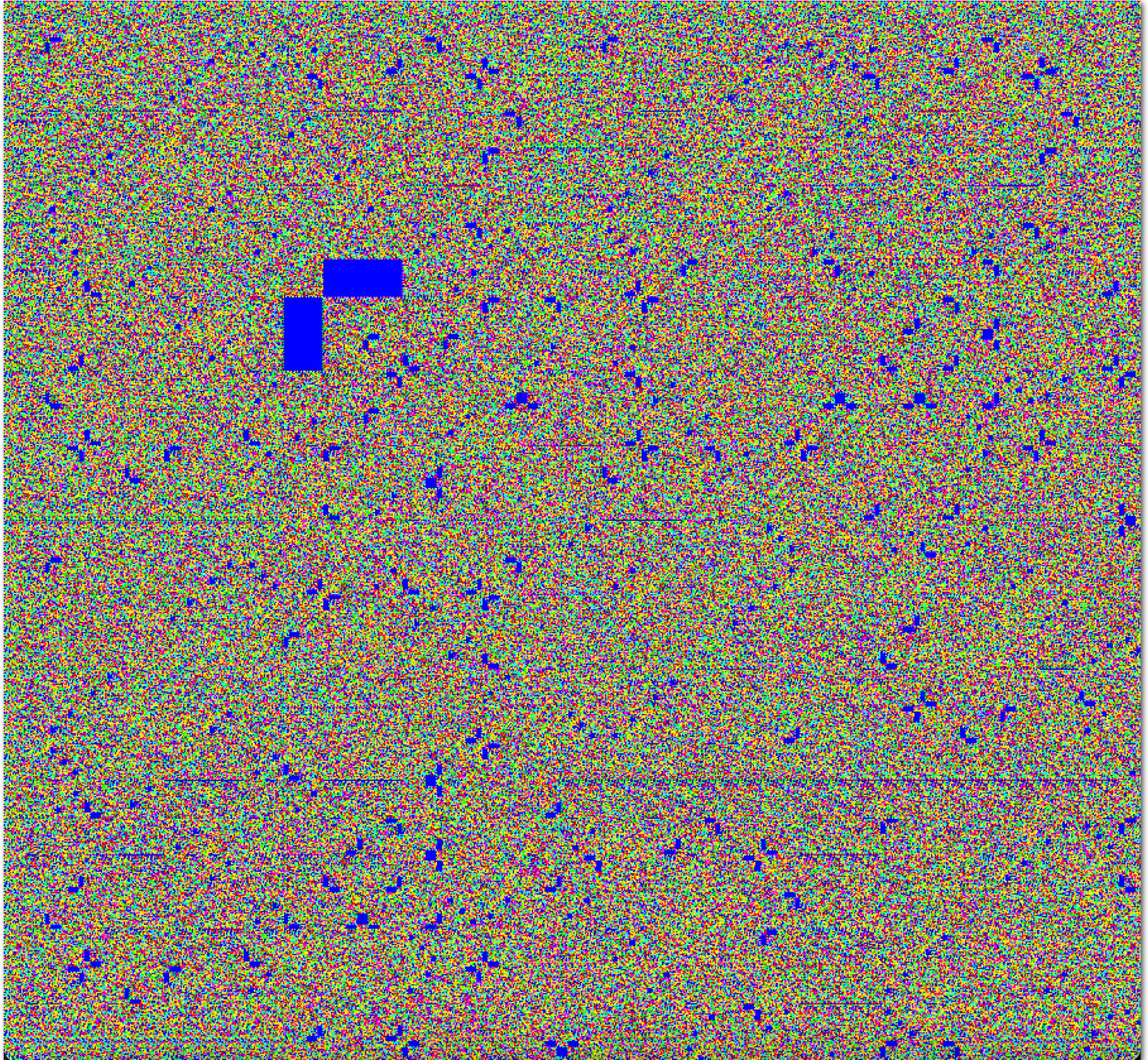


2 over 2 under
mod 5
difference carpet

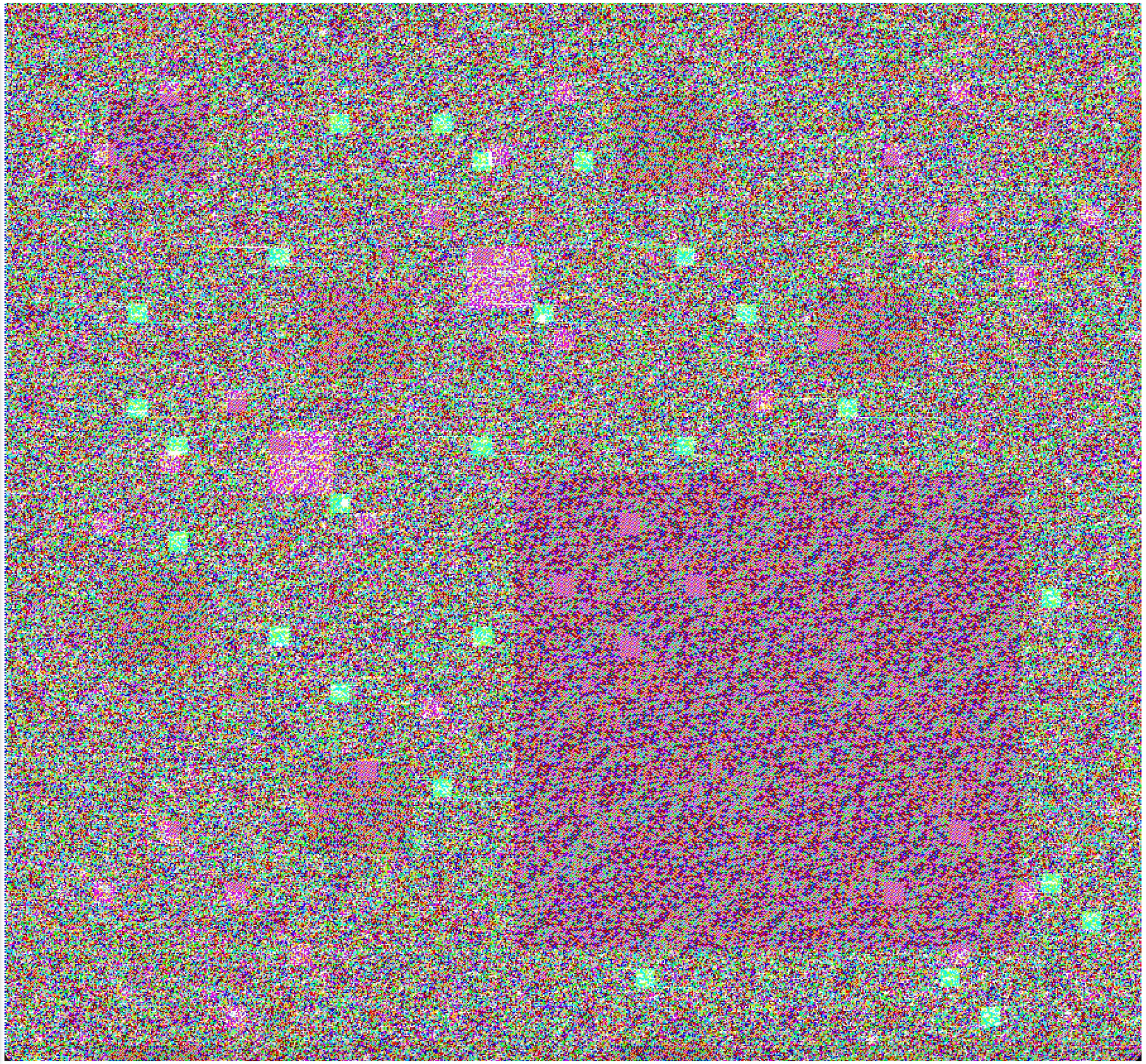


2 over 2 under
mod 5
difference carpet
(changed colors)

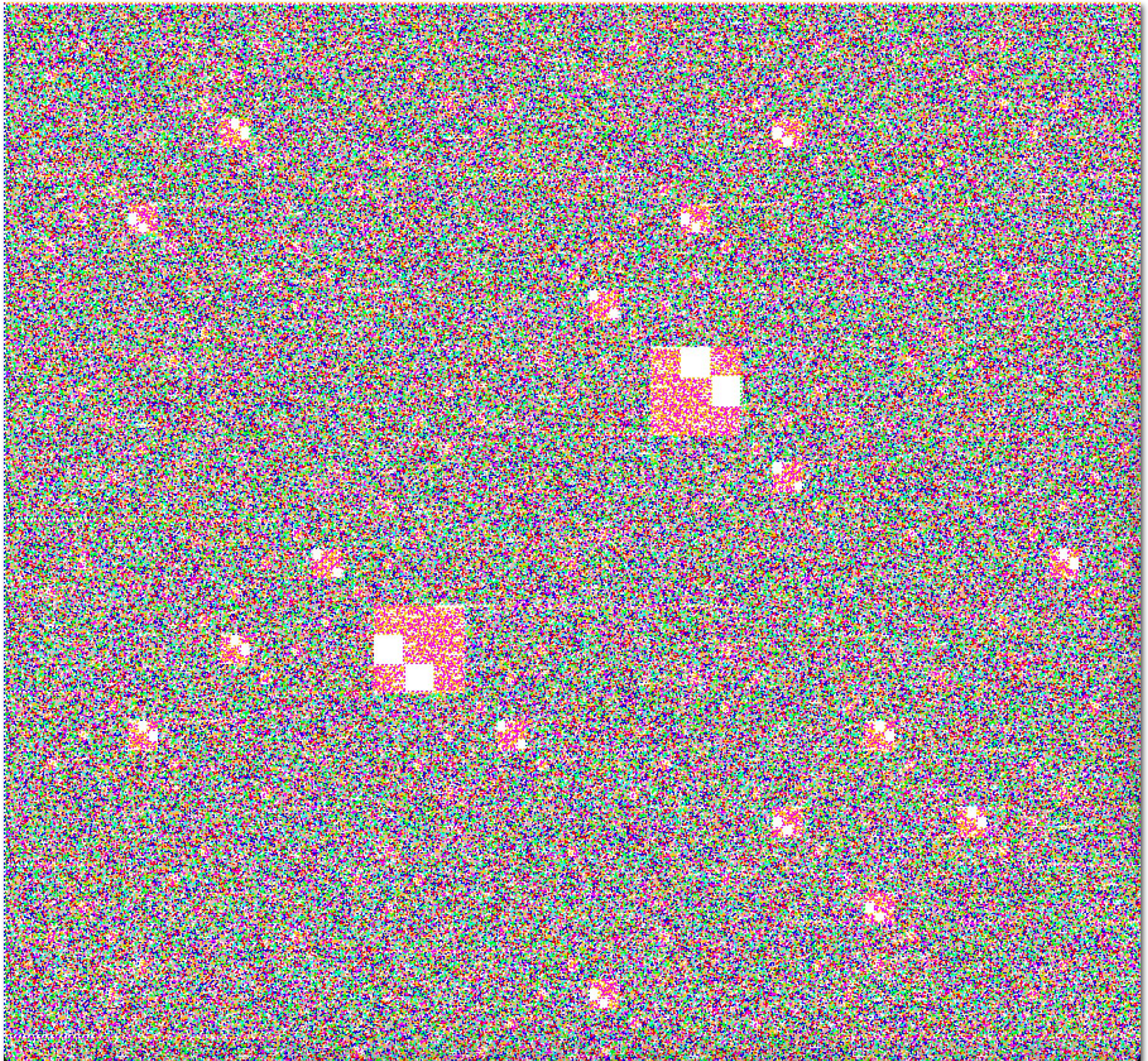
2 over 2 under
mod 7
difference carpet
(unthinned)



