The Home-Field Paradox: These Go To Eleven

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AMS Special Session
A Showcase of Recreational Mathematics
JMM Washington, January 4–7, 2026

Inspiration

jmm 2024 Joint Mathematics Meetings

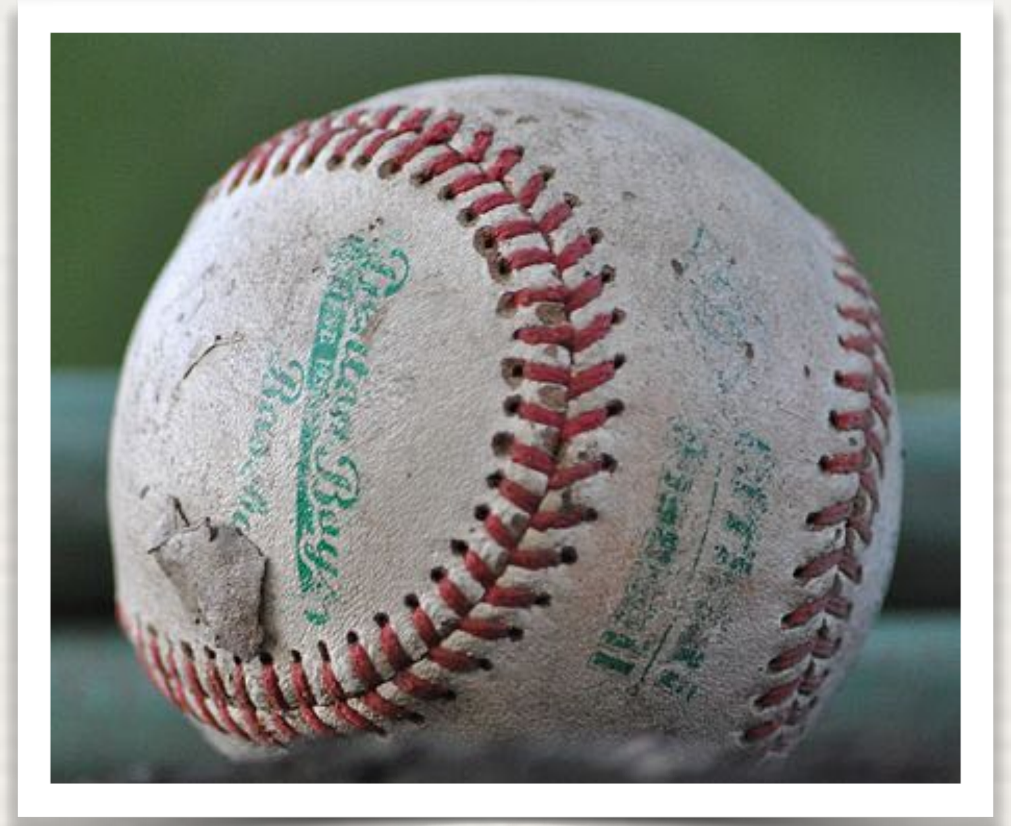
San Francisco • January 3–6

- **Saturday January 6, 2024, 1:00 p.m.-5:00 p.m.**
  AMS Special Session on Serious Recreational Mathematics, IV
  Celebrating the 50th anniversary of the Rubik's cube in 2024, this session explores serious mathematical research on playful topics such as puzzles, toys, games, origami, and juggling. History has shown that recreational roots can lead to serious discoveries, such as probability, graph theory, and the aperiodic monotile of 2023. The session aims to showcase both the joy and depth of recreational mathematics to the global mathematical community, and share/solve open problems.
  Room 024, The Moscone Center
  Organizers:
  Erik Demaine, Massachusetts Institute of Technology edemaine@mit.edu
  Robert A. Hearn, Gathering 4 Gardner
  Tomas Rokicki, California

- 3:30 p.m.
  Probabiility and Intuition
  Peter M Winkler*, Dartmouth College
  (1192-10-30965)

- P. Winkler, The Home-Field Paradox, *Math in the Time of Corona*, A. Wonders, ed., Springer (2021), pp. 17–20.

Home-Field Paradox



- Suppose two teams meet for a best-of-seven series: the Albany AlleyCats and the Binghamton Bumblebees.
- By tradition, first three at Albany, then up to four at Binghamton.
- Evenly matched, but home team wins with probability $p > 1/2$.
- Which team is expected to do better?

Intuition?

- Maybe Albany!
  - Series often doesn't require seven games. Maybe 5 or 6?
  - Plausible that most games are in Albany.
  - Albany should usually win the series!
- Maybe Binghamton!
  - Independence: may as well play all seven games.
  - Binghamton should usually win the series!

Historical Data

- MLB (1974–2025): home team wins 53.8% of the time.
  - Regular season average (over 52 years): 53.8%.
  - Best: 1978, 57.4%. Worst: 1994, 51.6%.
  - Postseason average: 54.8%.
  - Best: 1982, 80.0%. Worst: 2023, 36.6%.
- 1974–2025: mean games played in 7-game postseason series: 5.740.

Expected Series Length

- Let $p =$ home field advantage (0.54).
- Pr(4 game series): $p^3(1-p) + p(1-p)^3$.
- Pr(5 game series): $p^4(1-p) + 3p^2(1-p)^3 + 3p^3(1-p)^2 + p(1-p)^4$.
- $\mathbb{E}(\text{series length}) = \sum_{k=4}^{7} k \Pr(k \text{ game series})$

  $= 7 - 9p + 31p^2 - 64p^3 + 82p^4 - 60p^5 + 20p^6$.
- When $p = 0.54$, expected length is 5.8153.
- When $p = 0.50$, expected length is 5.8125.
- Plausible that A should win more games!

Expected Series Winner

- No advantage for either team for series ending in at most six games.
- B has advantage in a seventh game.
- Let q_7 = probability seven games are needed.
- $\Pr(\text{B wins series}) = \frac{1}{2}(1 - q_7) + pq_7 = \frac{1}{2} + \left(p - \frac{1}{2}\right) q_7.$
- When $p = 0.54$: 0.512548.
- B should win more series!

Home-Field Paradox

- A wins more games, and B wins more series!
- Simulation: 10^6 series, $p = 0.54$.

$p = 0.54$	**Game Wins**	**Series Wins**
Albany	2914280 (50.11%)	487133
Binghamton	2901013 (49.89%)	512867

- If $p = 1$, B would win more games. Where is point of no advantage?

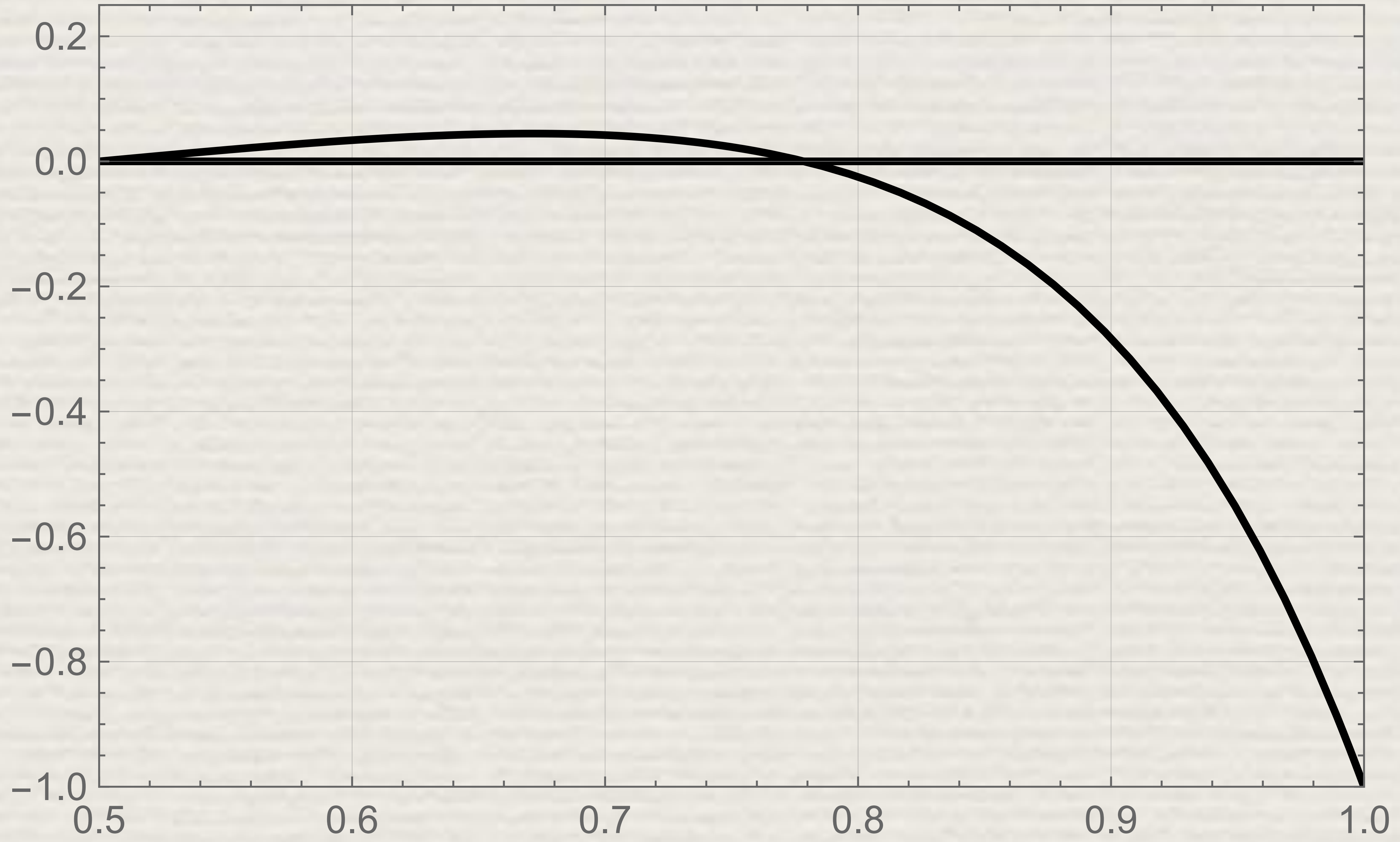
$p = 0.78$	**Game Wins**	**Series Wins**
Albany	2999212 (49.96%)	387431
Binghamton	3003545 (50.04%)	612569

Expected Game Advantage

- Compute expectation of (#A wins – #B wins).
- 4 (Pr(A in 4) – Pr(B in 4)) + 3 (Pr(A in 5) – Pr(B in 5)) + 2 (Pr(A in 6) – Pr(B in 6)) + Pr(A in 7) – Pr(B in 7)

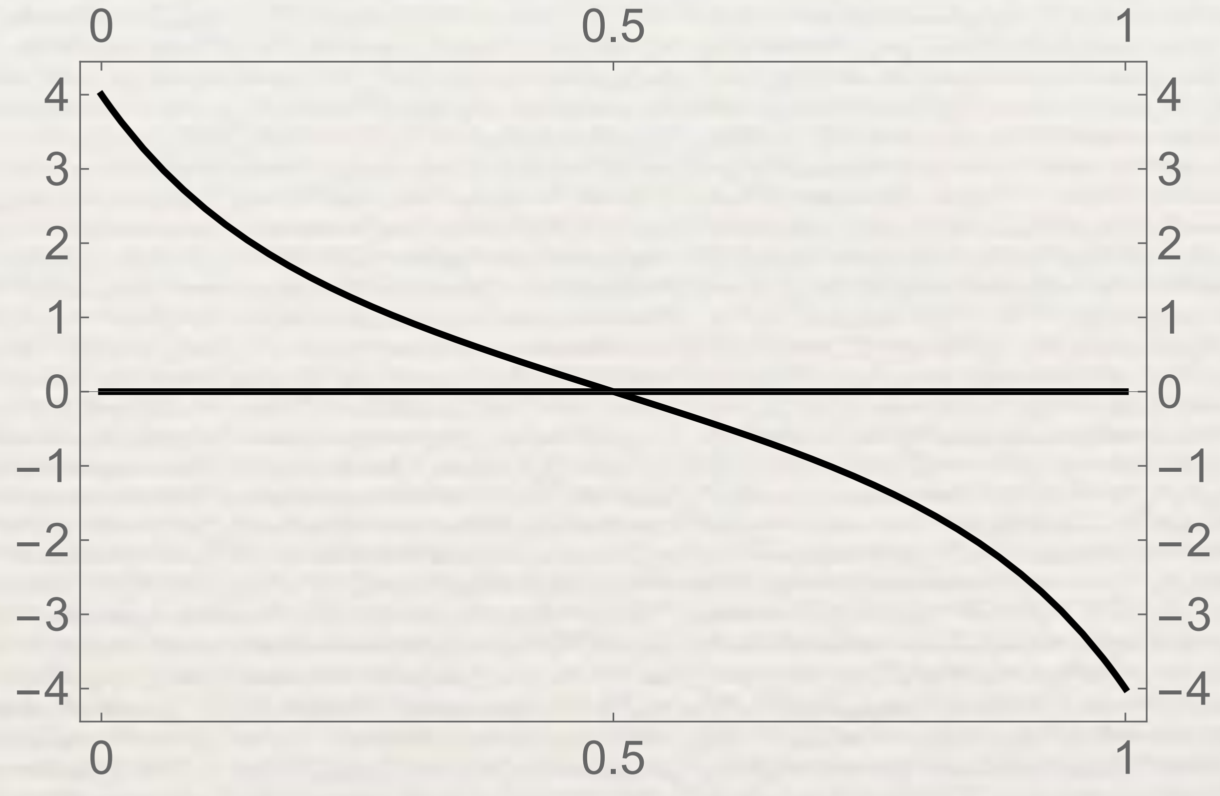
$$= (1 - 2p)\left(1 - 9p + 31p^2 - 64p^3 + 82p^4 - 60p^5 + 20p^6\right).$$

- $p = 0.54$: 0.01477.
- $p = 1$: -1.
- Break-even $p = 0.7787$.
- Optimal $p = 0.6712$: 0.04441.