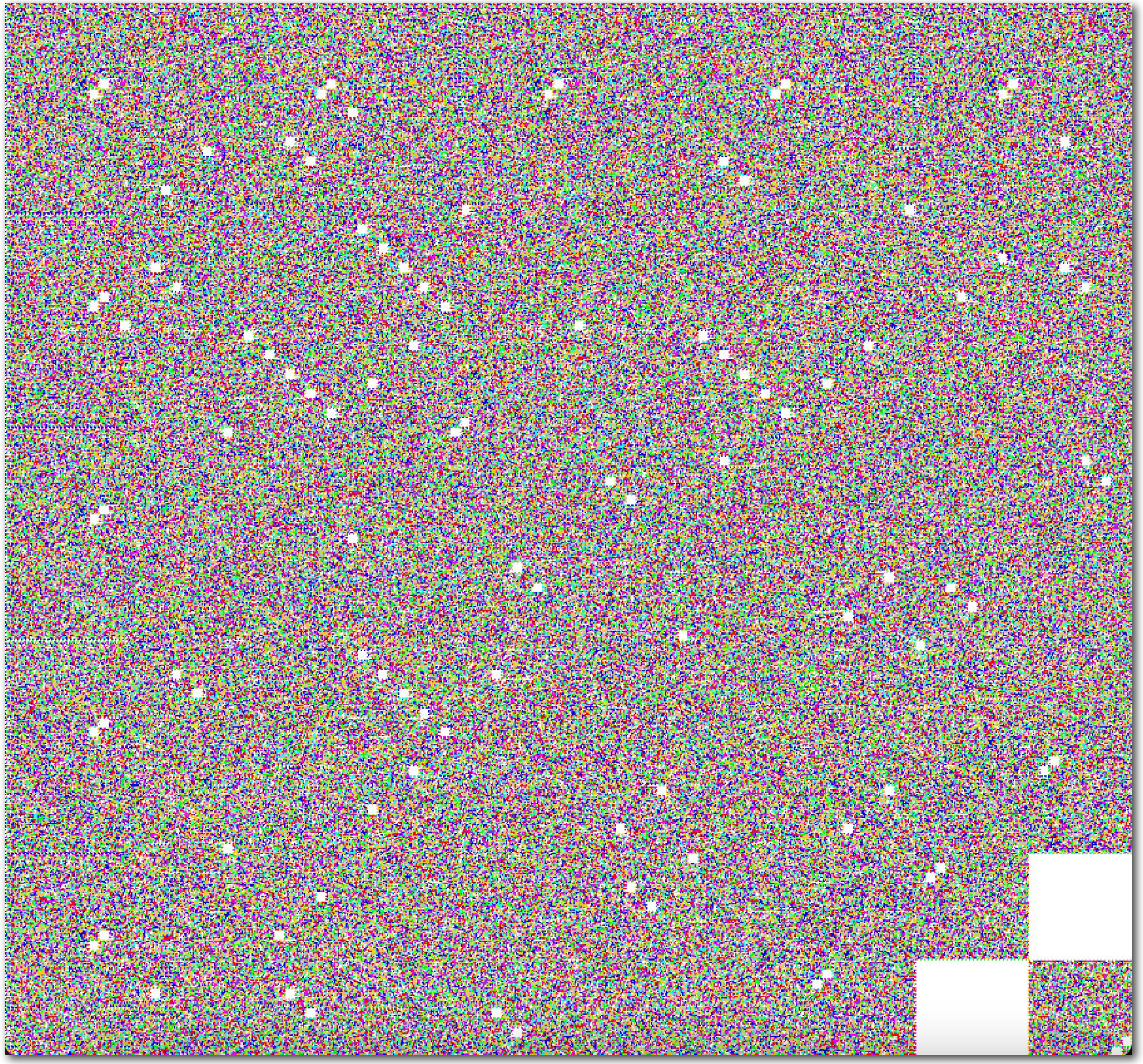
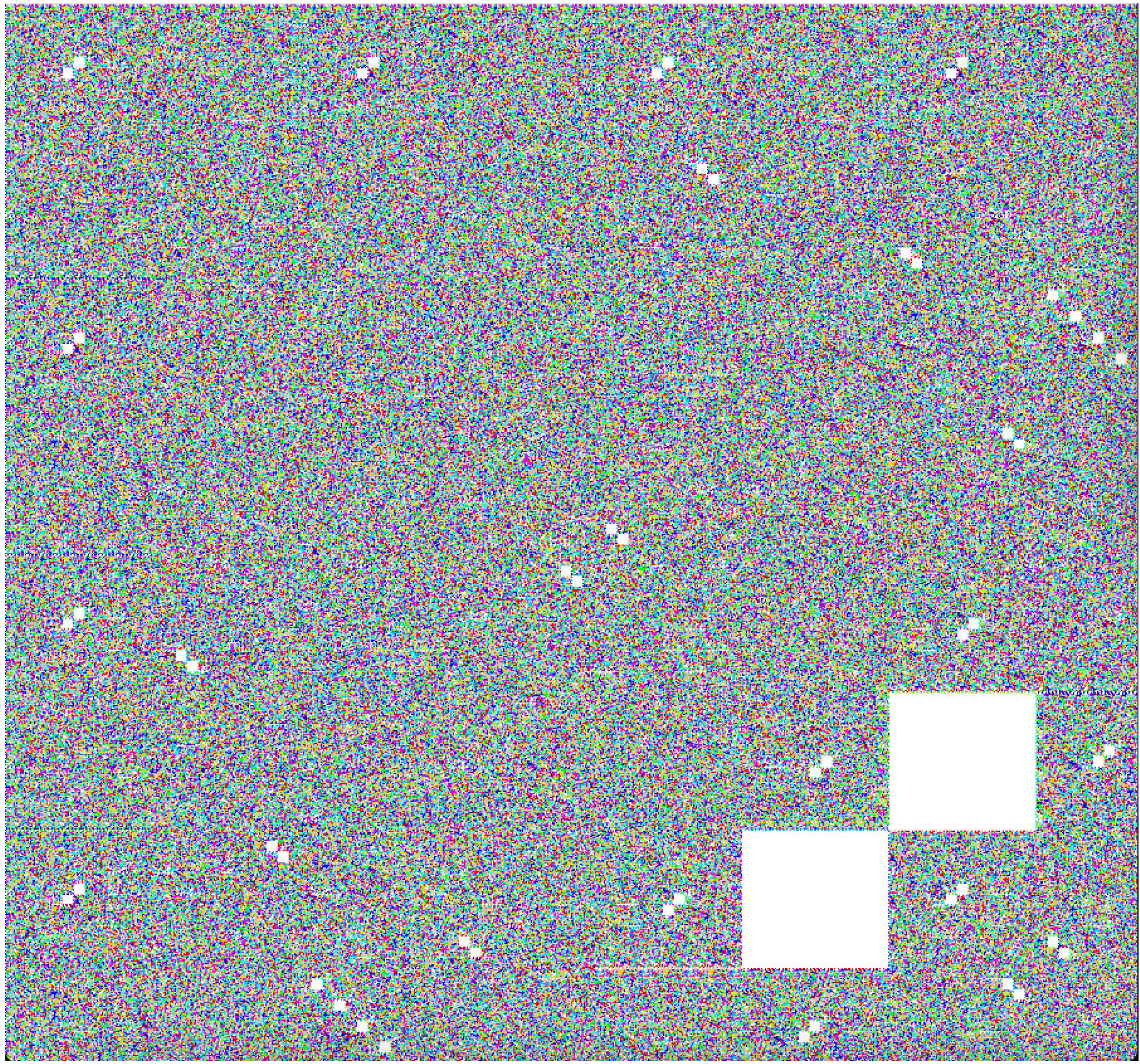
2 over 2 under
mod 7
difference carpet
(changed colors)



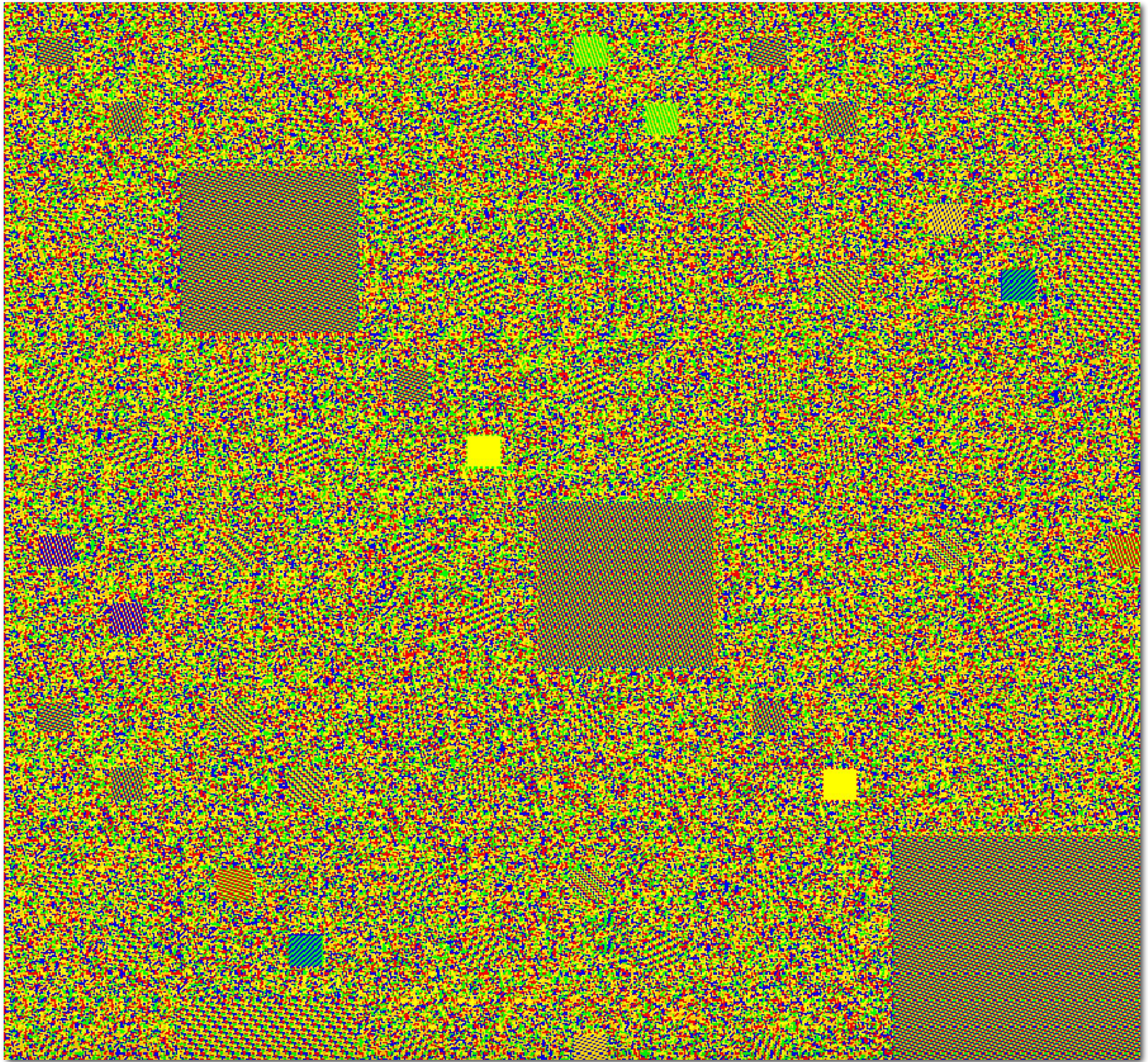
2 over 2 under mod 15 difference carpet



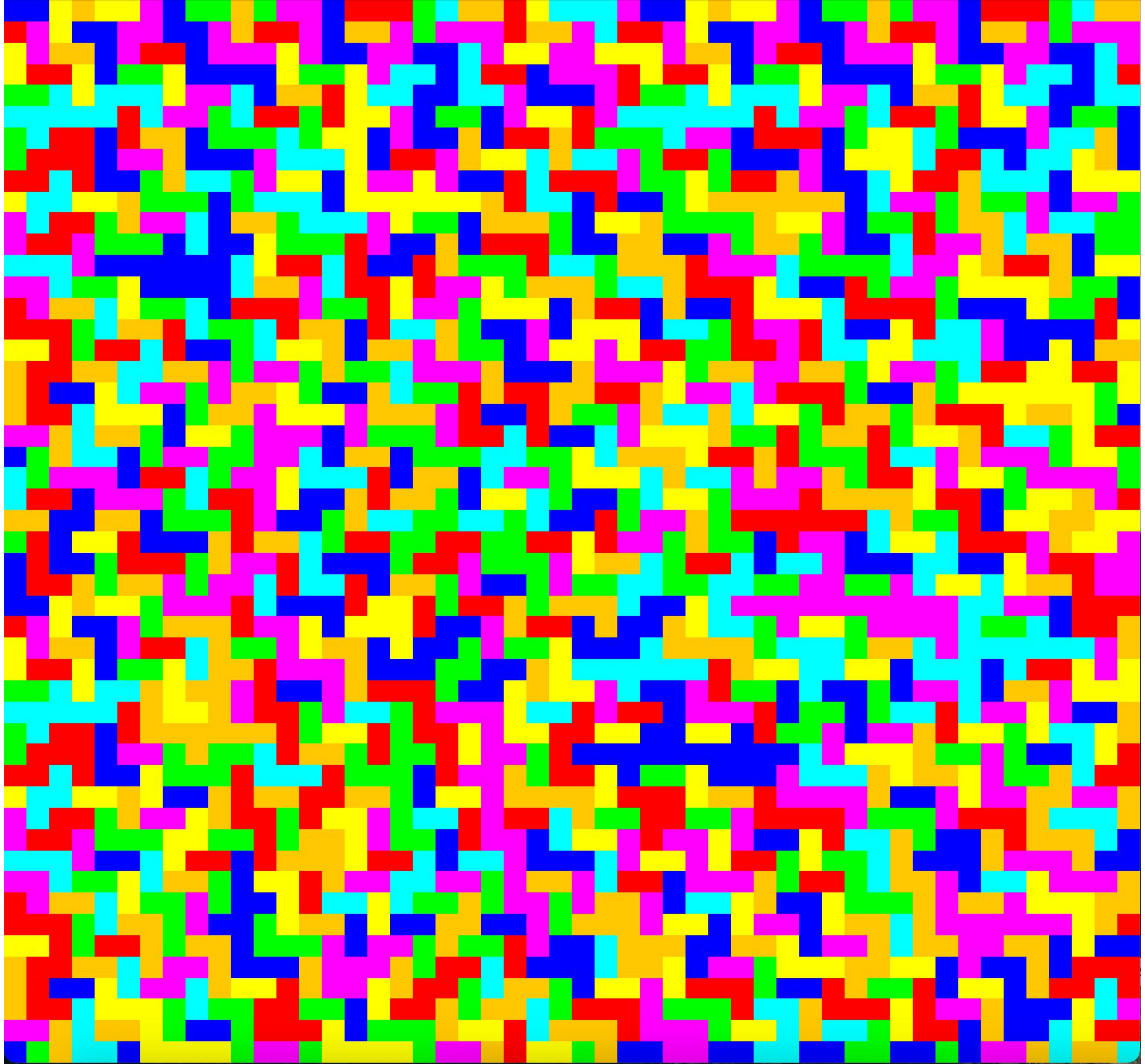
2 over 2 under
mod 9
difference carpet

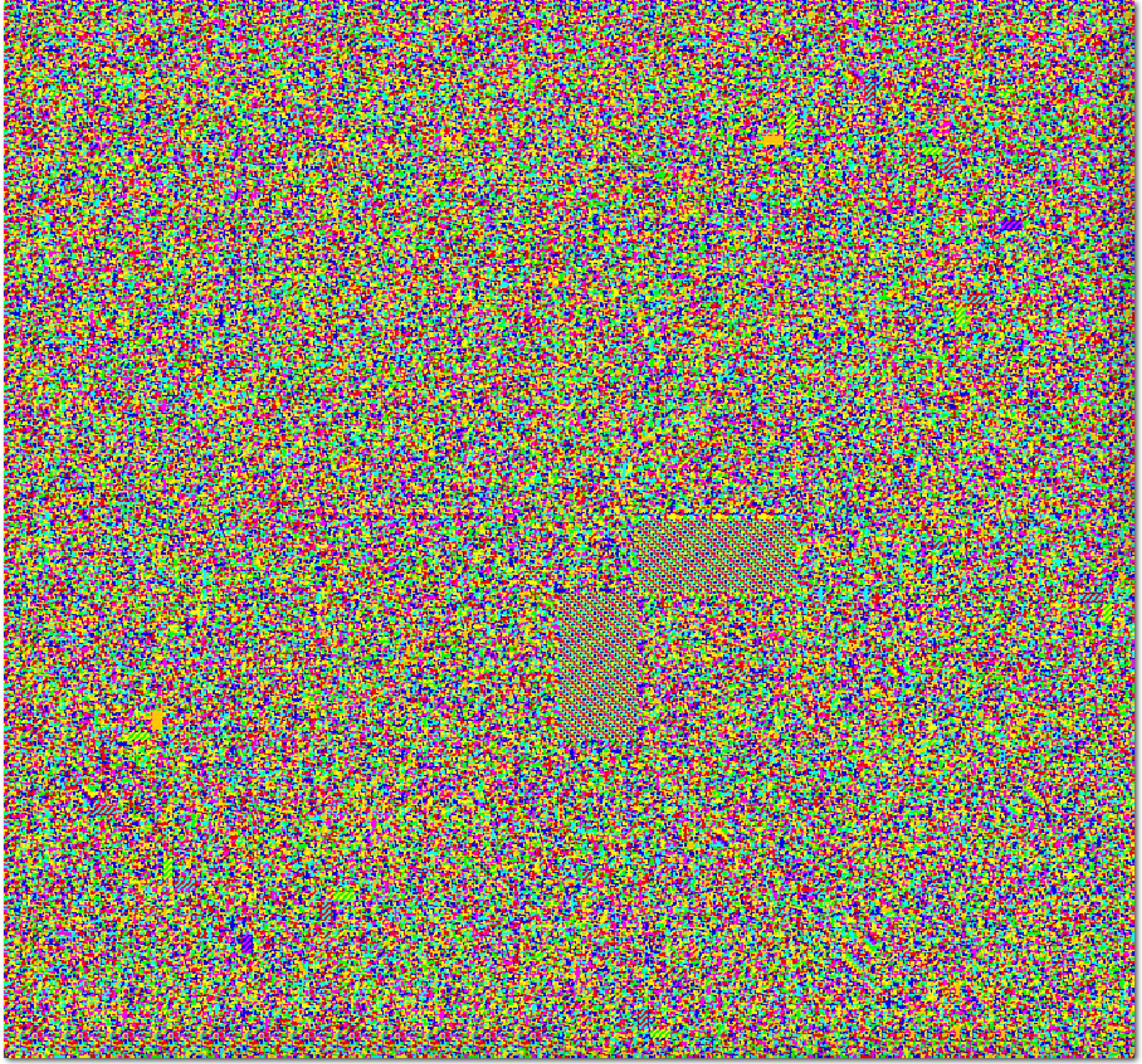
2 over 2 under mod 11 and 13
difference carpets



2 over 2 under
mod 5
periodic carpet



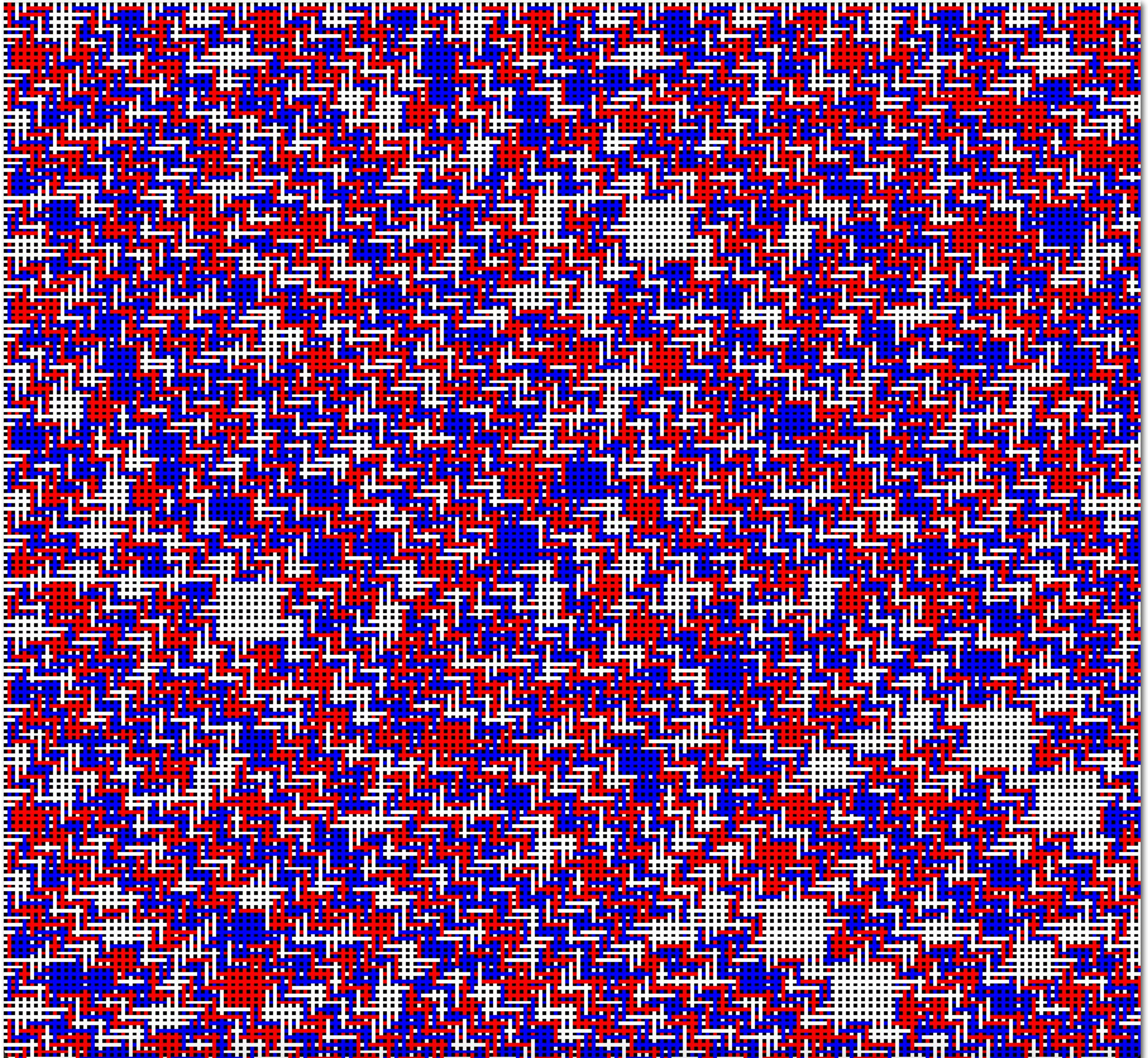
2 over 2 under
mod 7
periodic carpet



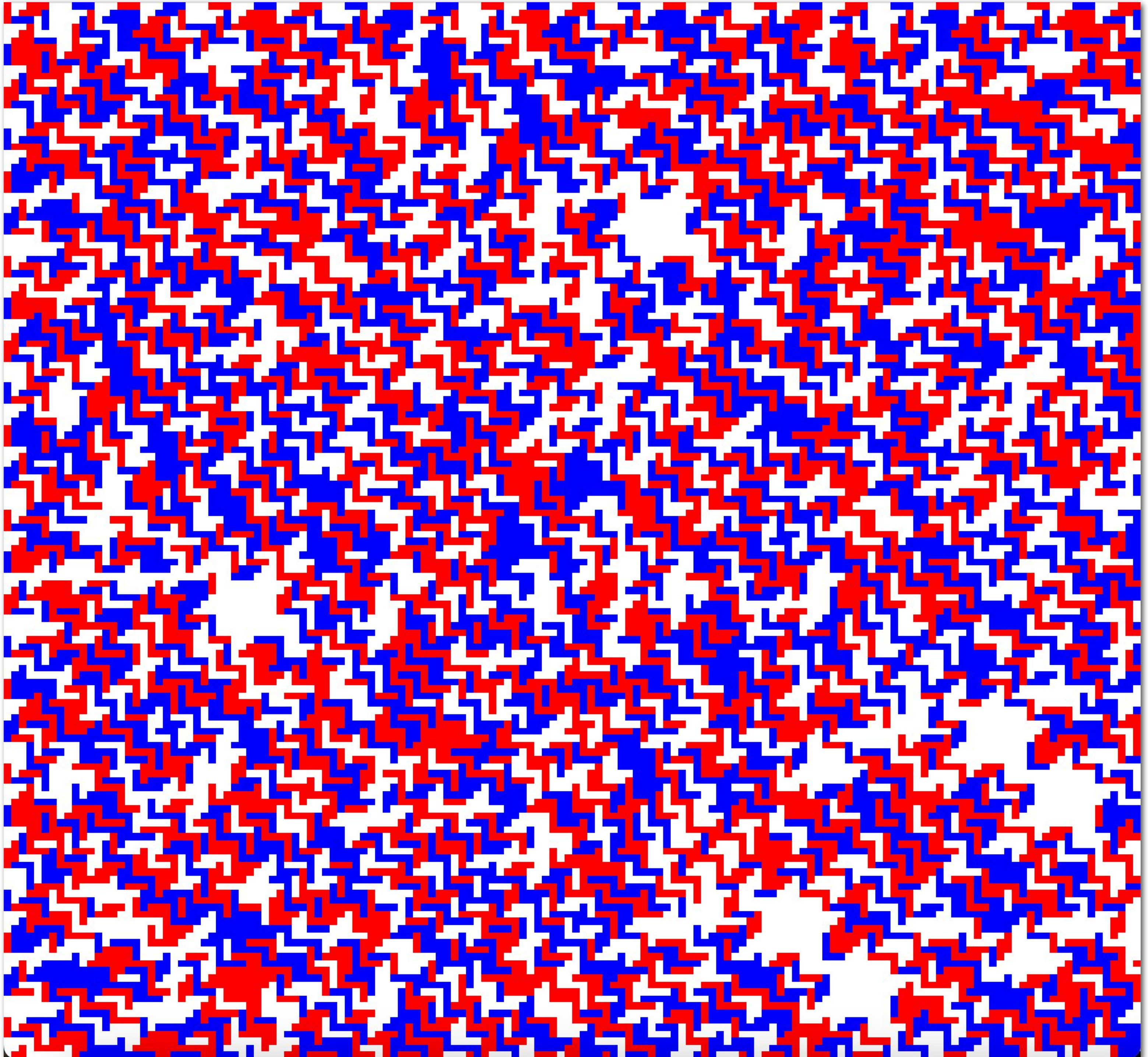
2 over 2 under
mod 7
periodic carpet

3 over 3 under carpets

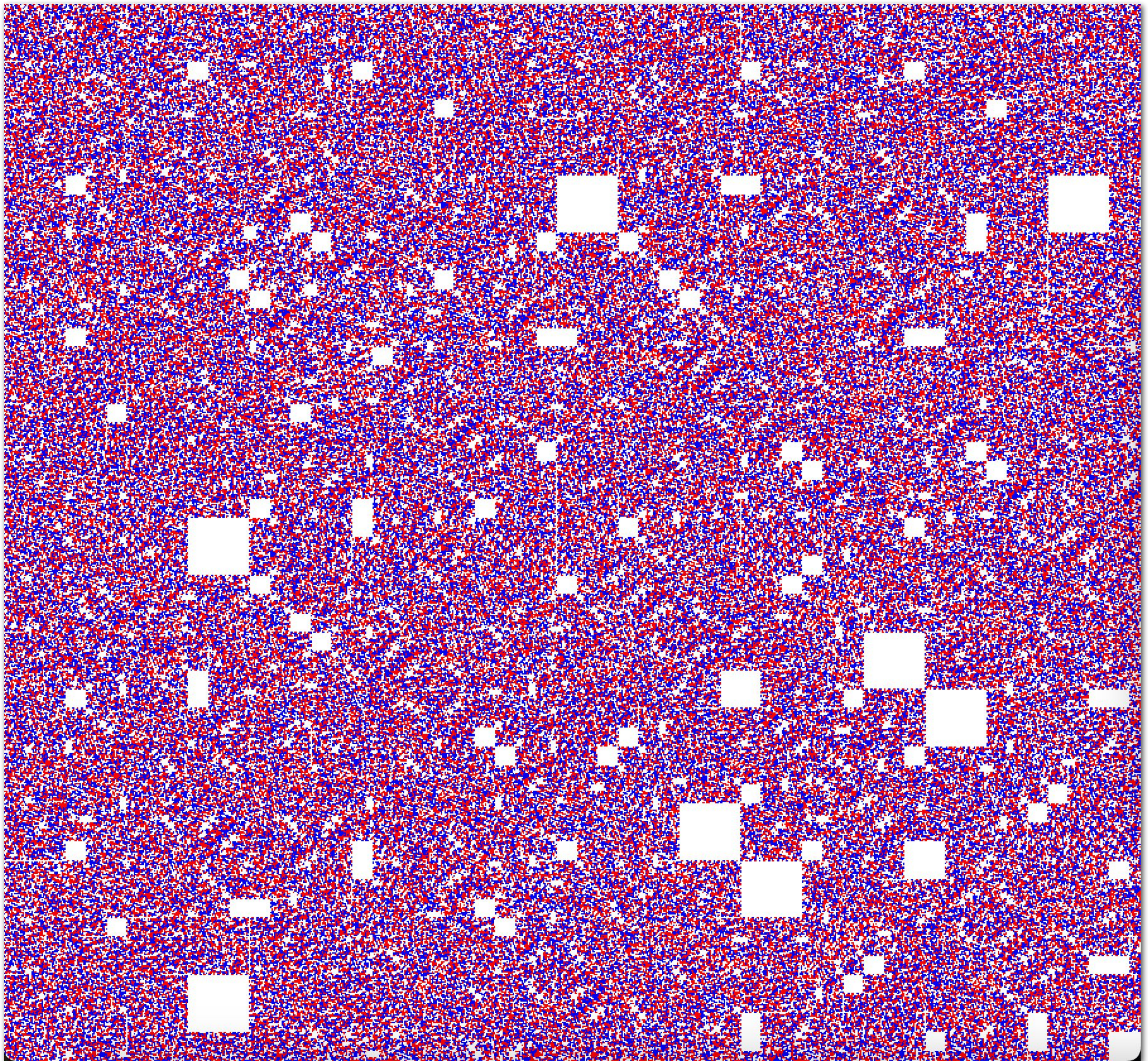
3 over 3 under
mod 3
difference carpet
(unthinnned)