Other Home-Away Patterns?

- Game advantage for other home-away seven-game patterns?



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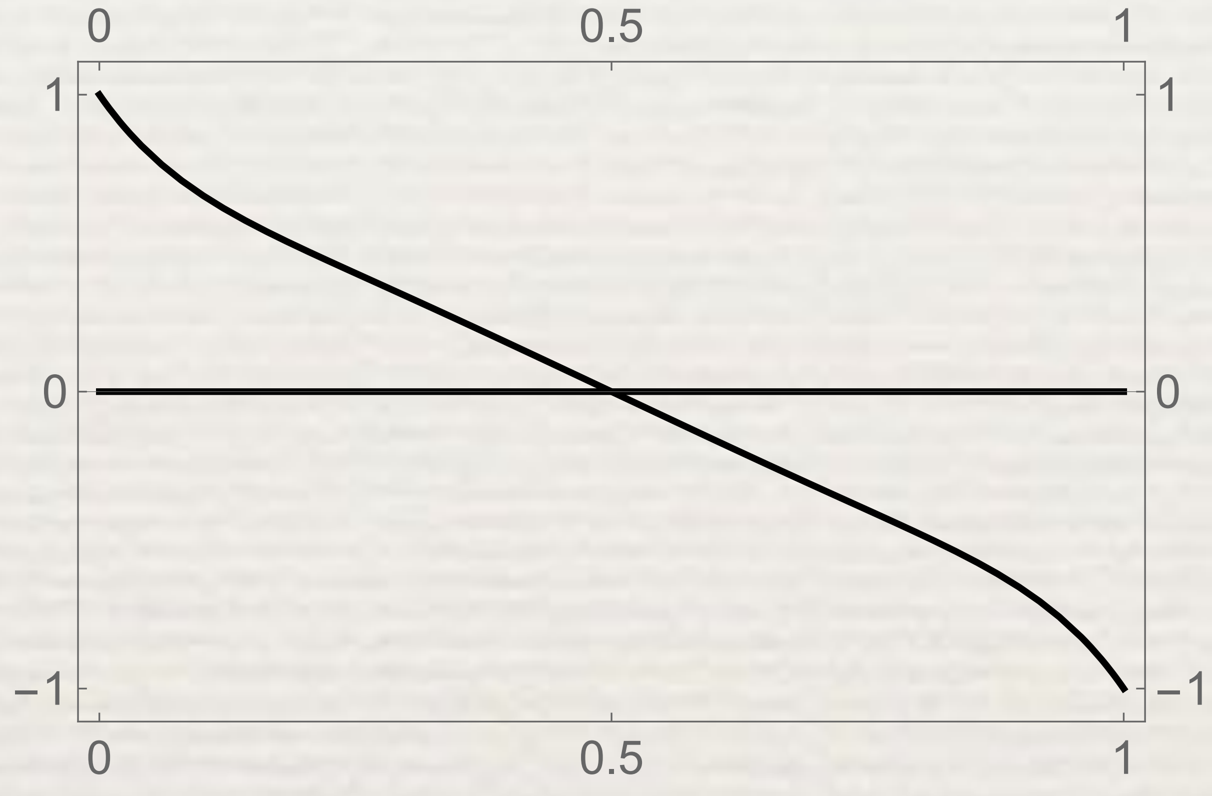

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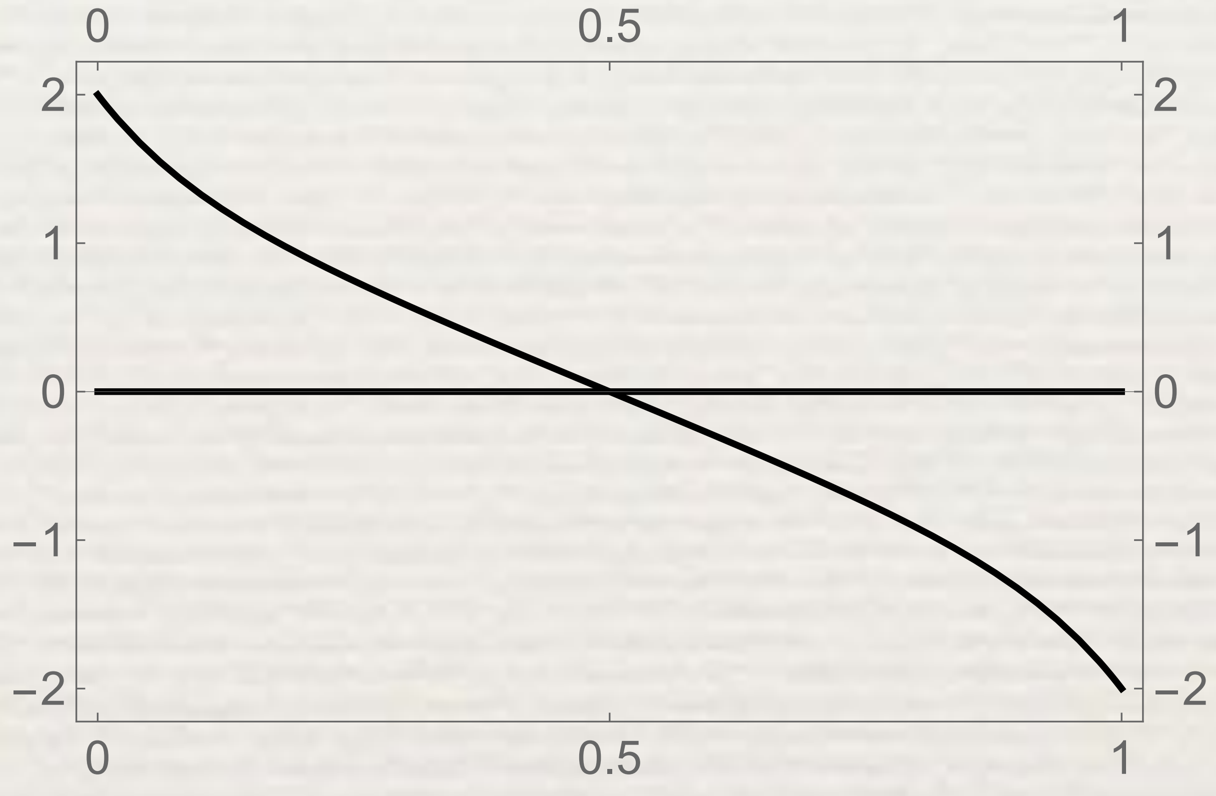

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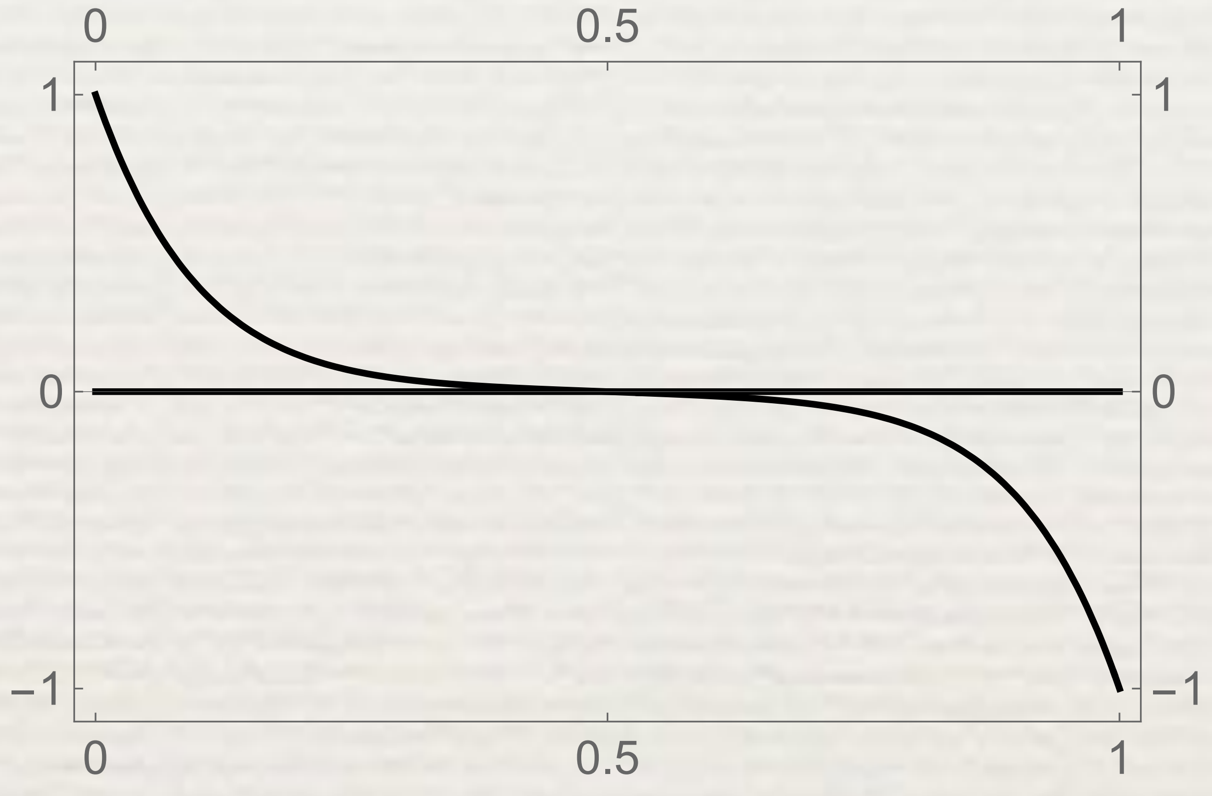