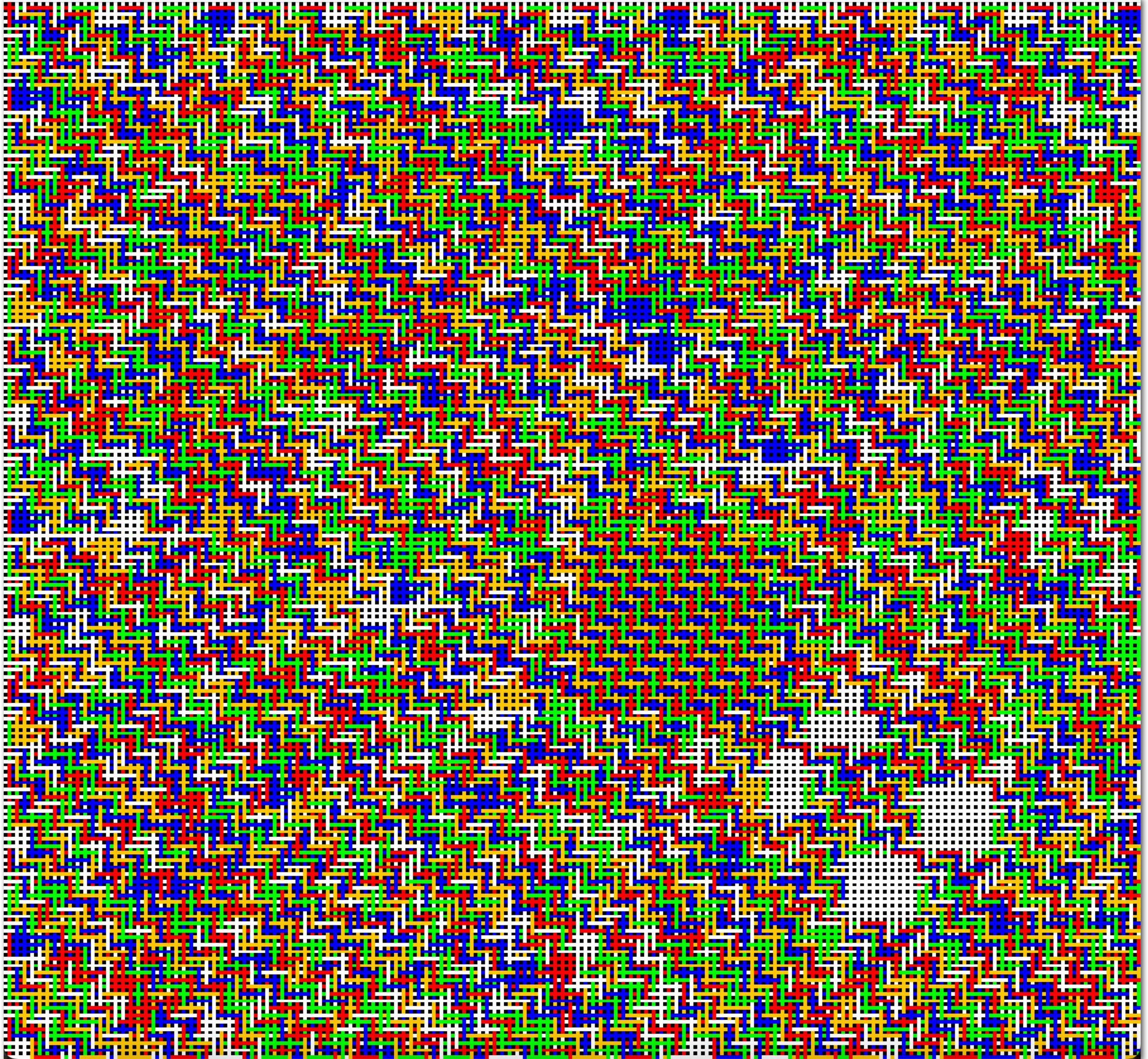
3 over 3 under
mod 3
difference carpet



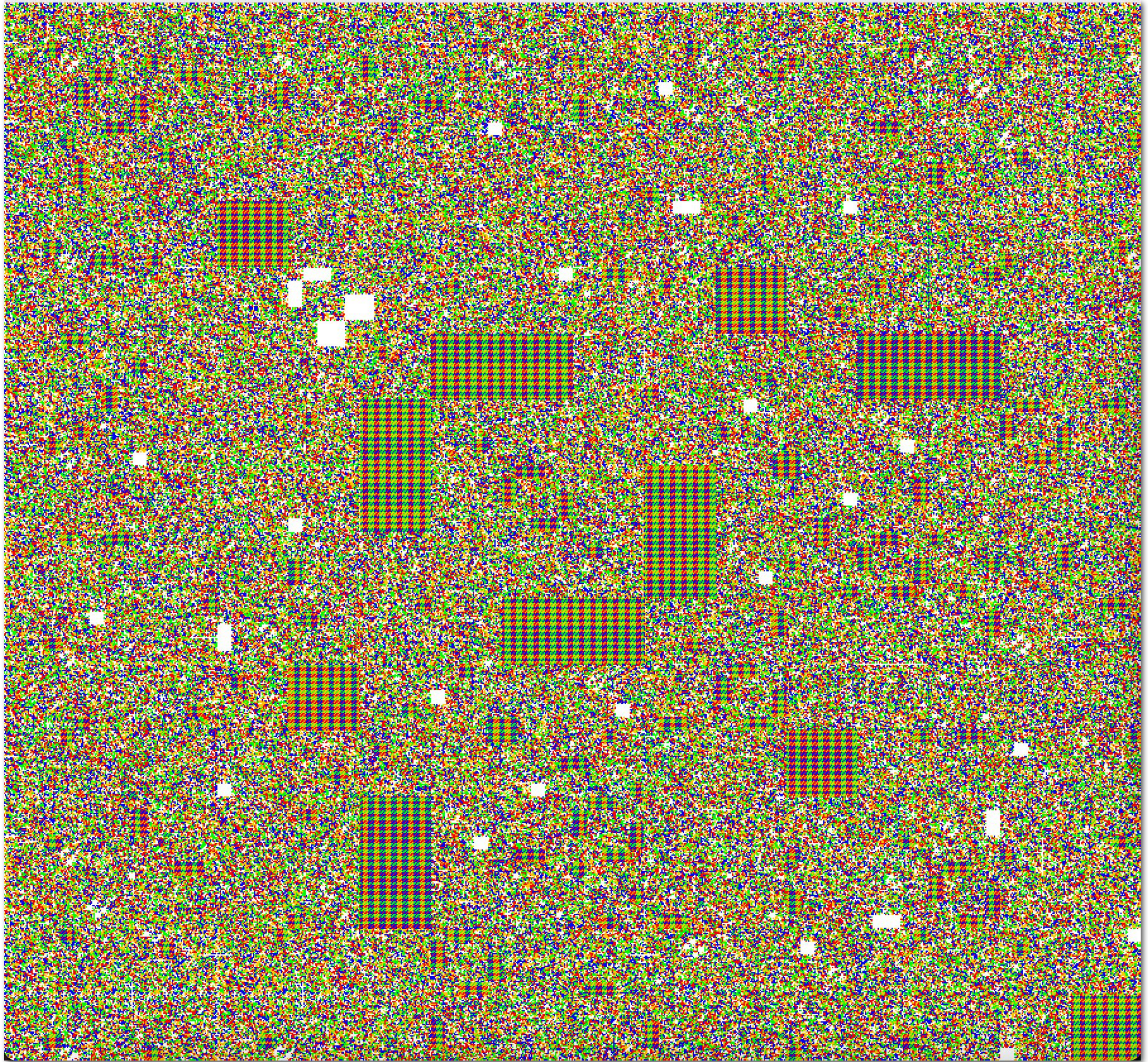
3 over 3 under
mod 3
difference carpet



3 over 3 under
mod 5
difference carpet
(unthinnned)

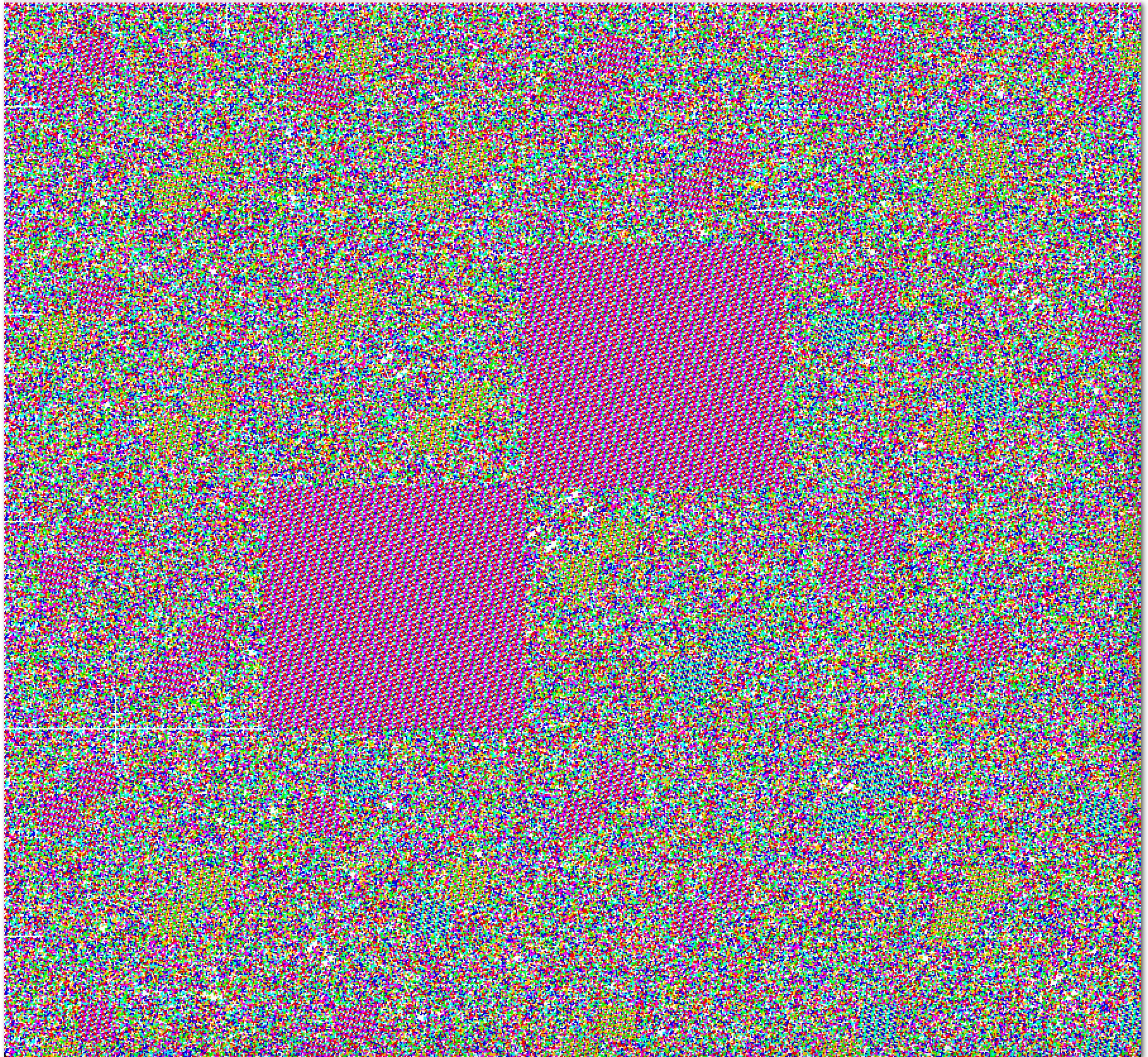


3 over 3 under
mod 5
difference carpet

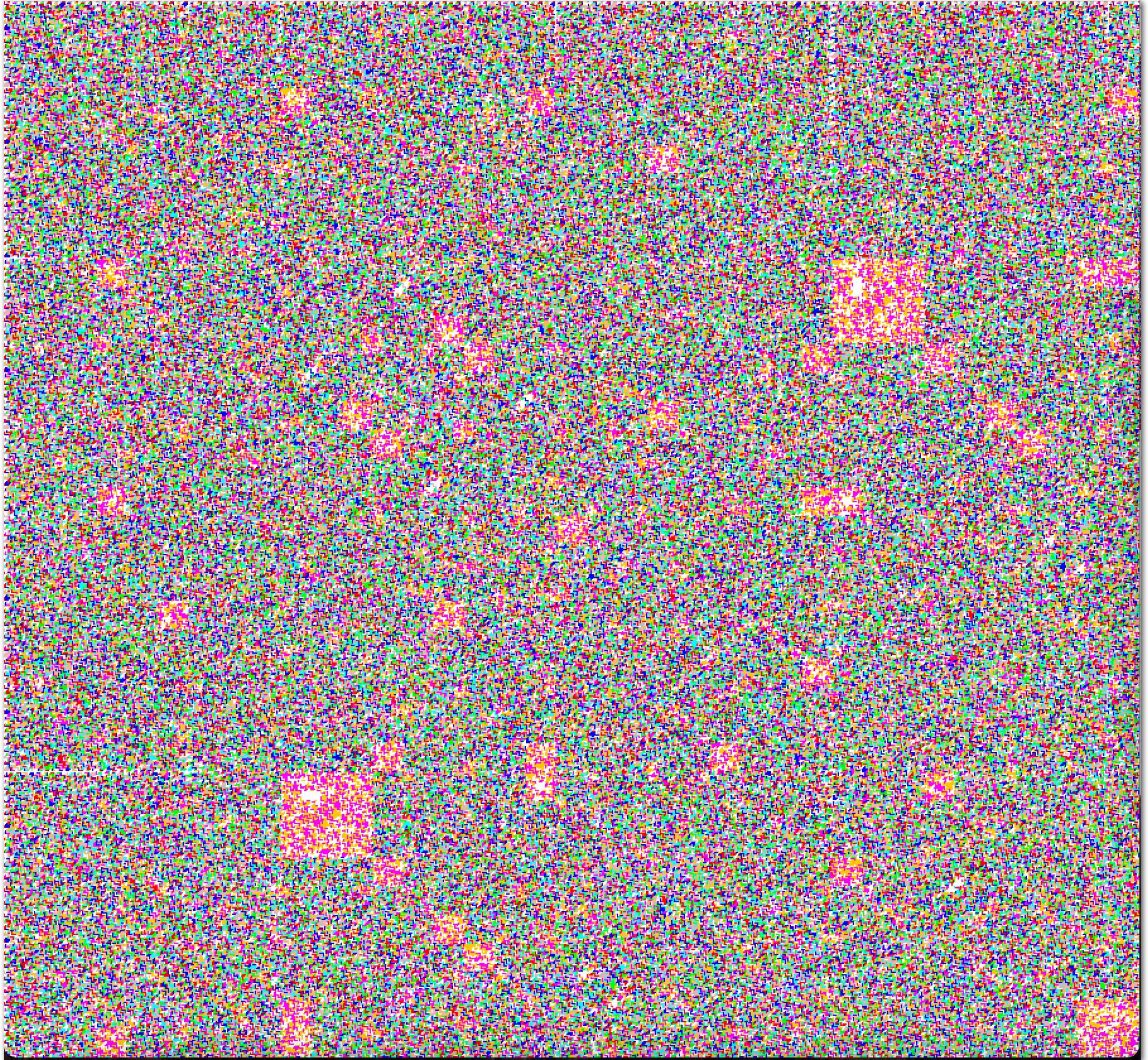


3 over 3 under
mod 7
difference carpet
(unthinnned)

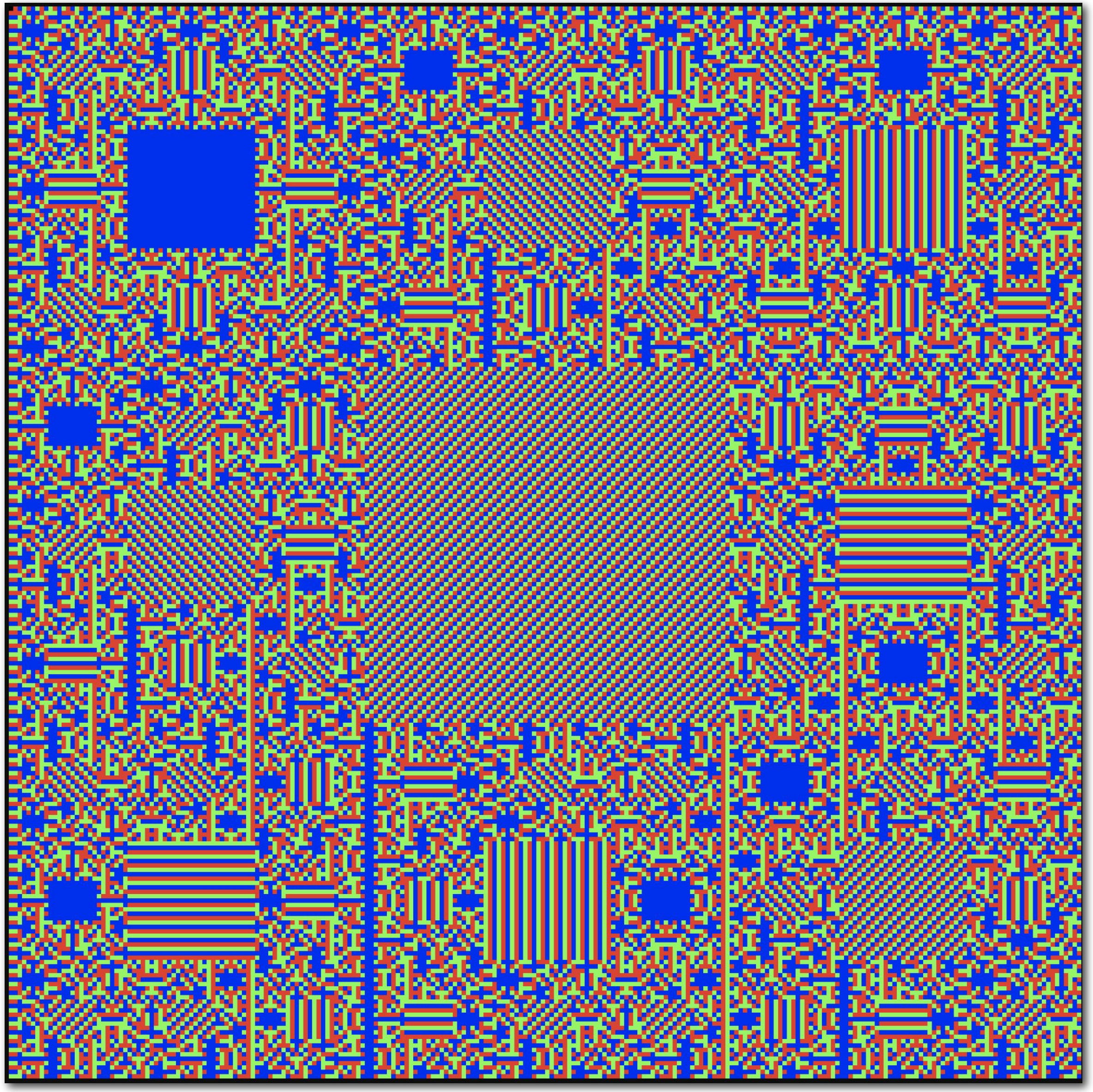
3 over 3 under
mod 7
difference carpet



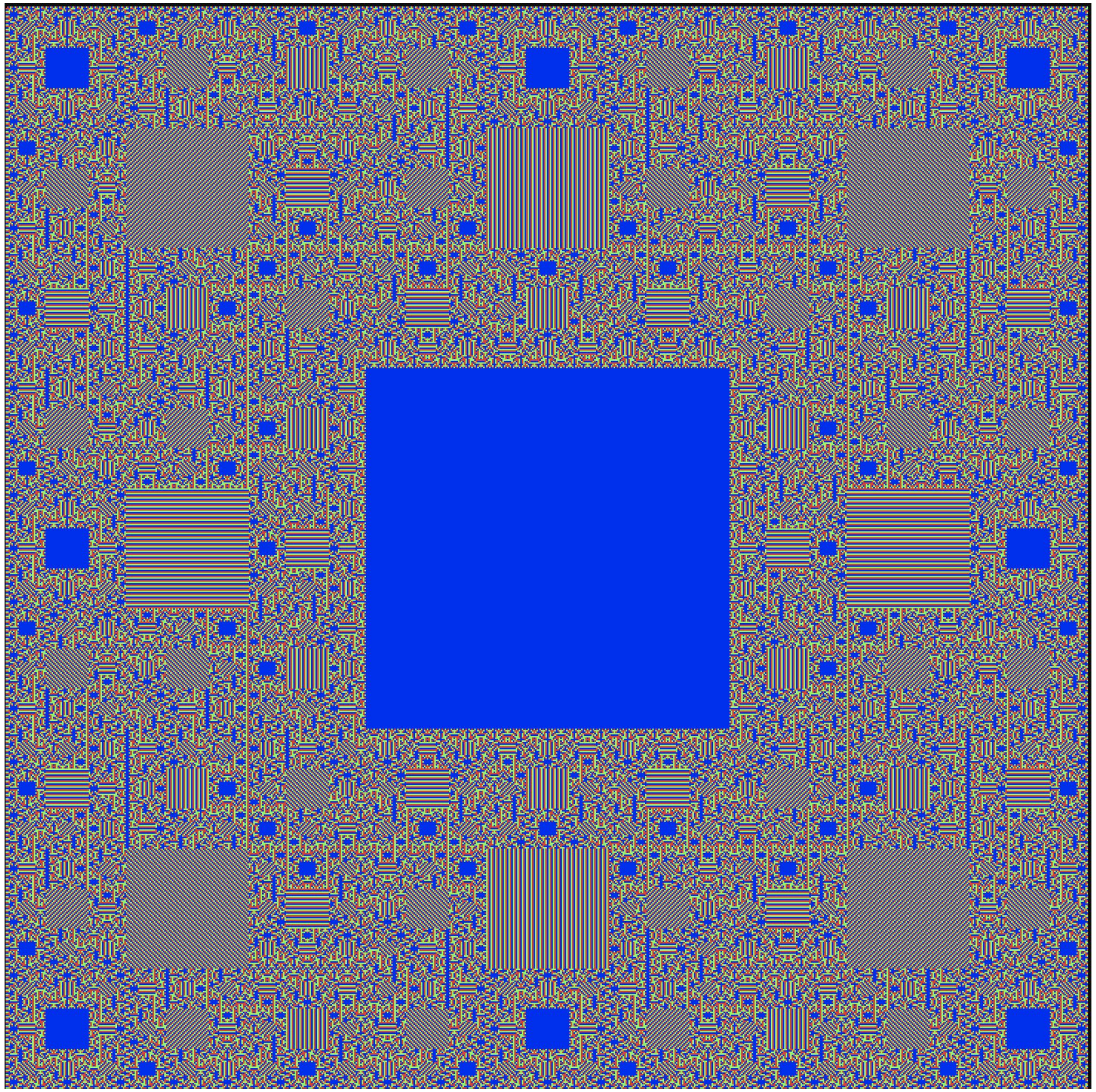
3 over 3 under
mod 9
difference carpet



Carpets with Periodic Fringes (back to alternating)