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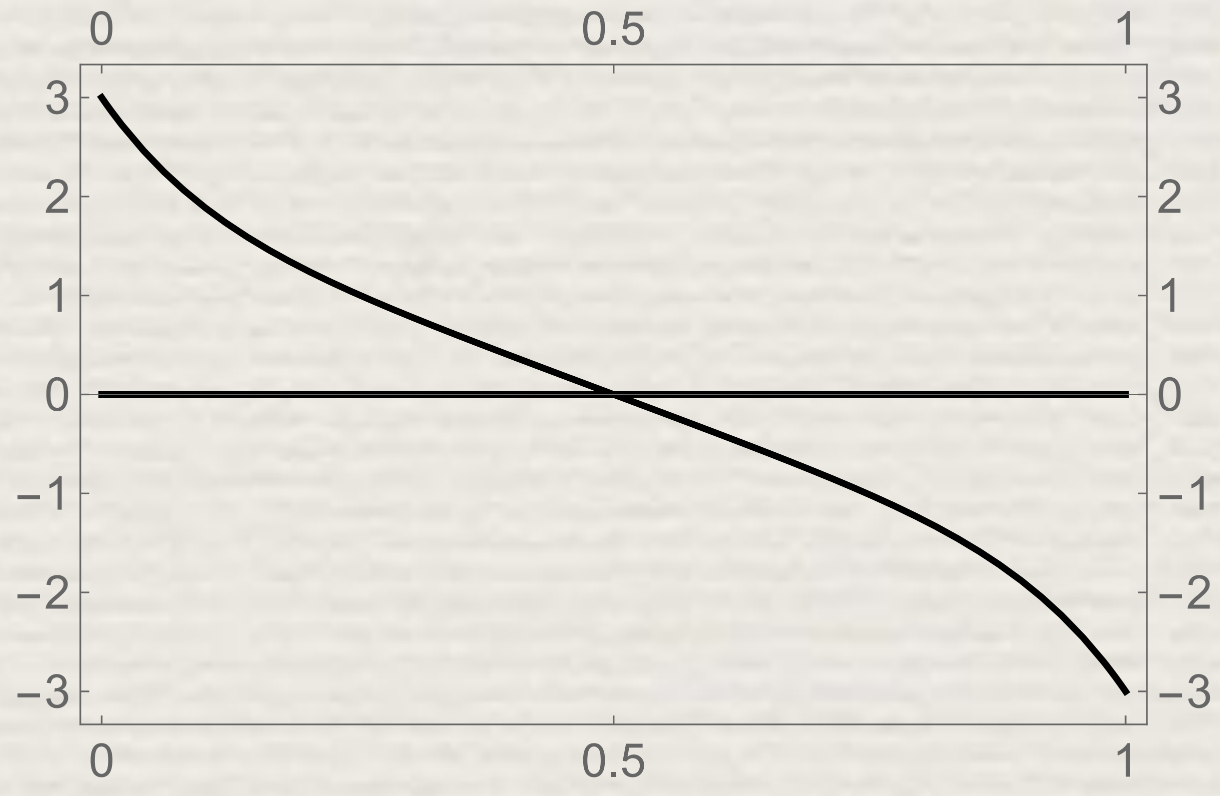

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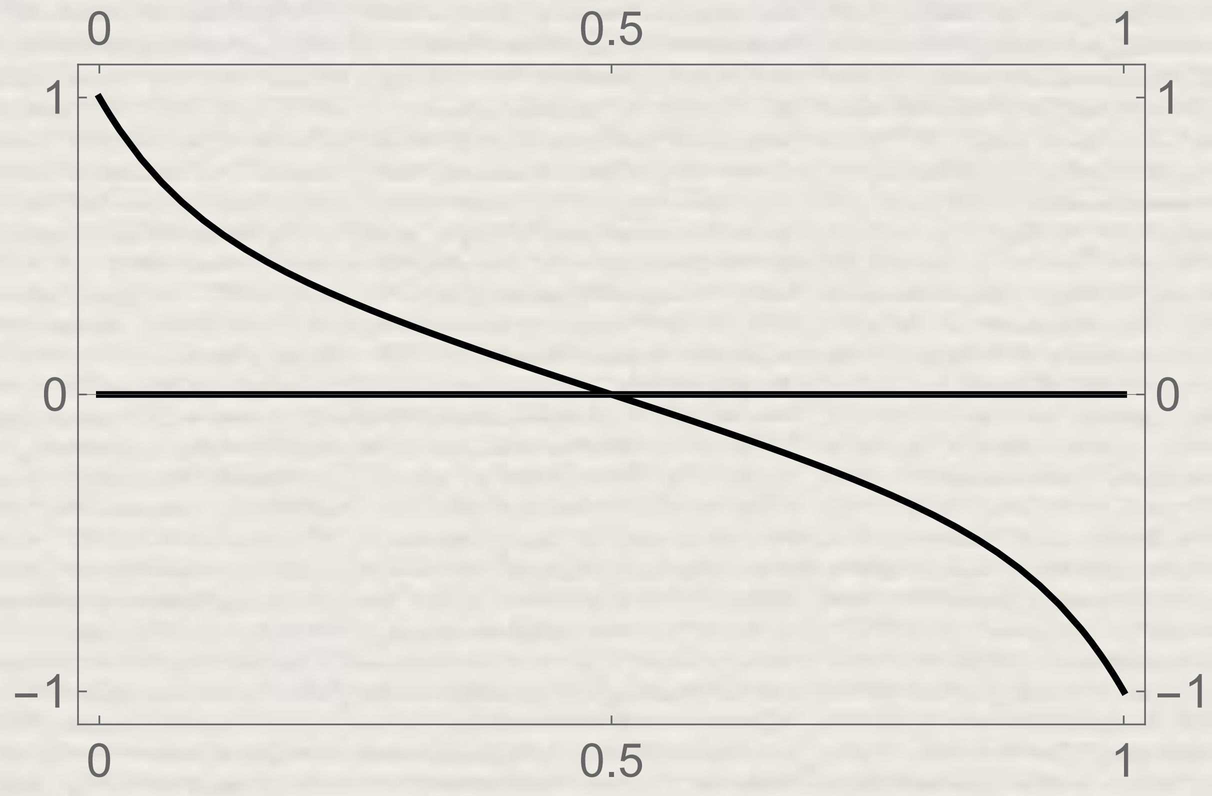

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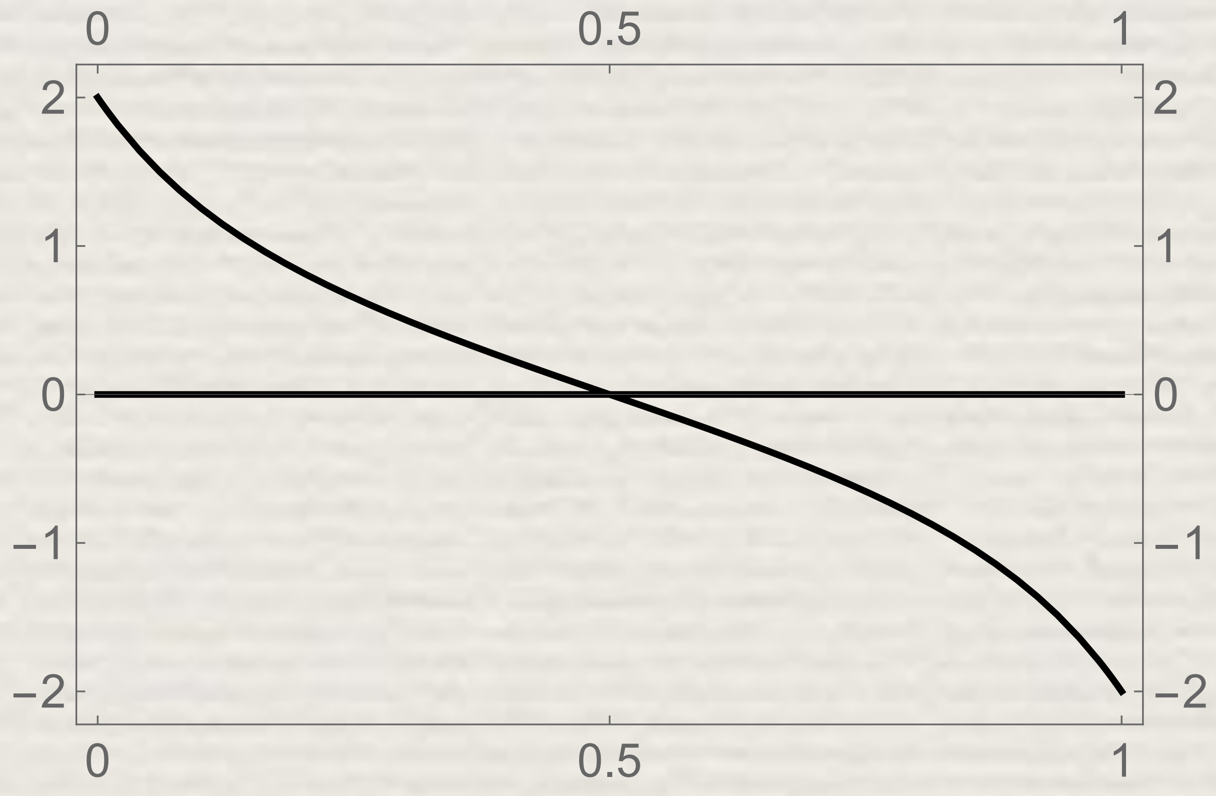

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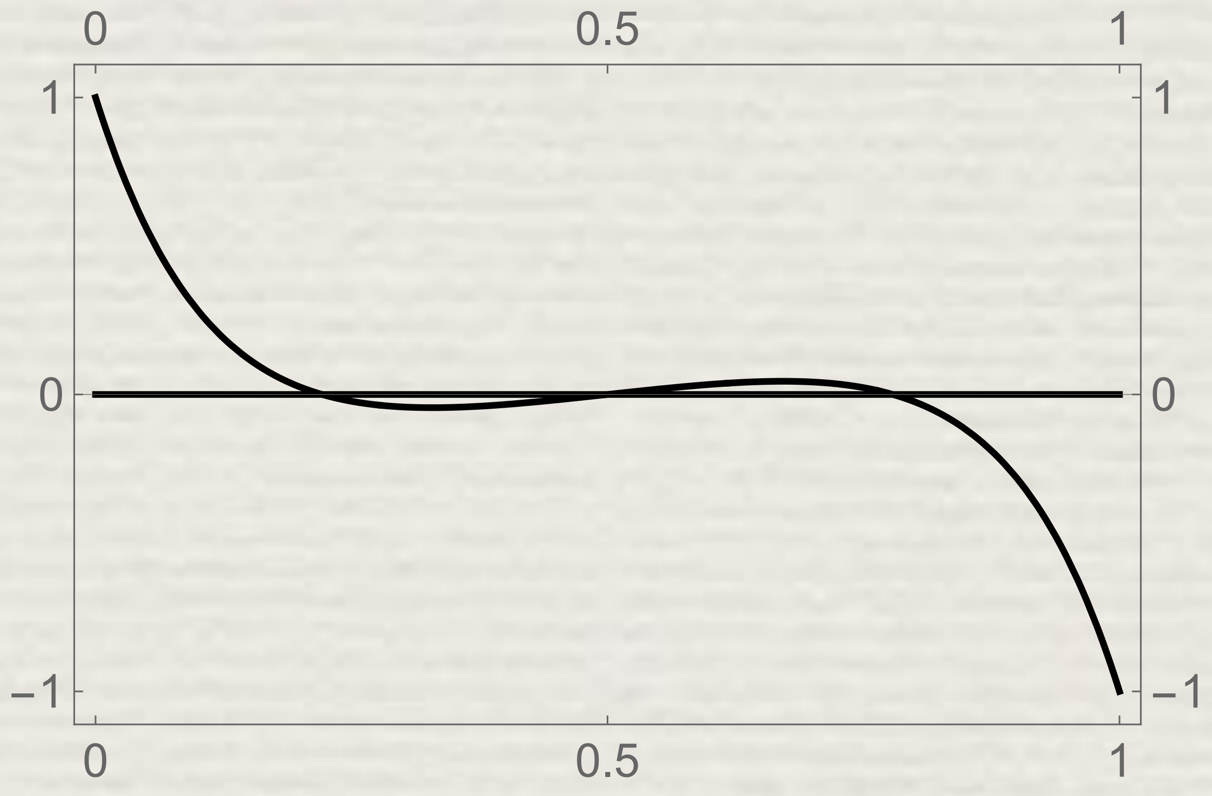