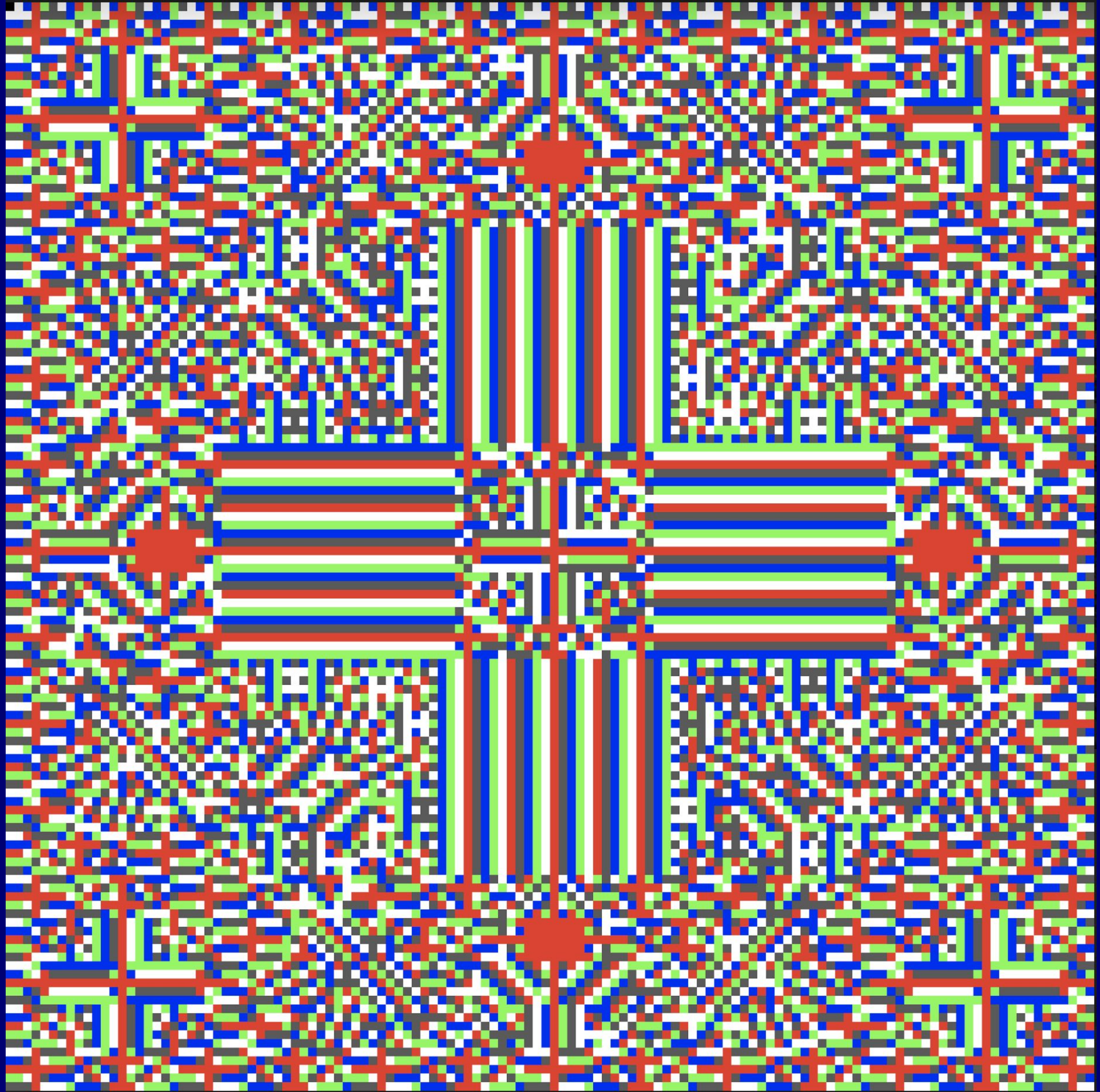


243 x 243

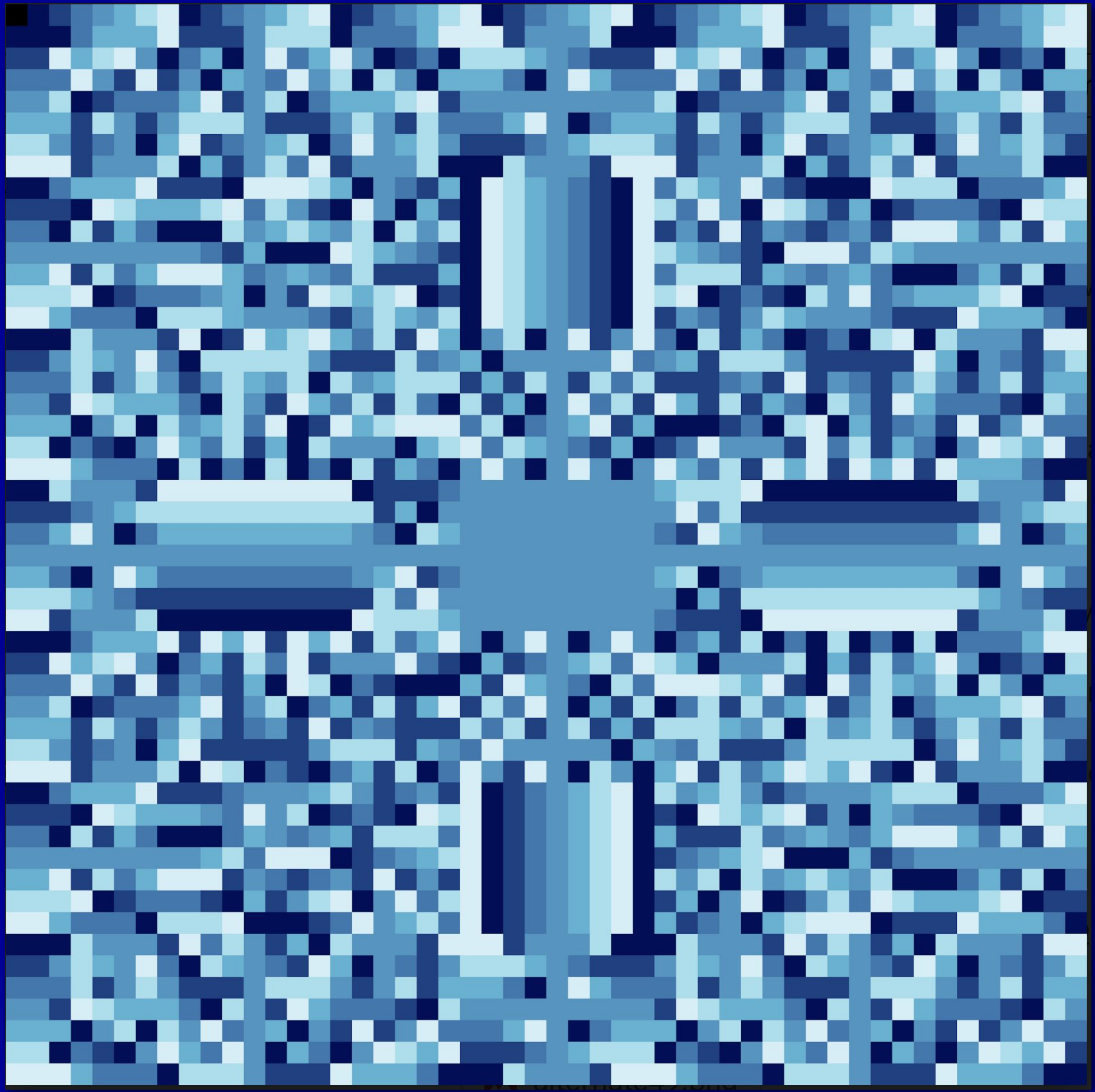


729 x 729

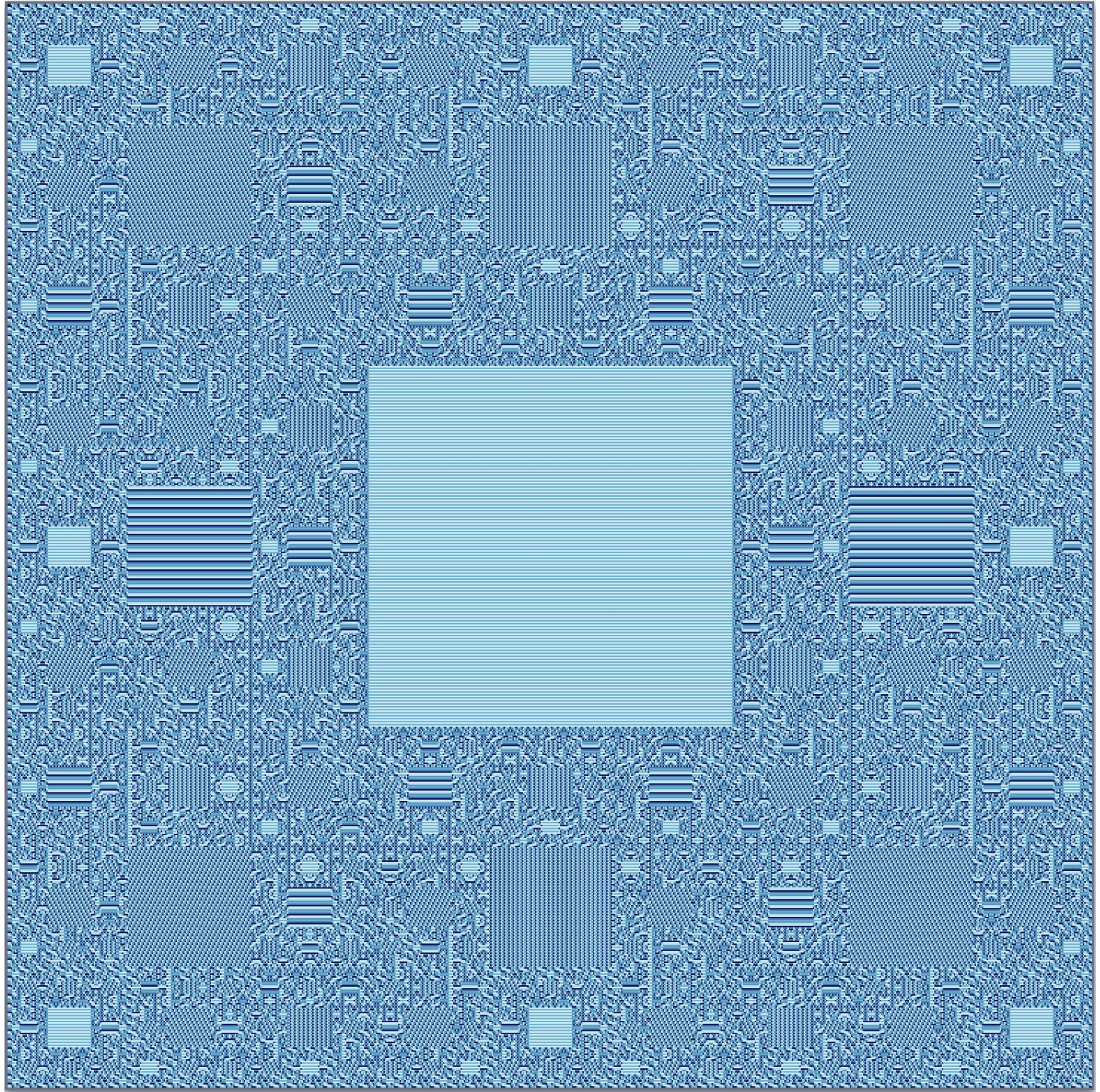
p = 3,
periodic fringe



p = 5,
periodic fringe

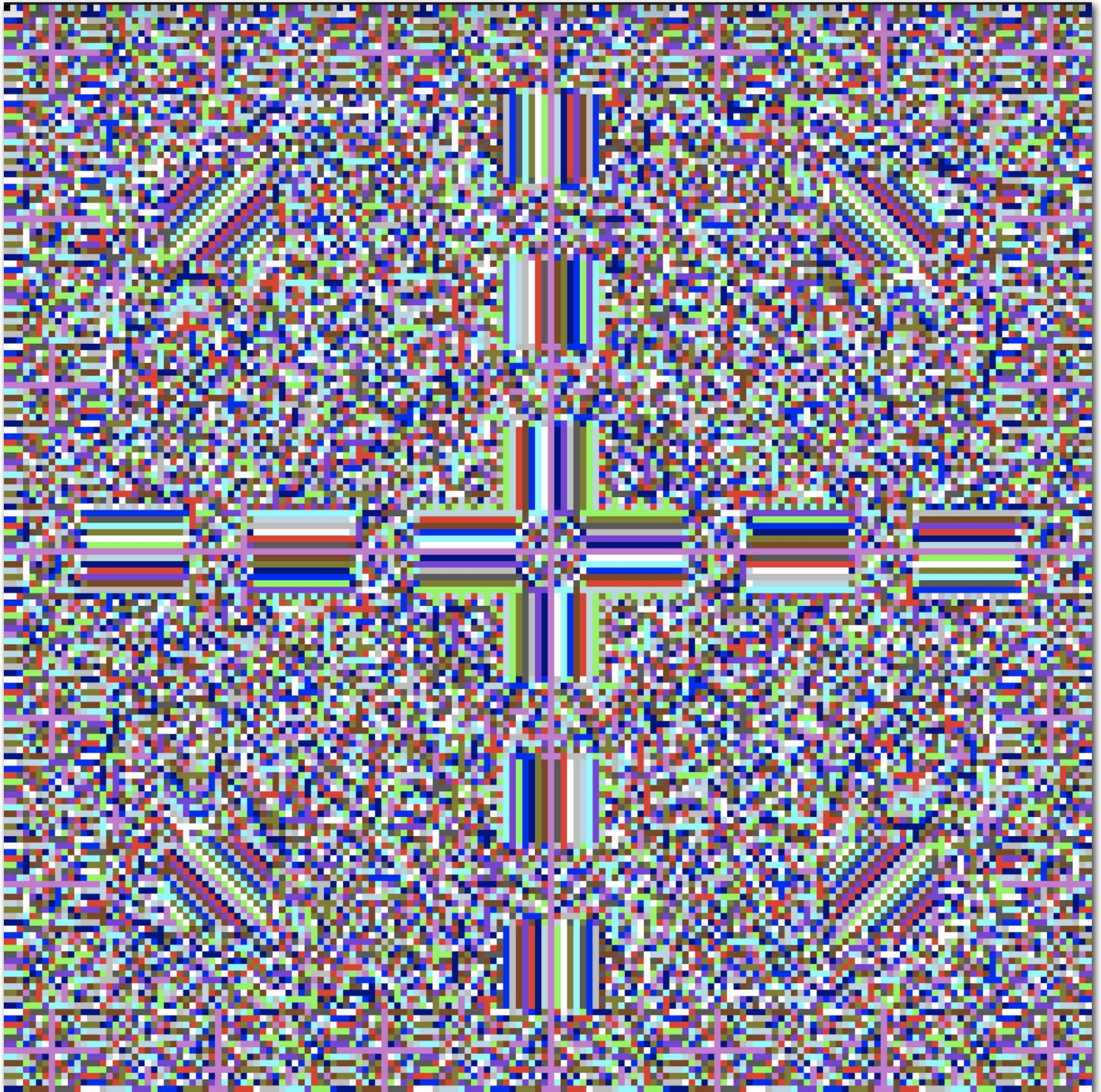


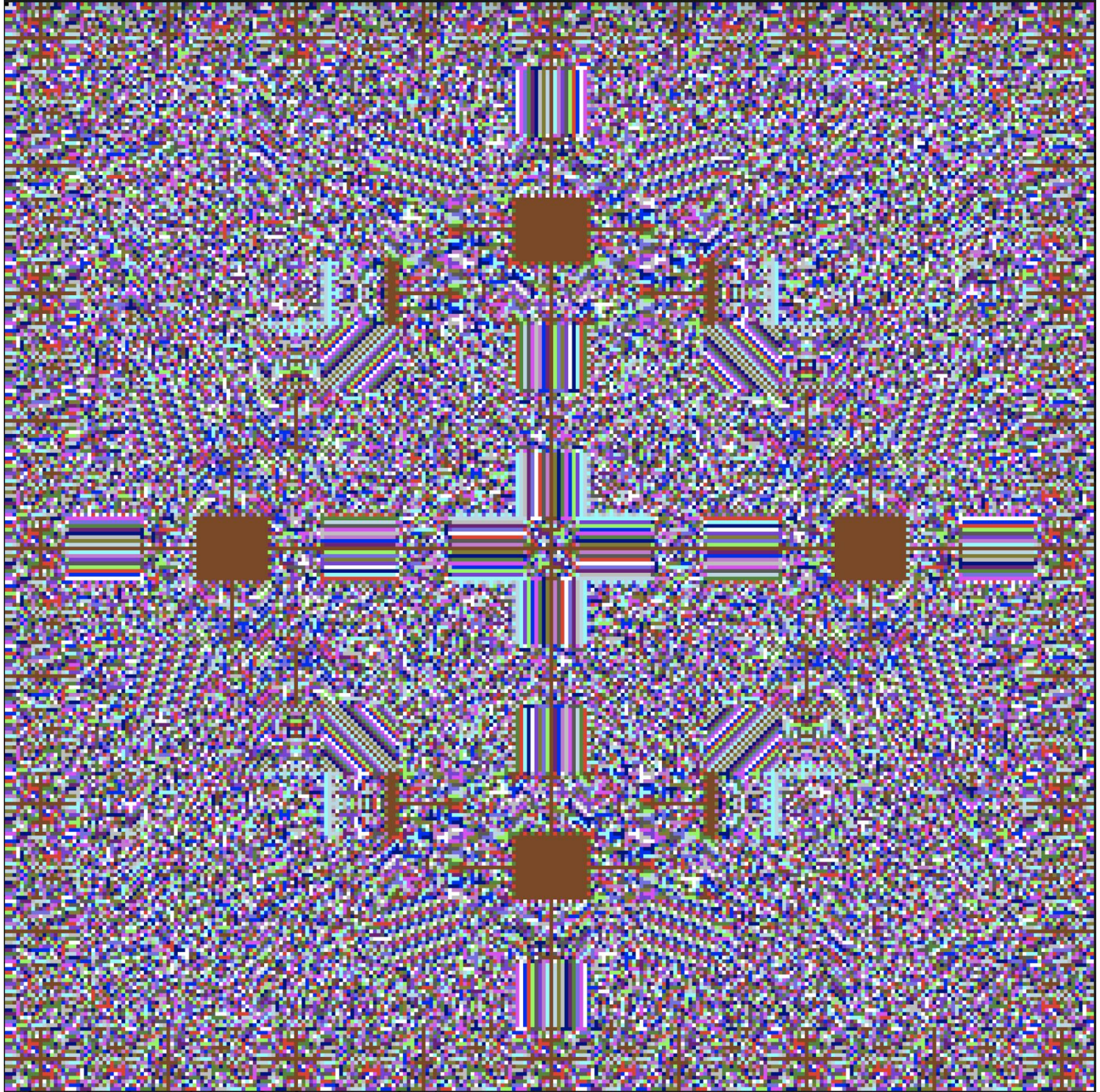
p = 7,
periodic fringe



p = 9,
periodic fringe