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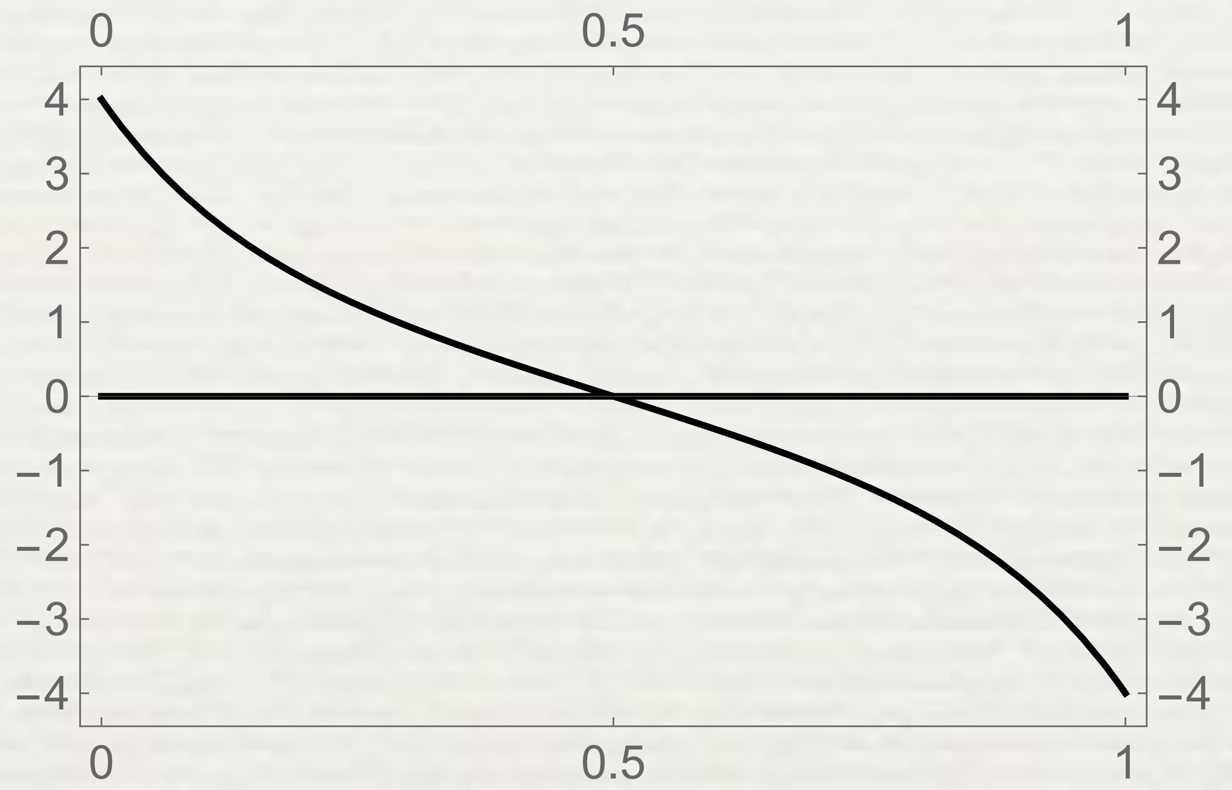

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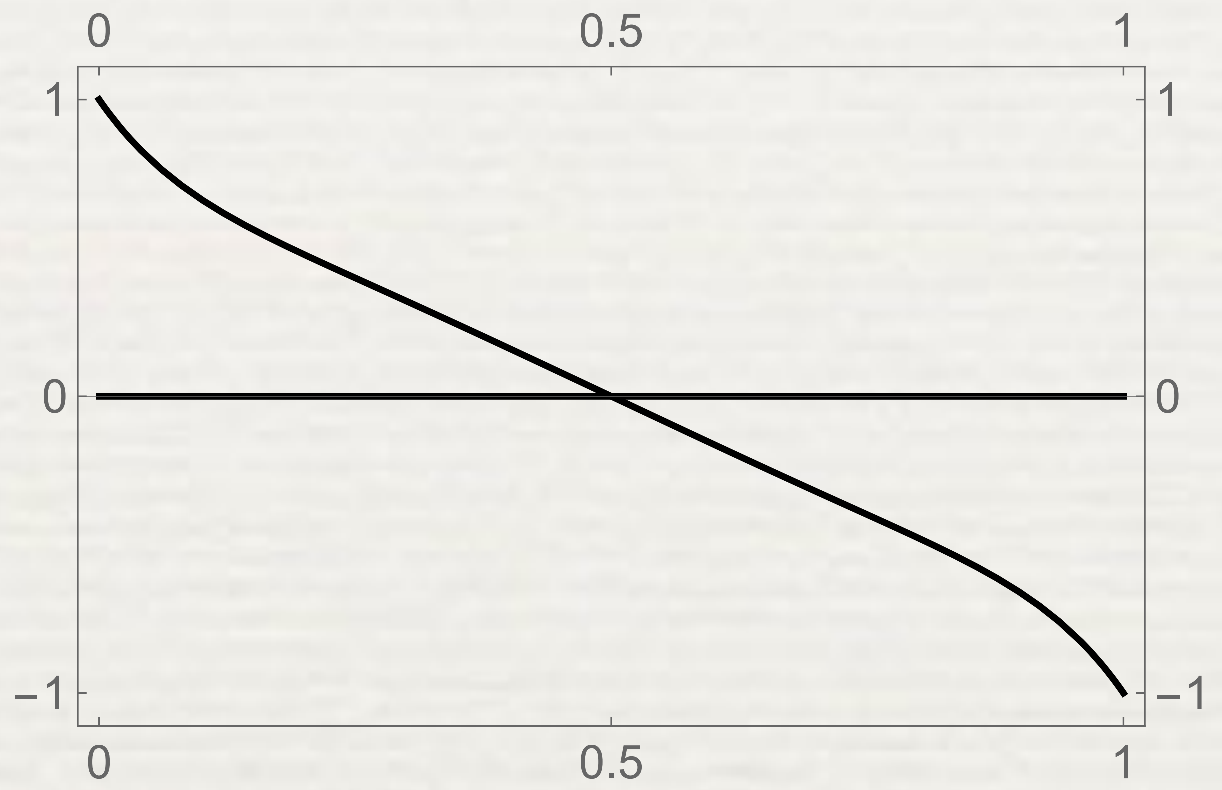

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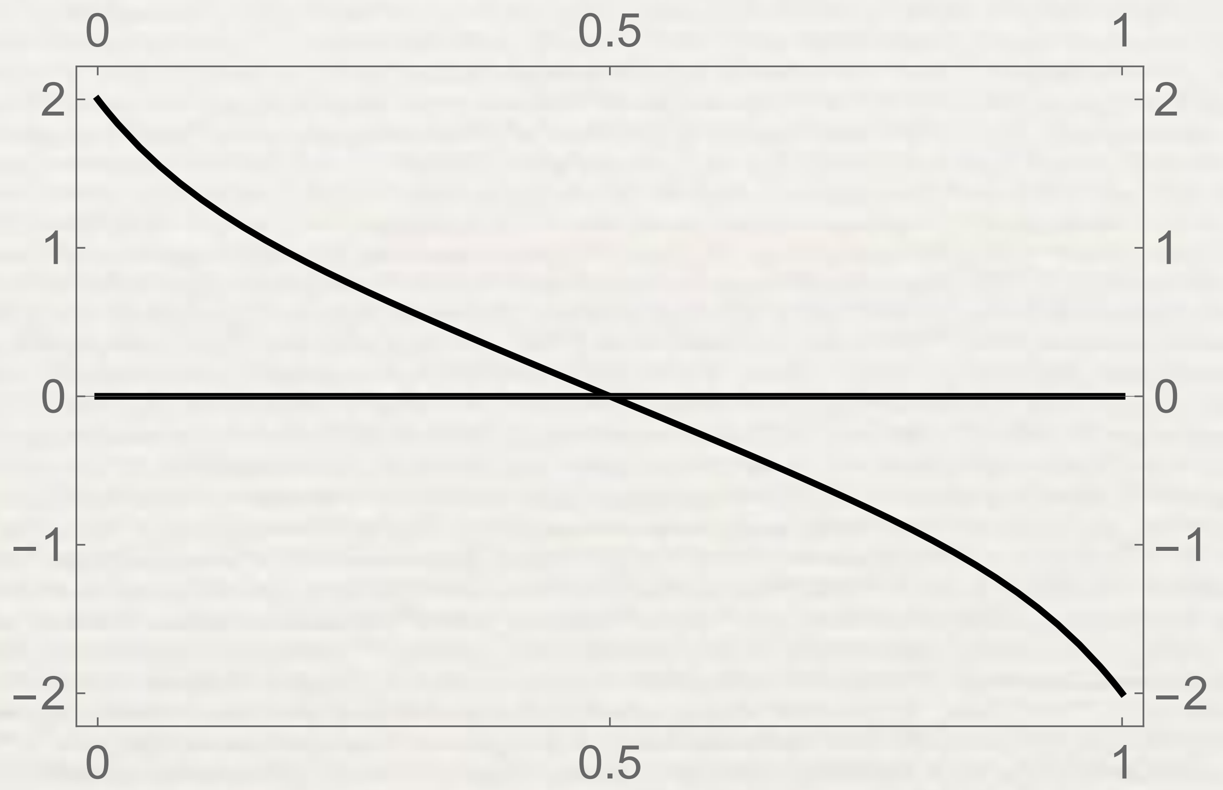

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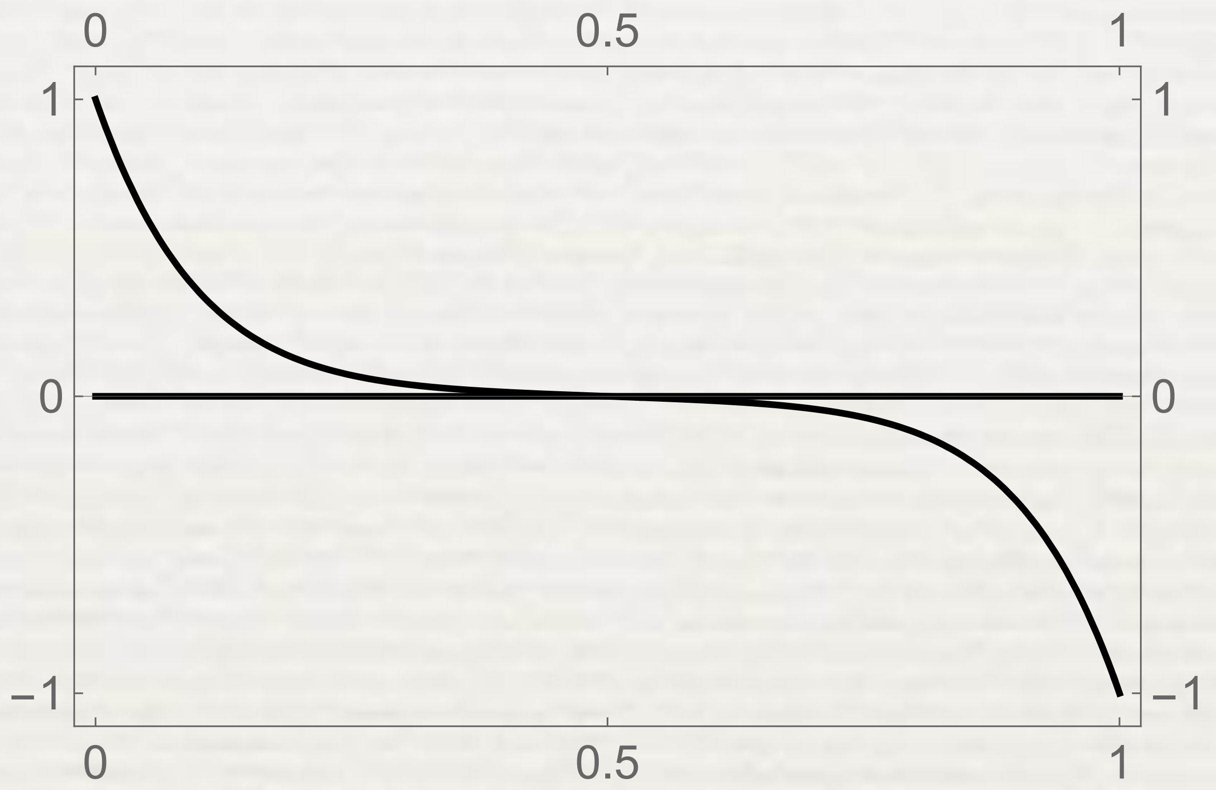

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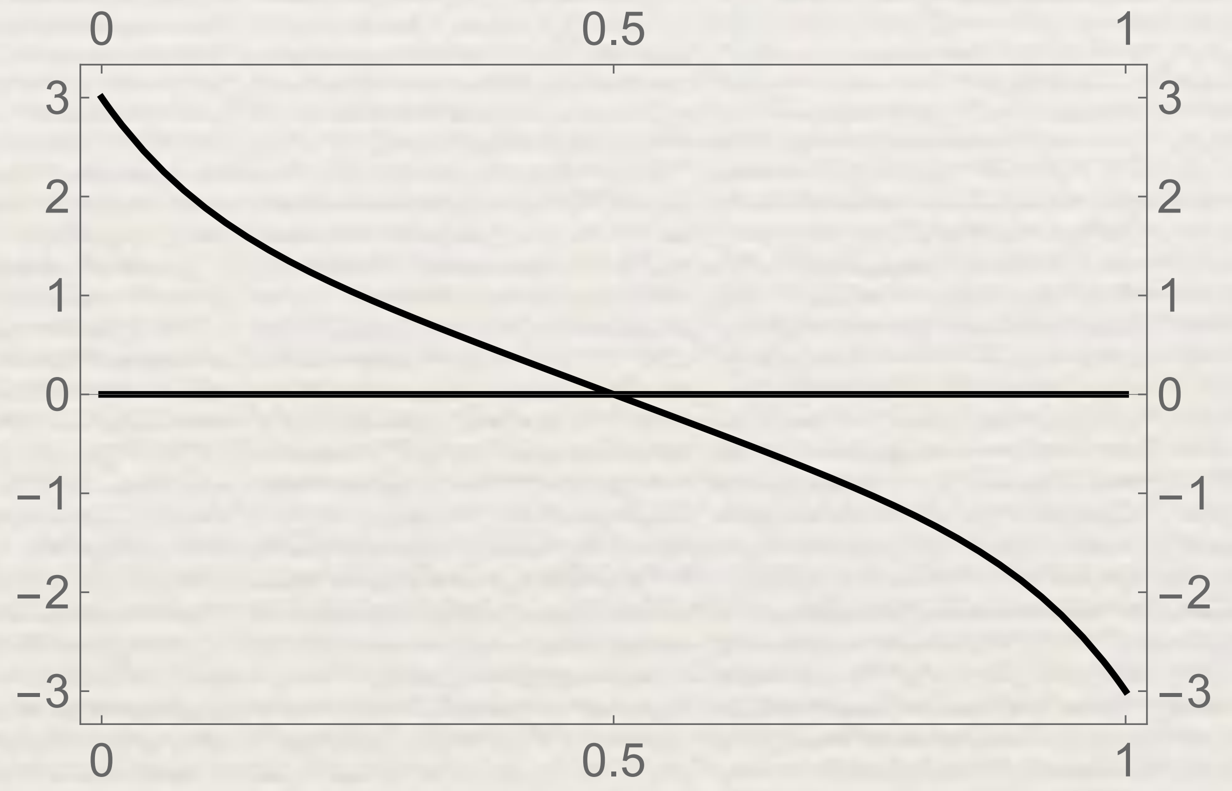

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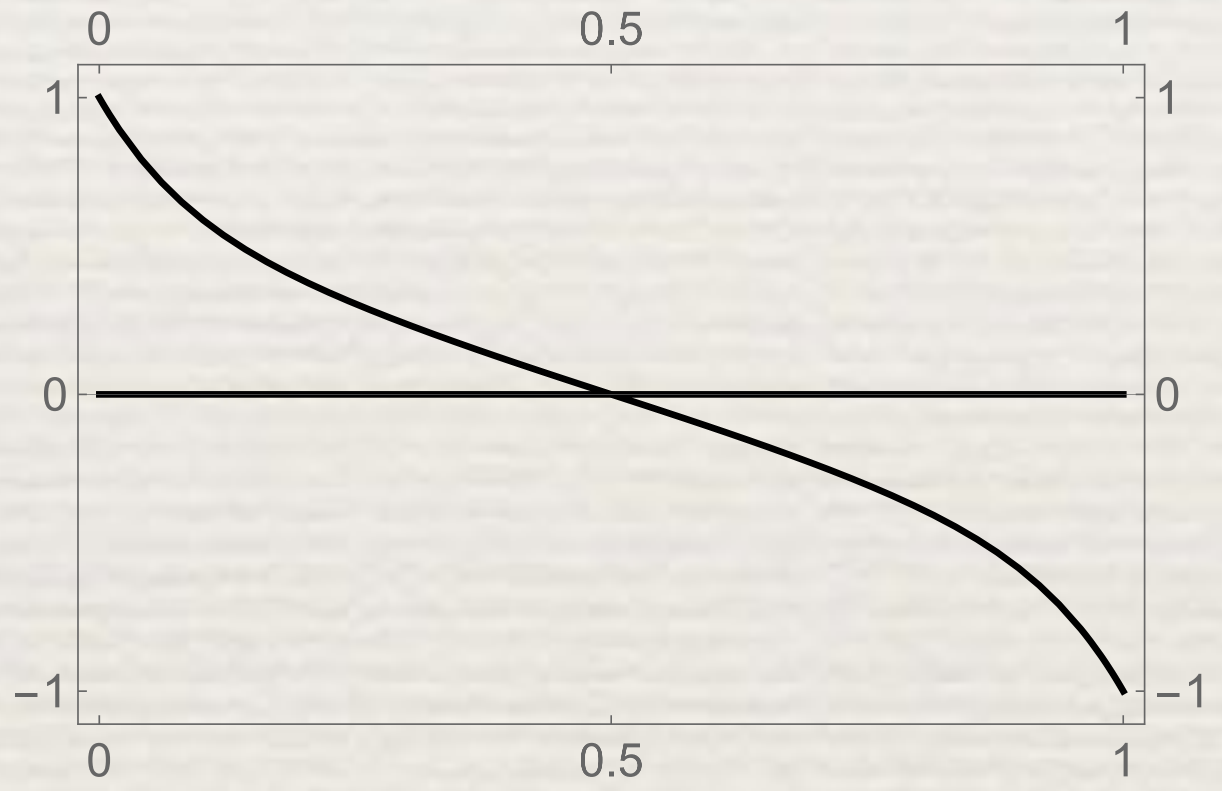

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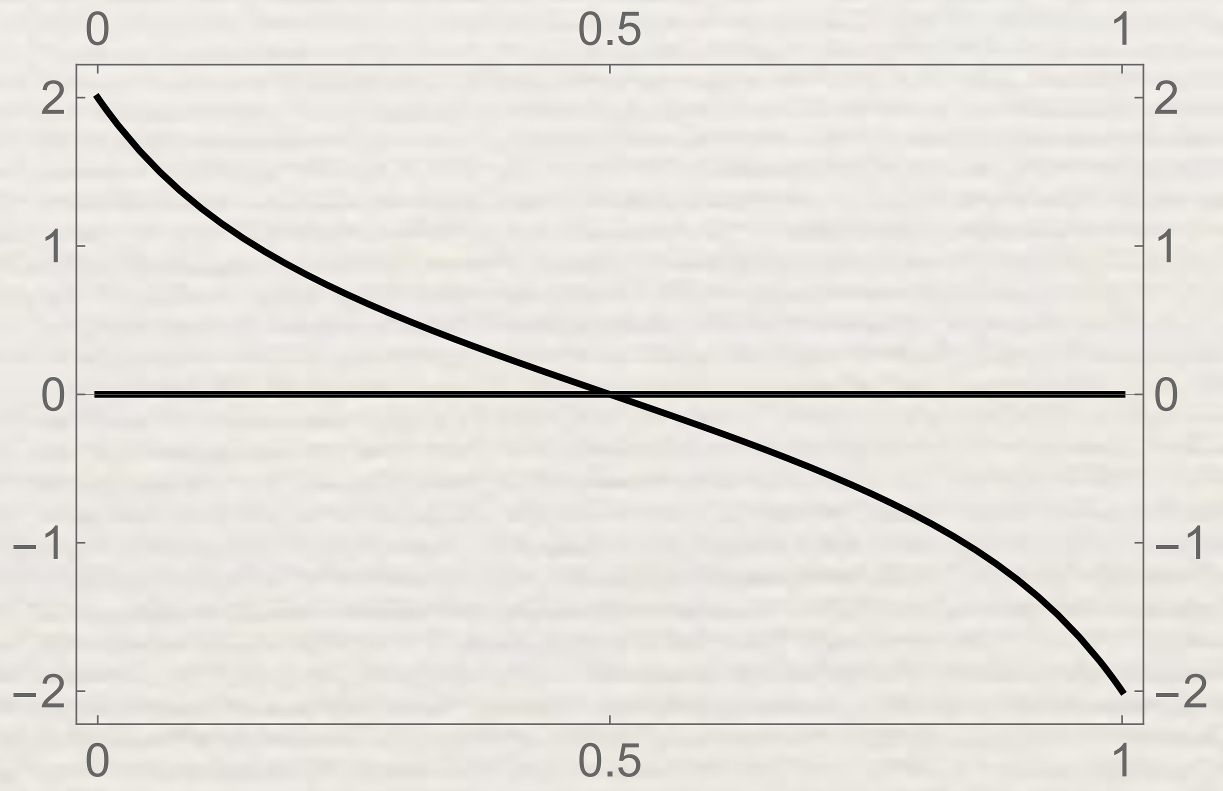

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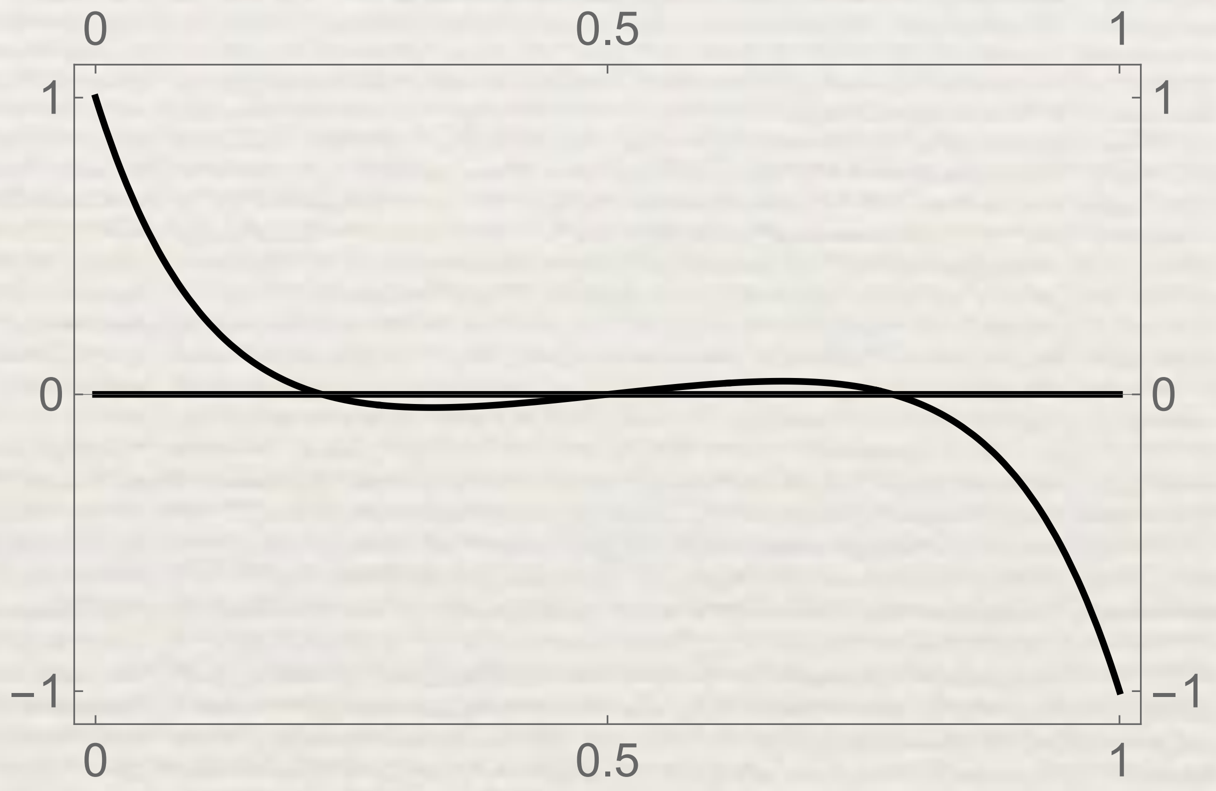