p = 13,
periodic fringe





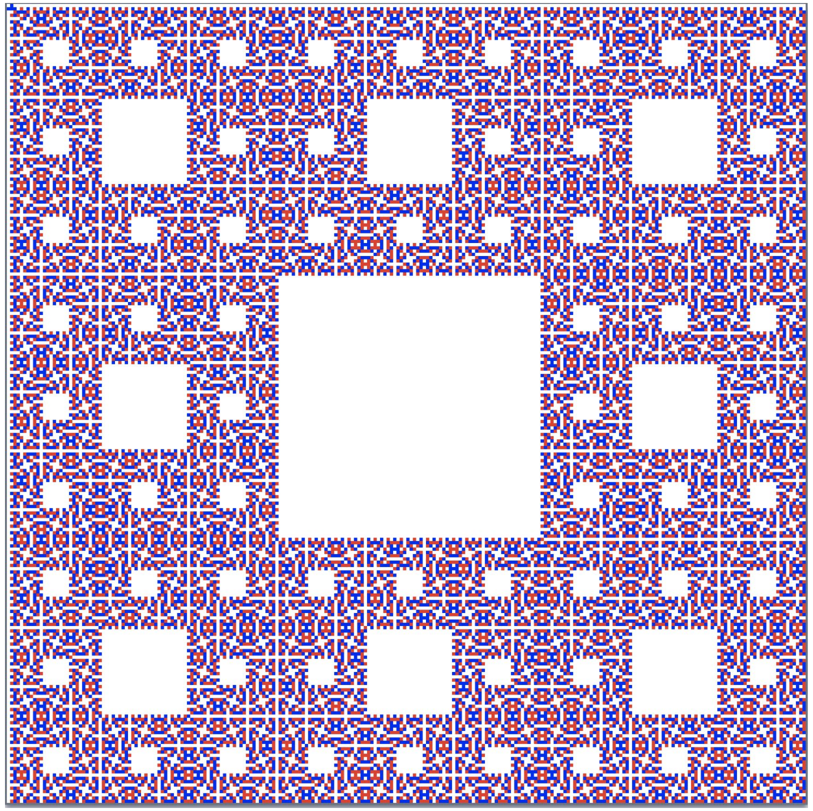
p = 17,
periodic fringe

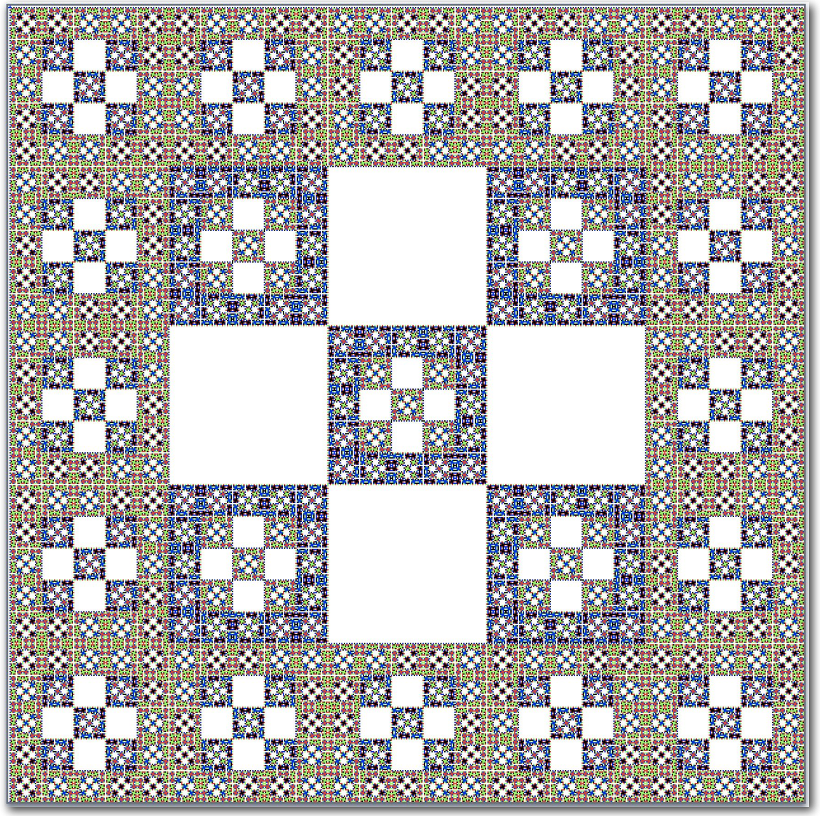
Come to humorous math theater put on by the Mobiusbandaid Players,

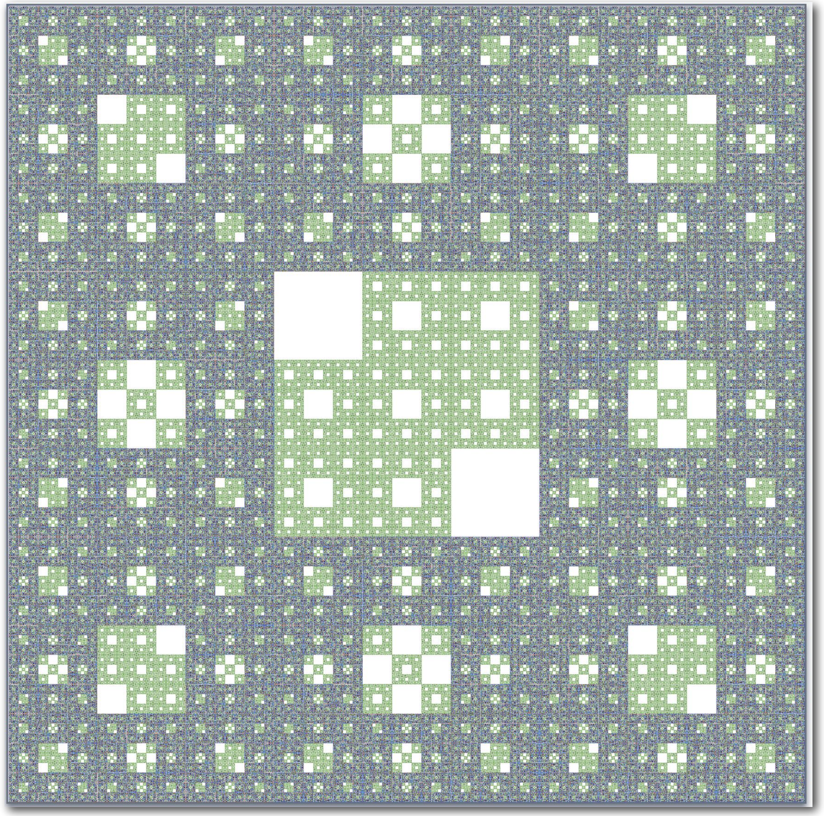
Tues. Jan. 6, 6:00-7:00 pm in Ballroom A

(including an appearance by the President…)