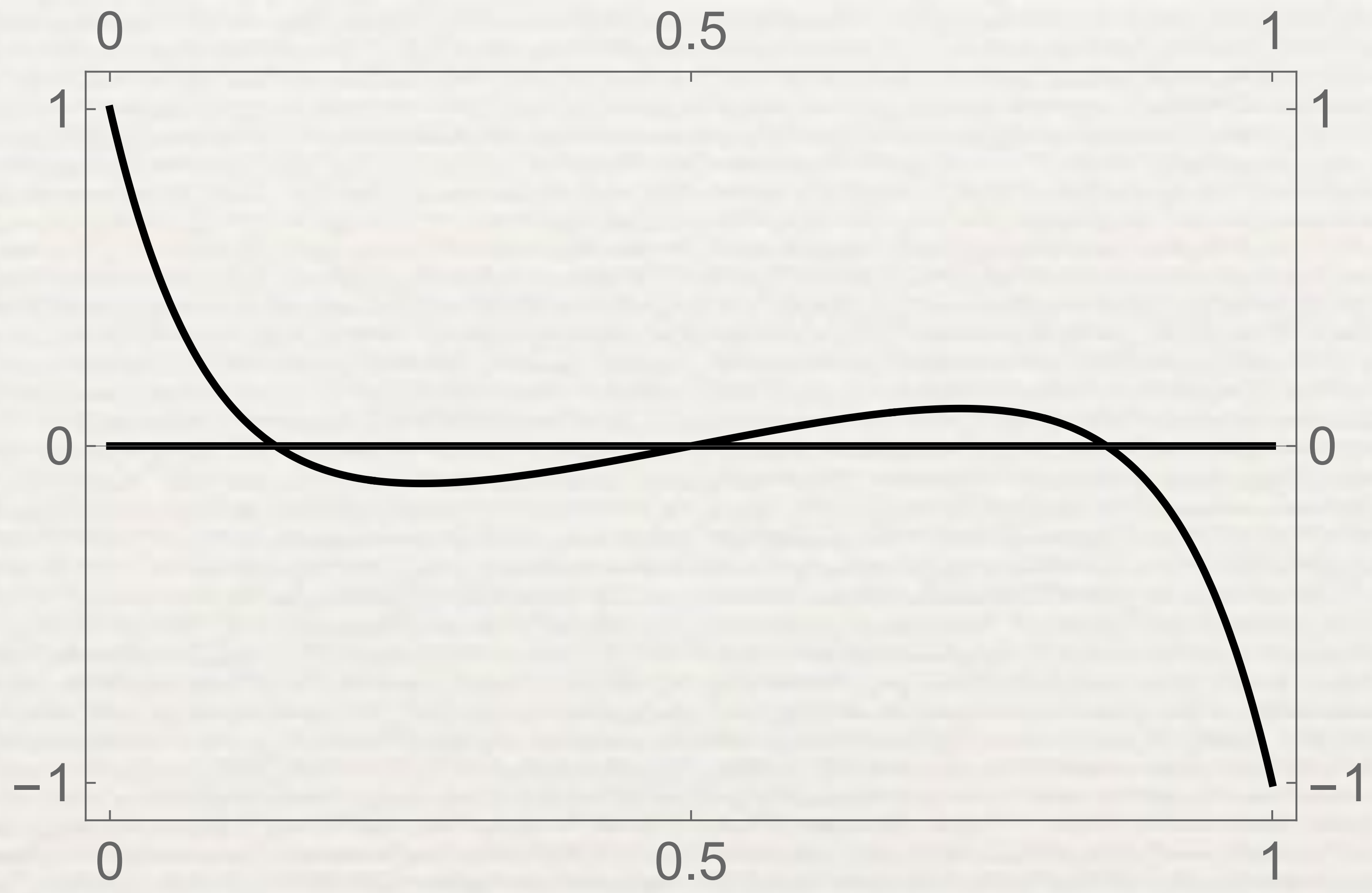
AAABBBB

- A priori: $\binom{7}{3} = 35$ patterns.
- Expectation depends only on last 3: must play ≥ 4 games, so only 8.

Persistent Paradox

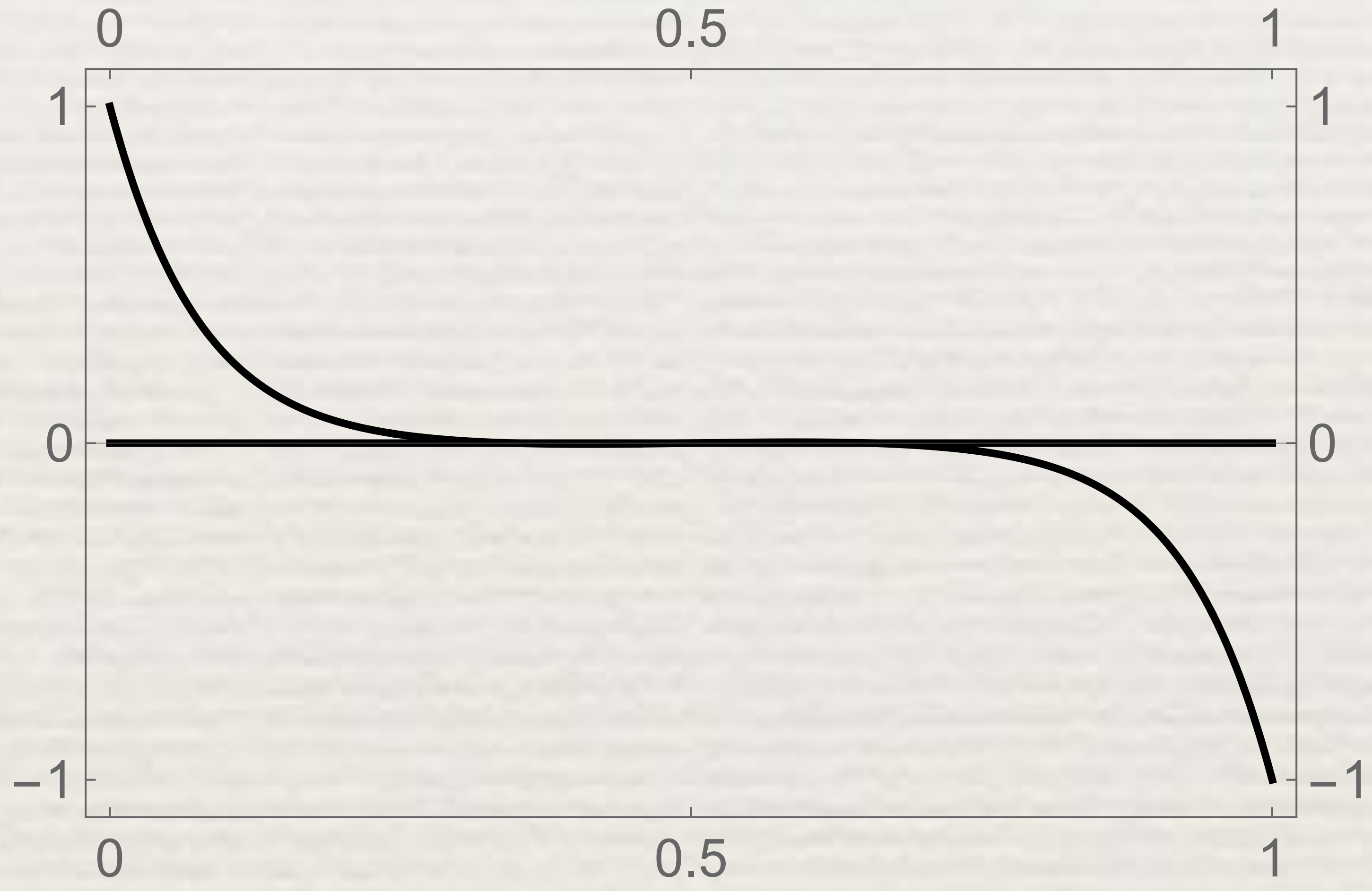
- Suppose the series has odd length s.
- How often does the home-field paradox appear? Can it be worse?
- Check all $2^{(s-1)/2}$ different patterns.
- Home-field paradox first appears at $s = 7$, only for AAABBBB.
- Appears three times when $s = 9$.

HostPattern	MaxGameBias	p
AAABBABBB	0.1112	0.7317
AAABBBABB	0.0028	0.5898
AAAABBBBB	0.1683	0.7504



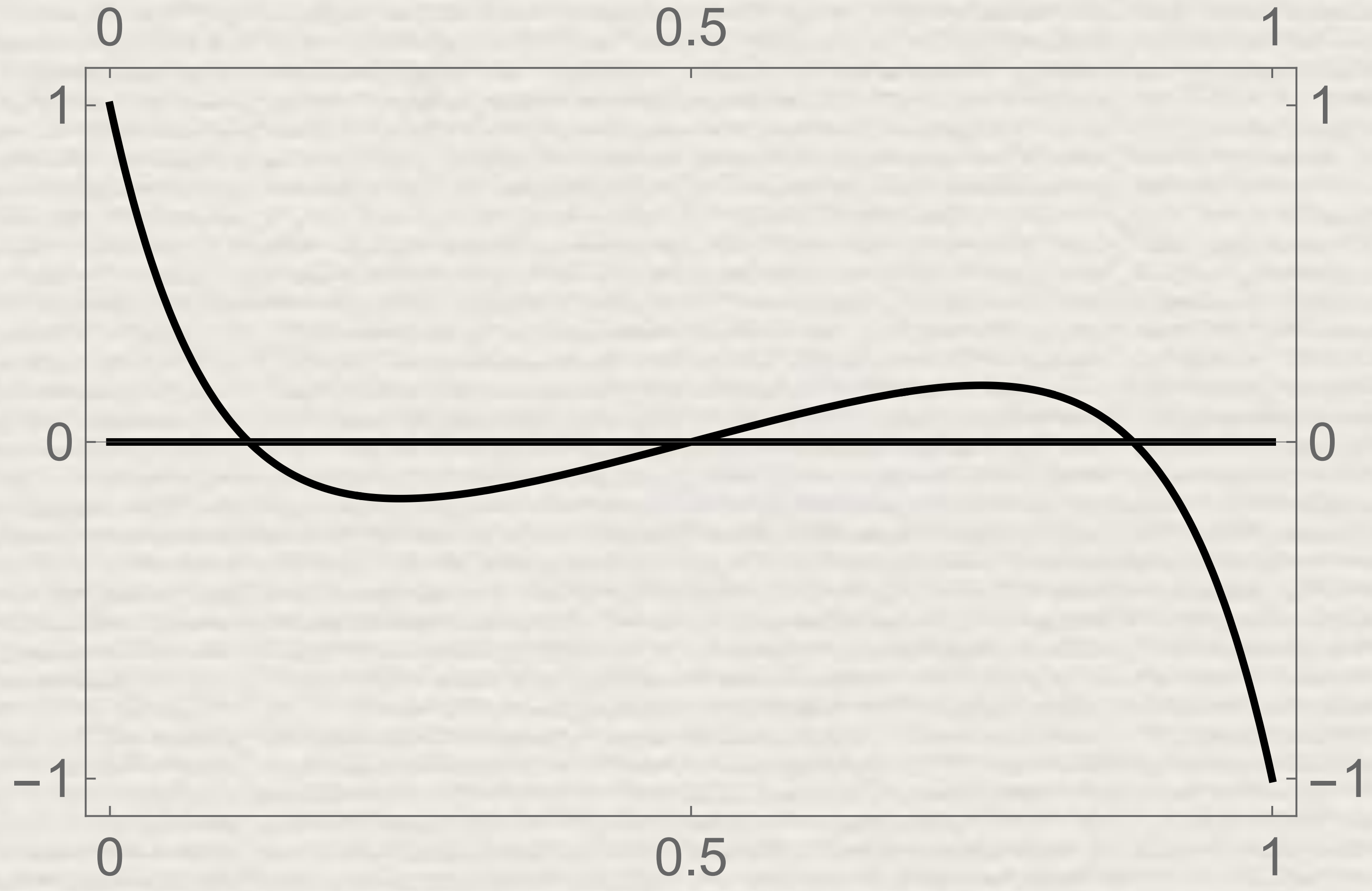
AAABBABBB — MaxGameBias 0.1112, p = 0.7317

$$(1-2p)\left(1-13p+63p^2-195p^3+405p^4-565p^5+515p^6-280p^7+70p^8\right)$$



AAABBBABB — MaxGameBias 0.0028, p = 0.5898

$$(1-2p)\left(1-6p+13p^2-14p^3+7p^4\right)\left(1-5p+15p^2-20p^3+10p^4\right)$$



AAAABBBBB — MaxGameBias 0.1683, p = 0.7504

$$(1-2p)\left(1-14p+66p^2-199p^3+407p^4-565p^5+515p^6-280p^7+70p^8\right)$$

AMS Fall Sectional, Albany

- **Sunday October 20, 2024, 8:00 a.m.-11:00 a.m.**
  Special Session on Topics in Recreational Math and Finite Geometry, III
  HU 20, Humanities Building
  Organizers:
  Lauren L Rose, Bard College rose@bard.edu
  Kelly Isham, Colgate University
  Liz McMahon, Lafayette College
  Gary Gordon, Lafayette College

- 9:30 a.m.
  Baseball-bany: Winkler's home-field paradox revisited
  John M. Harris, Furman University
  Michael J. Mossinghoff*, Center For Communications Research
  (1200-10-39362)
- 10:00 a.m.
  The Blob and the Blob Game
  James A. Schmidt, Dartmouth
  Peter M Winkler*, Dartmouth College
  (1200-51-39948)

Algorithm

- Given series length s, home site sequence H, home field advantage p. Compute game bias for A: $P_A(H, p)$.

- Suppose series ends at game g, say this is at A.

- Let $n_A = \#$ games at A over first $g - 1$ games; similarly n_B.

- Suppose series winner won w_A games at A and w_B at B over the first $g - 1$ games, so $w_A + w_B = (s - 1)/2$.

- Either A or B won the series, so add to $P_A(H, p)$:

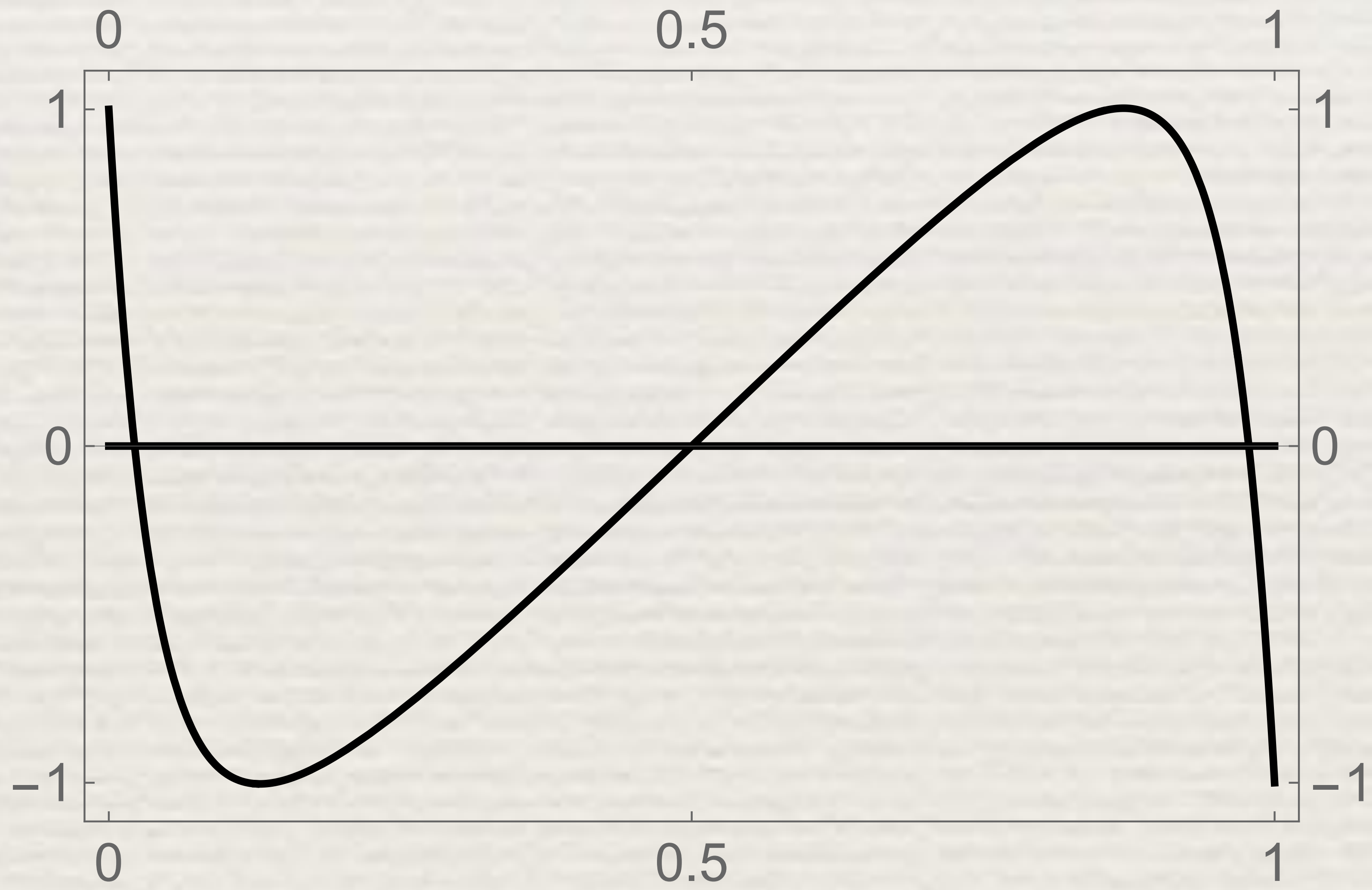
$$\left(\frac{s+1}{2} - \left(g - \frac{s+1}{2}\right)\right)\binom{n_A}{w_A}\binom{n_B}{w_B}\left(p^{w_A+1+n_B-w_B}(1-p)^{n_A-w_A+w_B} - p^{w_B+n_A-w_A}(1-p)^{w_A+n_B-w_B+1}\right).$$

Algorithm

- For each g with $(s+1)/2 \leq g \leq s$, set n_A and n_B appropriately, and iterate over all possible w_A and w_B, subject to $w_A + w_B = (s-1)/2$, $0 \leq w_A \leq n_A$, $0 \leq w_B \leq n_B$.
- Accumulate coefficients in array C indexed by g and j so that at conclusion $P_A(H, p) = \sum_{g,j} C_{g,j} p^j (1-p)^{g-j}$.
- Speedy! $>$ 5500 polys/sec at $s = 51$ (one core).

Bigger Game Bias

- Can we find series where the game bias is worse than its value when $p = 1$?
- Yes! First occurs when $s = 21$.



Max game bias: 1.00345

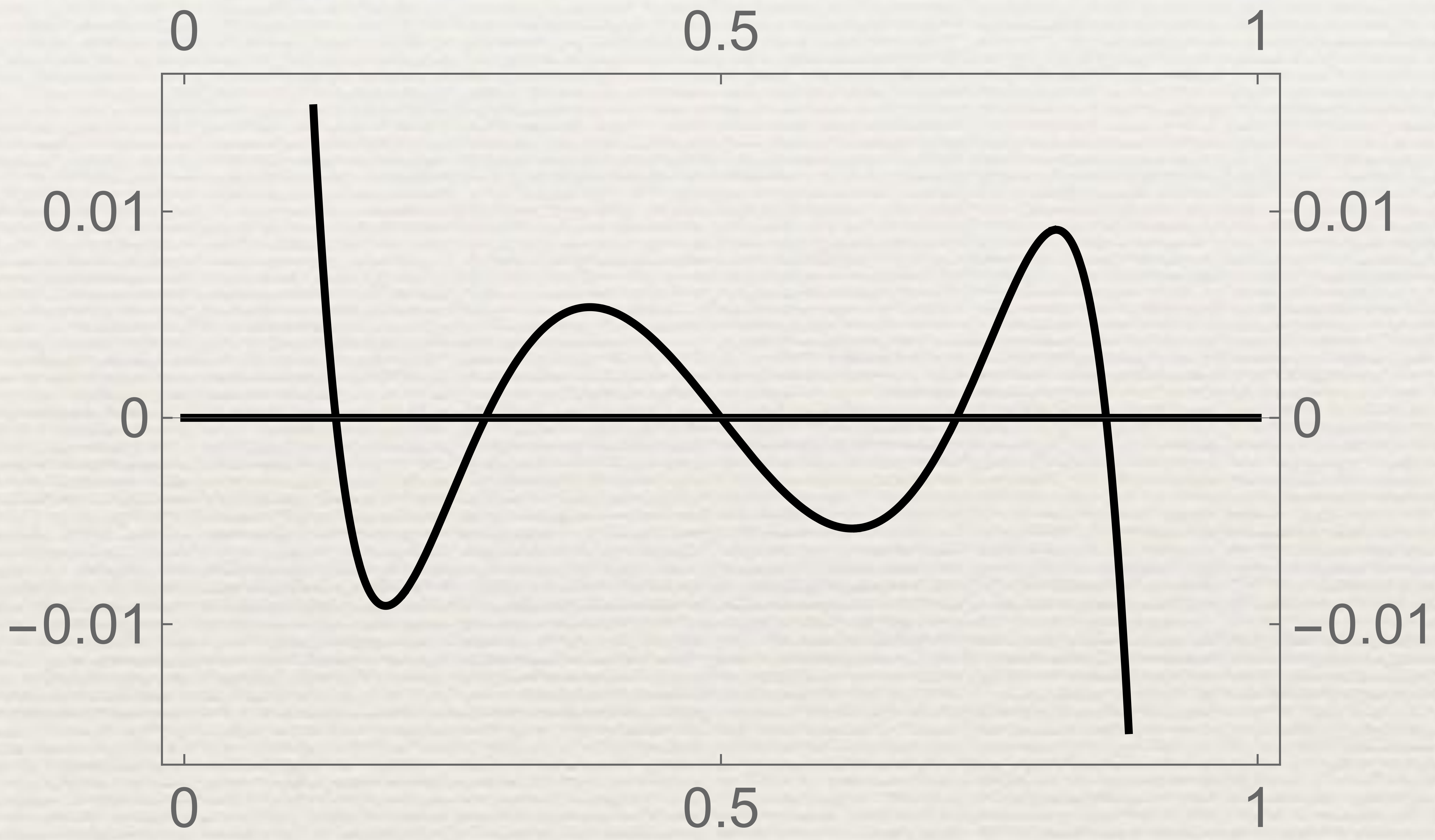
$p = 0.87047$

$H =$ ...ABABBBBBBB

$$(1-2p)\big(1 - 59p + 754p^2 - 6731p^3 + 45106p^4 - 236762p^5 + 1000552p^6 - 3466085p^7 + 9956853p^8 - 23879728p^9 + 47959310p^{10} - 80642661p^{11} + 113154415p^{12} - 131613053p^{13} + 125539843p^{14} - 96623384p^{15} + 58570798p^{16} - 26940342p^{17} + 8843978p^{18} - 1847560p^{19} + 184756p^{20}\big)$$

More Complicated Bias

$s = 19$: $H = \ldots\text{AAAAABABB}$



- B expects to win more games for $p \in (1/2, 0.7194\ldots) \cup (0.8588\ldots, 1]$.
- A expects to win more games for $p \in (0.7194\ldots, 0.8588\ldots)$.

$$(1-2p)\big(1 - 26p + 333p^2 - 2889p^3 + 18724p^4 - 93870p^5 + 371319p^6 - 1175354p^7 + 3006255p^8 - 6249359p^9 + 10578512p^{10} - 14549172p^{11} + 16146141p^{12} - 14272258p^{13} + 9835254p^{14} - 5102240p^{15} + 1877590p^{16} - 437580p^{17} + 48620p^{18}\big)$$

Exhaustive Searches

- Count $\#H$ of length s where $P_A(H,p)$ has r roots in [0,1].