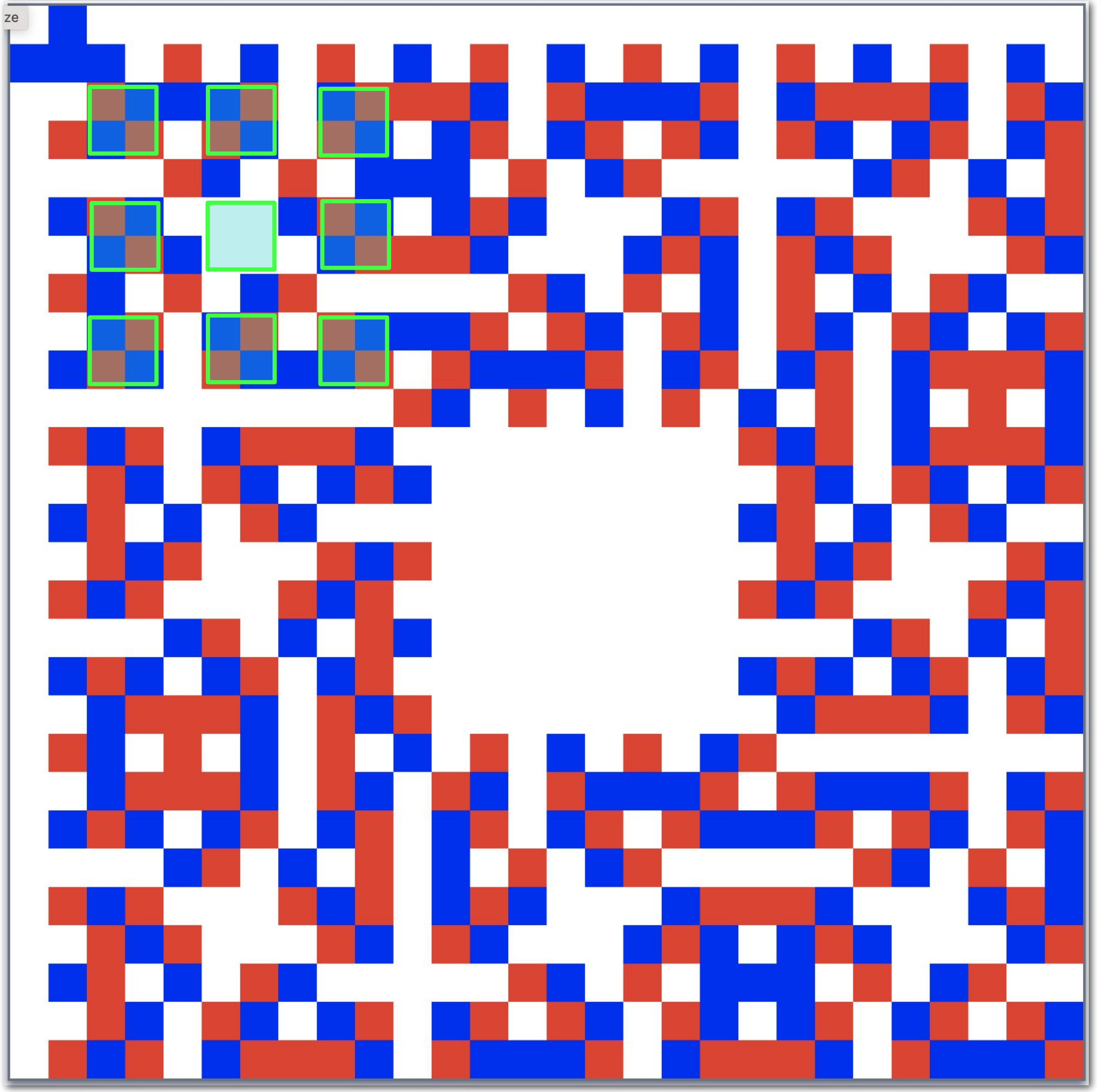
The End





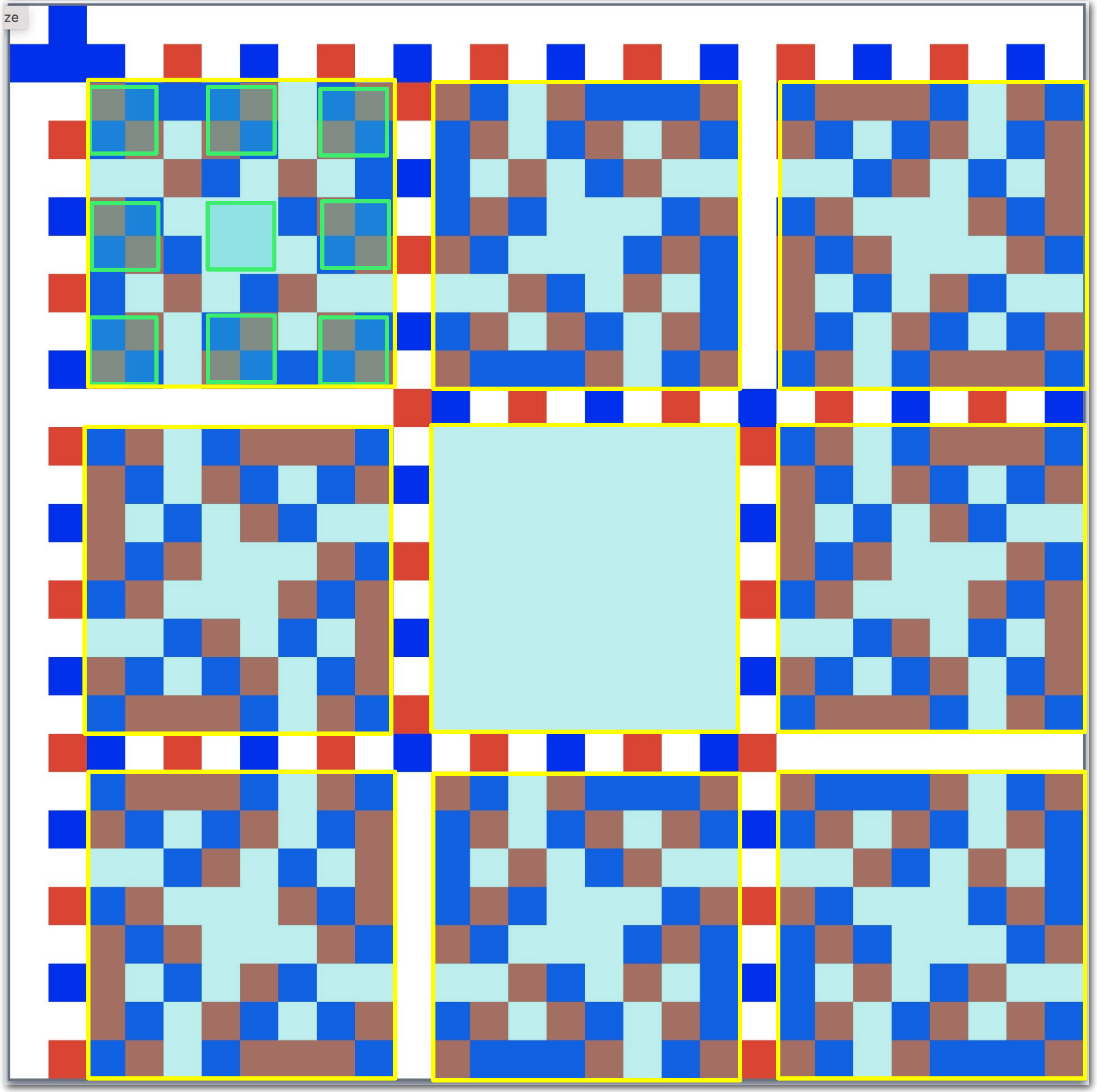


3-color difference carpet



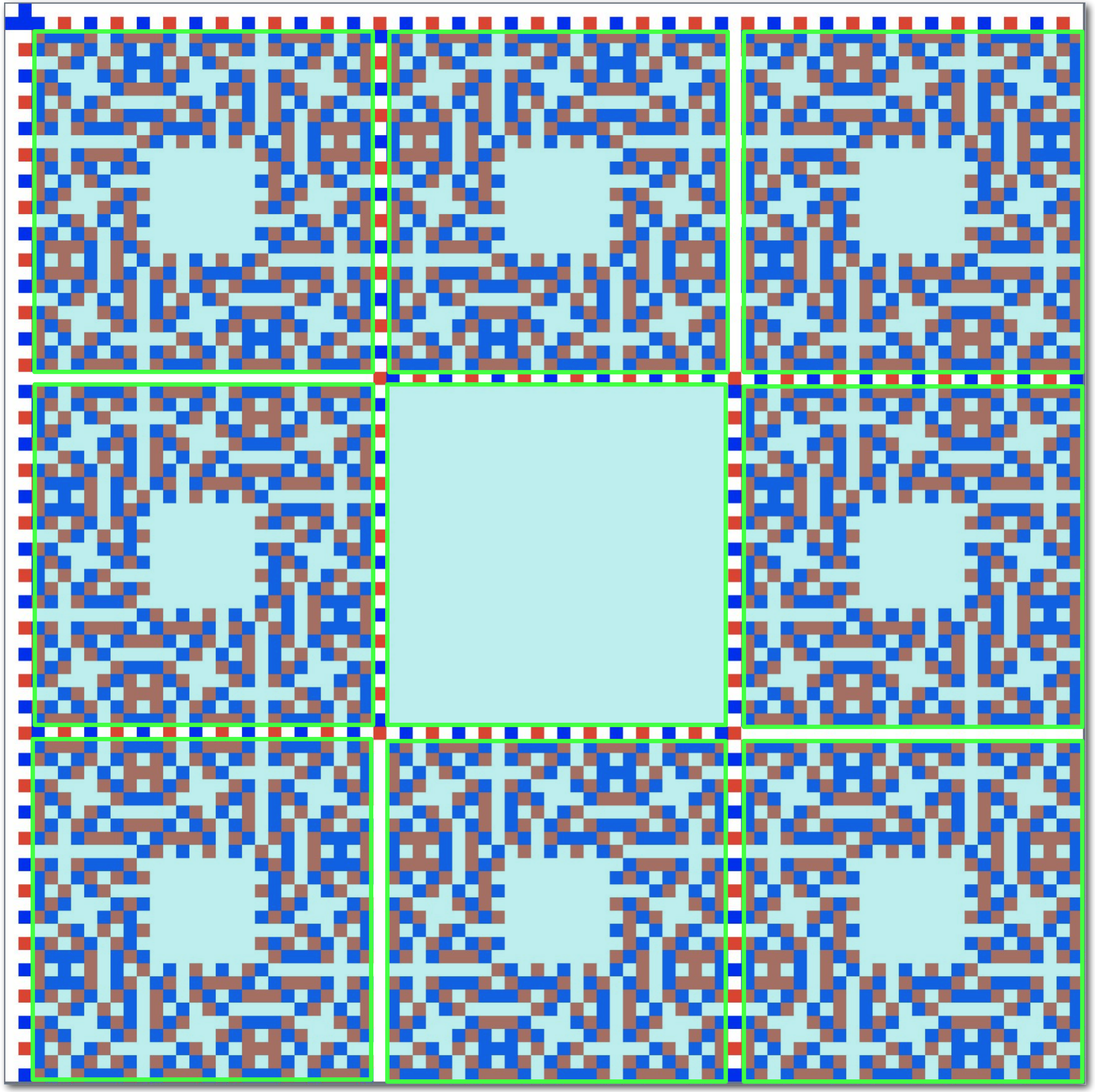
p = 3

3-color difference carpet

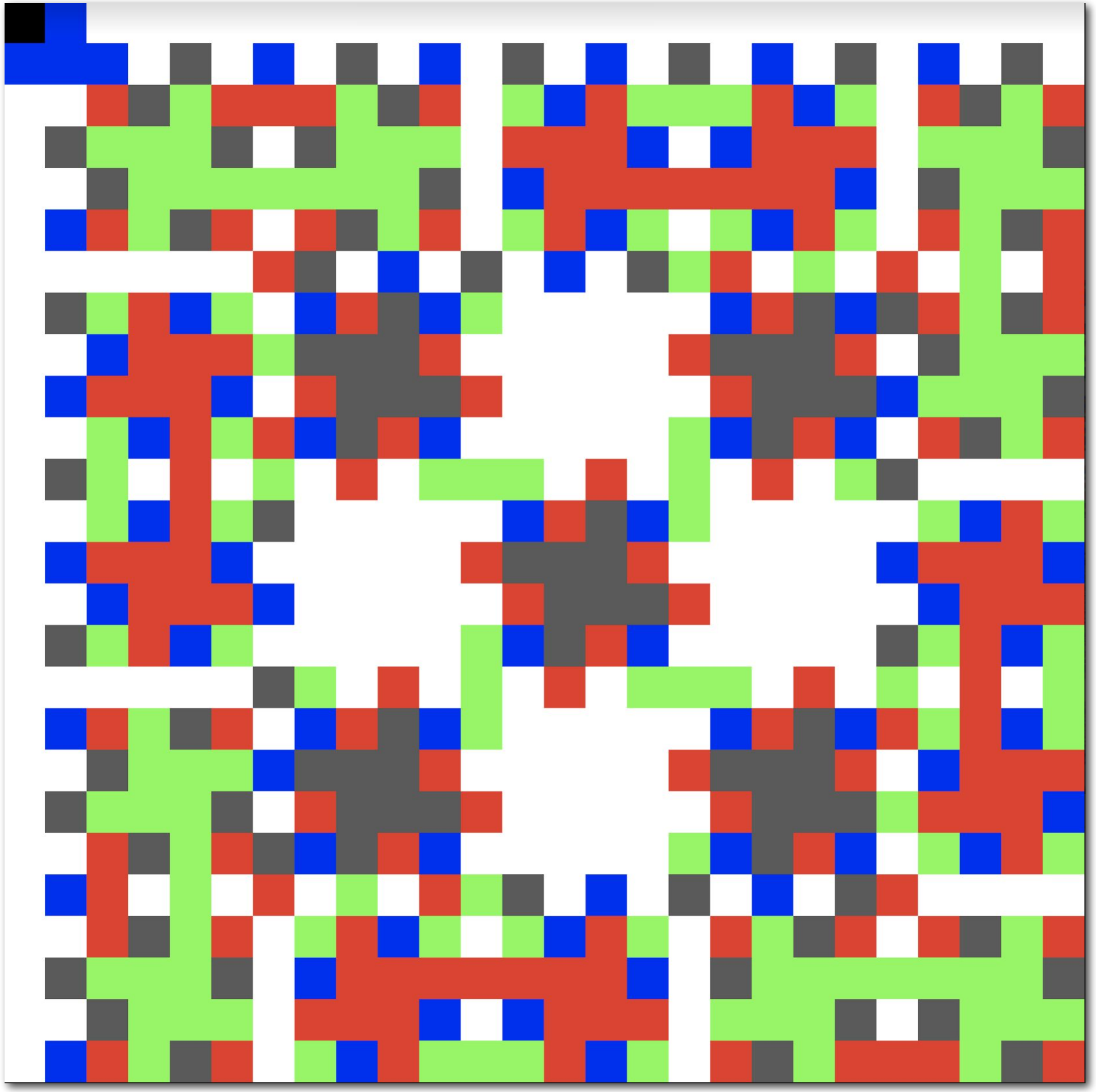


p = 3

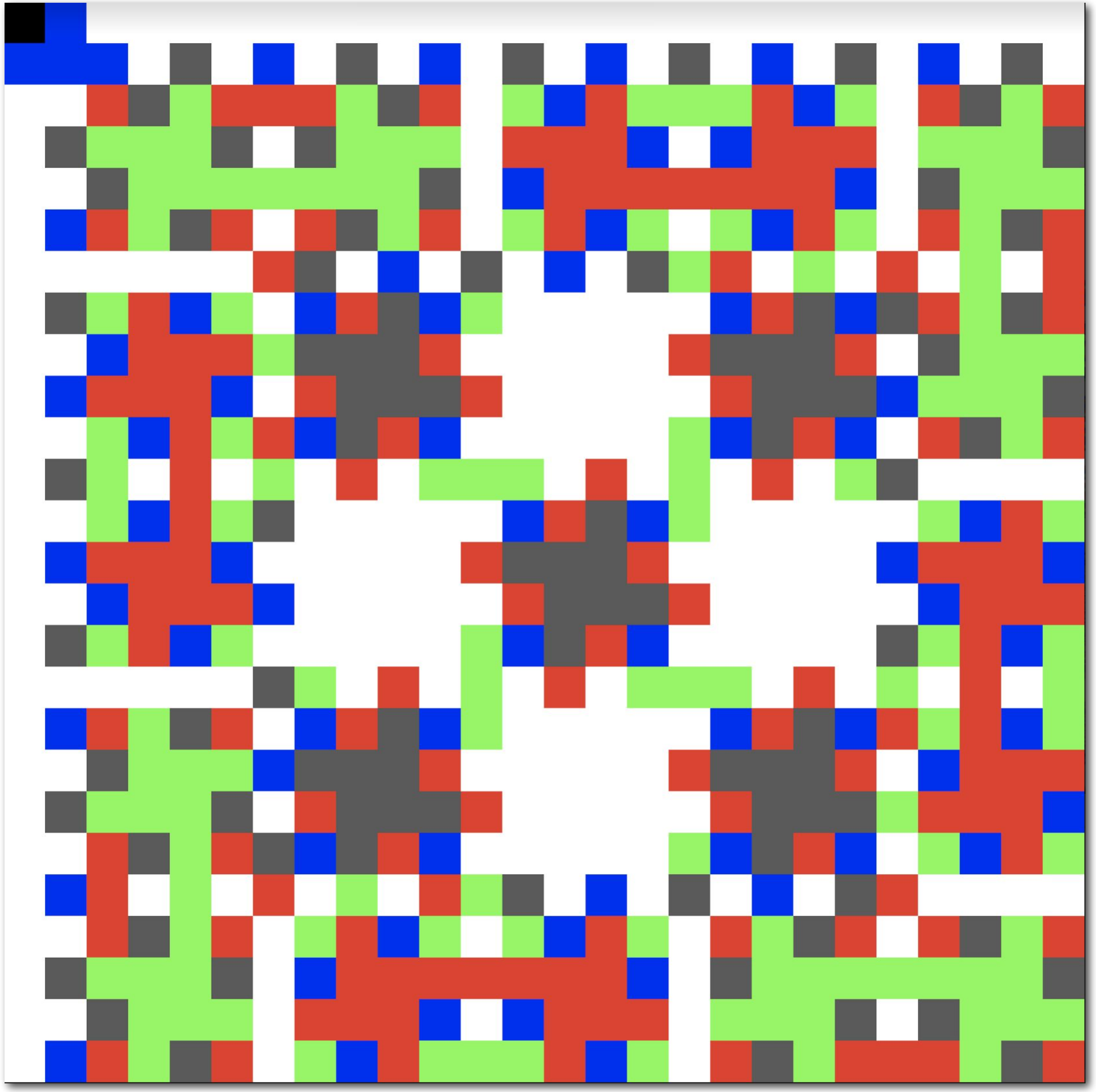
3-color difference carpet

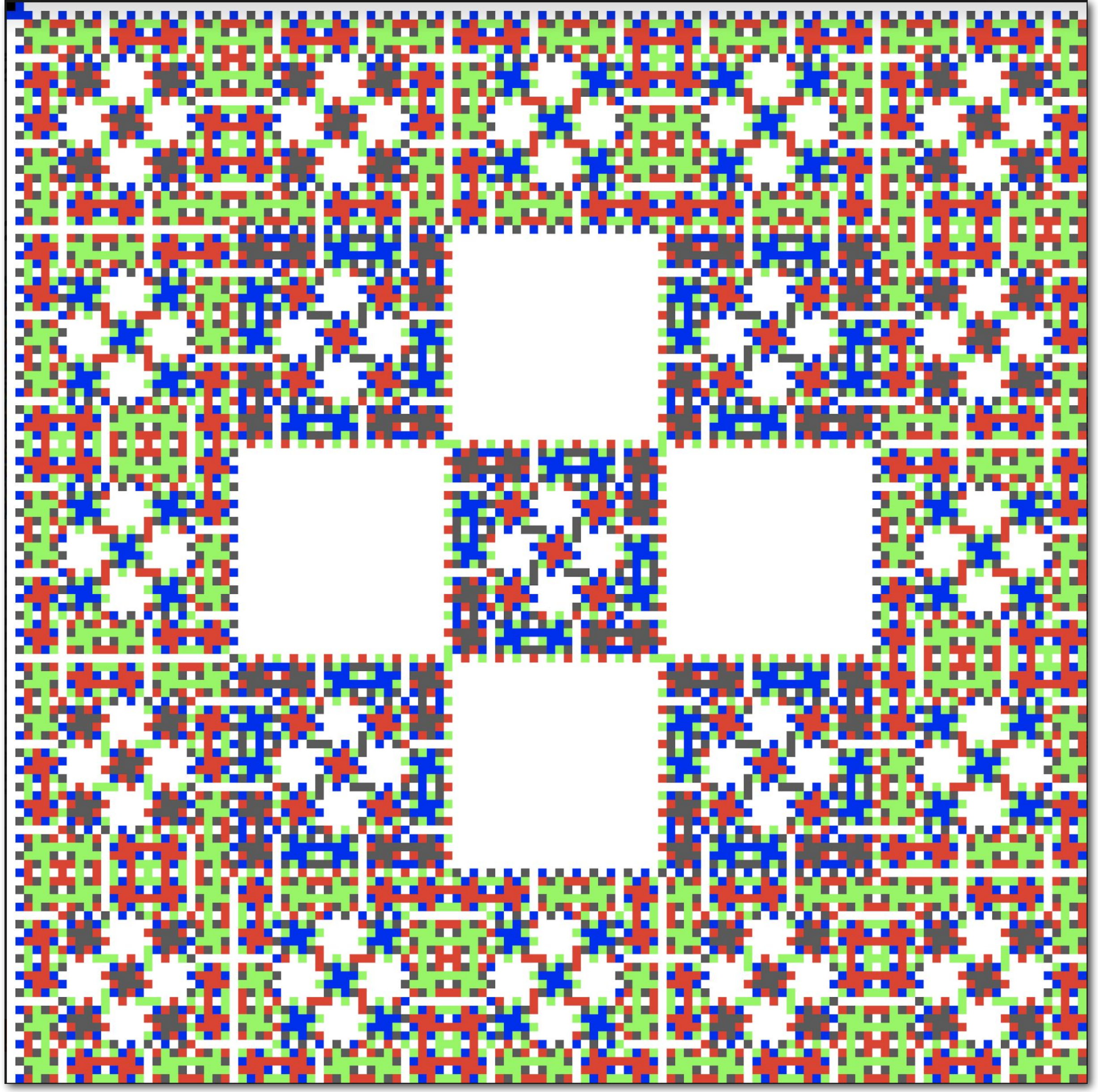


p = 3



p = 5





p = 5