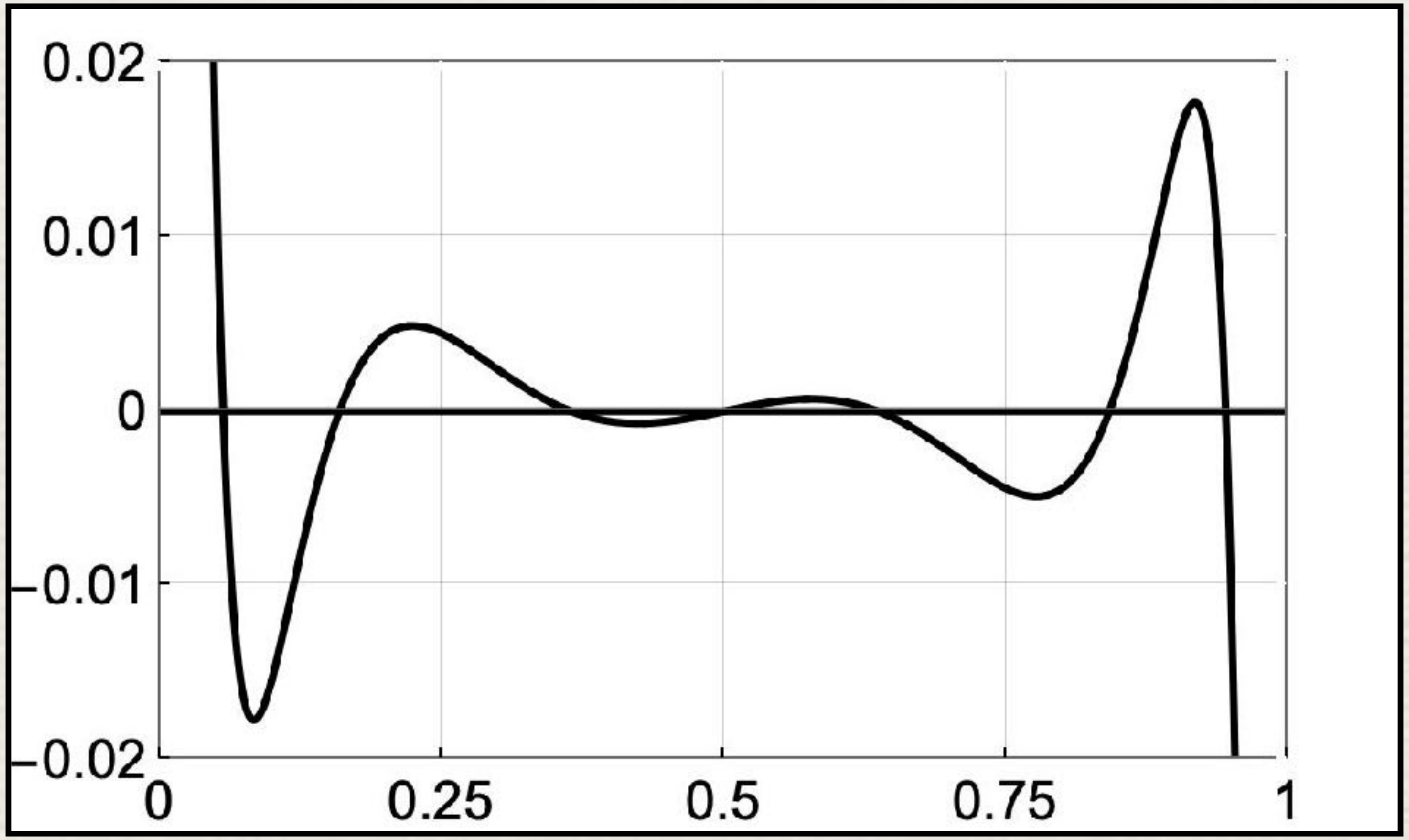
$\boldsymbol{s}$	$\boldsymbol{r=1}$	$\boldsymbol{r=3}$	$\boldsymbol{r=5}$
1	1		
3	2		
5	4		
7	**7**	**1**	
9	13	3	
11	26	6	
13	52	12	
15	104	24	
17	204	52	
19	**400**	**108**	**4**
21	790	228	6
23	1562	476	10
25	3091	989	16

Exhaustive Searches

s	$r = 1$	$r = 3$	$r = 5$	$r = 7$
27	6170	1986	36	
29	12236	4052	96	
31	24330	8194	244	
33	48352	16600	584	
35	95988	33756	1328	
37	190856	68336	2952	
39	379643	138337	6308	
41	755344	279550	13682	
43	1502278	565366	29508	
45	**2990776**	**1142190**	**61206**	**132**
47	5952078	2306626	129396	508
49	11848476	4653074	274386	1280
51	23590868	9385088	575468	3008

Seven Crossings

- $s = 45$, $H = $ ...BBBBBBBBBBBBABABAABABB.



More Crossings?

- Need a new strategy: exhaustive no longer feasible!
- Fix a suffix:
  - All ten H with $r = 5$ at $s = 19$ or 21 end AAABABB.
  - All 1920 with $r = 7$ at $s = 45$, 47, or 49 end BABAABABB.
- Fix a prefix:
  - All 1024 prefix strings of length 10 appear among the 1280 H with $r = 7$ at $s = 49$.
  - "Best" one: BBBBBBBBBB.

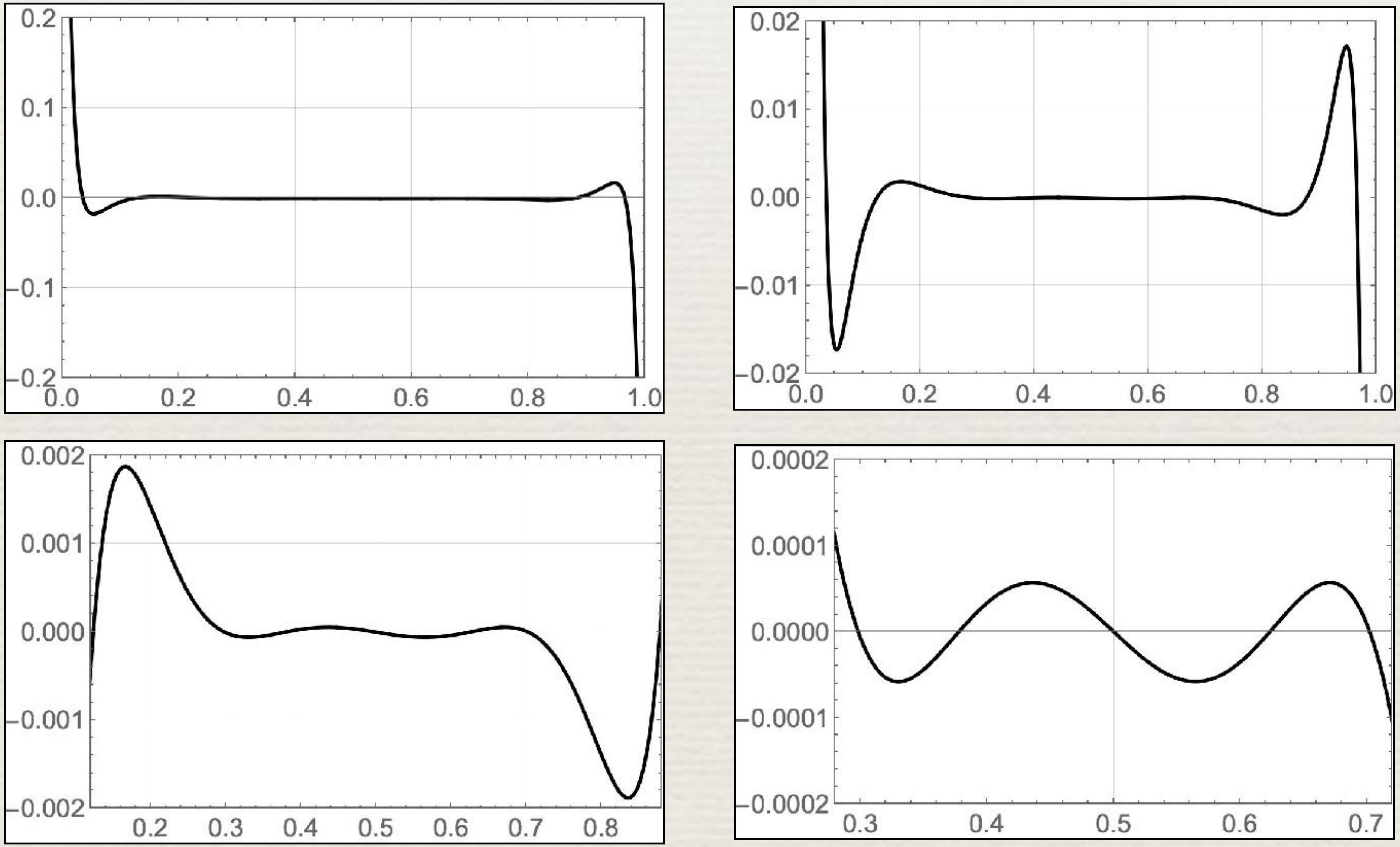
Targeted Searches

- Select a prefix of length α and suffix of length ω.

s	α	ω	$\lg(N)$	$r = 1$	$r = 3$	$r = 5$	$r = 7$	$r = 9$
53	8	4	14	2281	12435	1644	24	
55	9	4	14	2337	12373	1644	30	
57	9	4	15	4664	24628	3410	66	
59	9	4	16	9546	49031	6812	147	
61	10	4	16	9643	48840	6892	161	
63	10	4	17	19415	97318	14036	303	
65	10	5	17	38706	71000	20808	558	
67	11	5	17	38606	70741	21142	583	
69	11	5	18	77025	140992	42881	1246	
71	11	6	18	153789	73579	32336	2440	
73	**12**	**6**	**18**	**153432**	**73800**	**32399**	**2512**	**1**

Nine Crossings

- $s = 73$, ...BBBBBBBBBBBBAAAABAAAABBBAABBABAABABB.

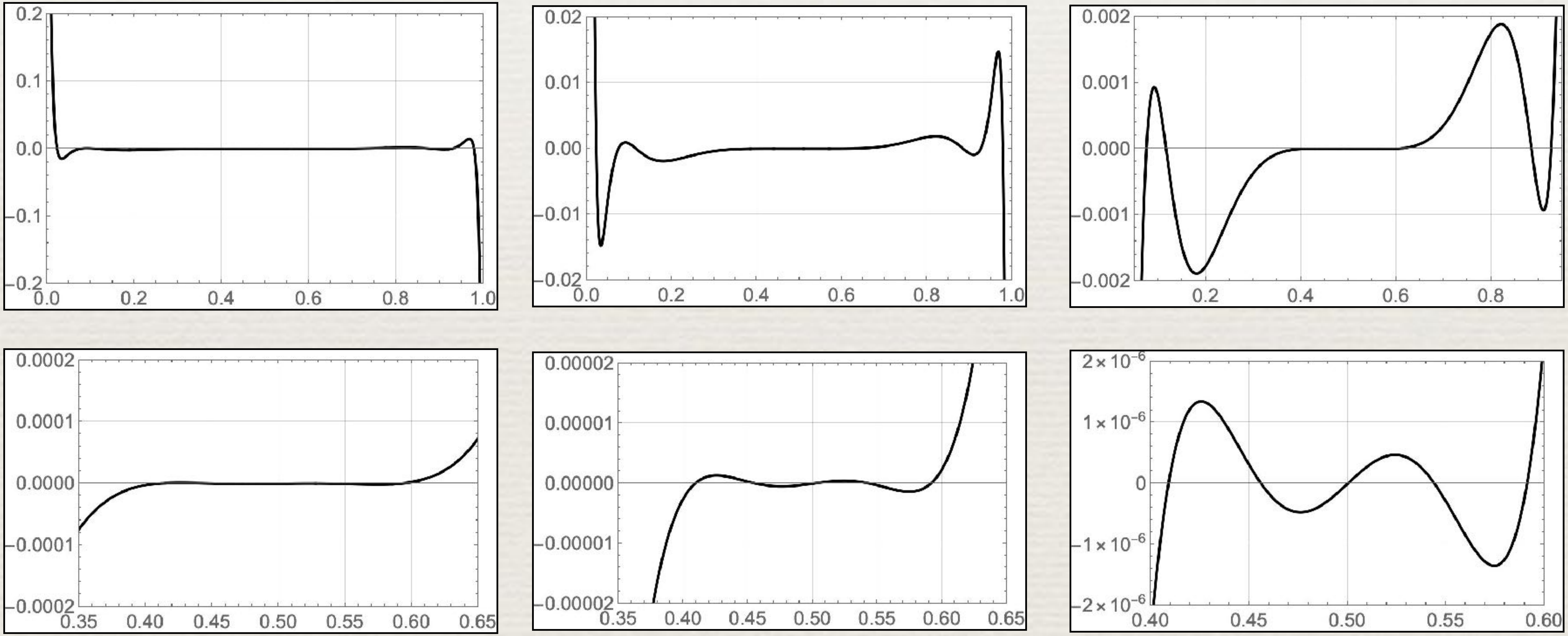


Targeted Searches

s	α	ω	$\lg(N)$	$r = 1$	$r = 3$	$r = 5$	$r = 7$	$r = 9$	$r = 11$
75	12	9	16	2048	24089	34470	4925	4	
77	12	9	17	4288	48052	68516	10207	9	
79	13	9	17	5264	48027	67252	10518	11	
85	14	14	14		7331	4982	3992	79	
91	15	15	15		28838	1245	2233	452	
97	16	16	16		51699	5302	6277	2258	
103	17	17	17		80696	22819	17040	10517	
109	18	18	18		85671	86635	42951	46887	
115	19	18	20		369367	370518	123535	185156	
121	21	19	20			693337	176411	178793	35
117	**19**	**24**	**15**				**465**	**32300**	**3**
119	19	24	16				1007	64482	47

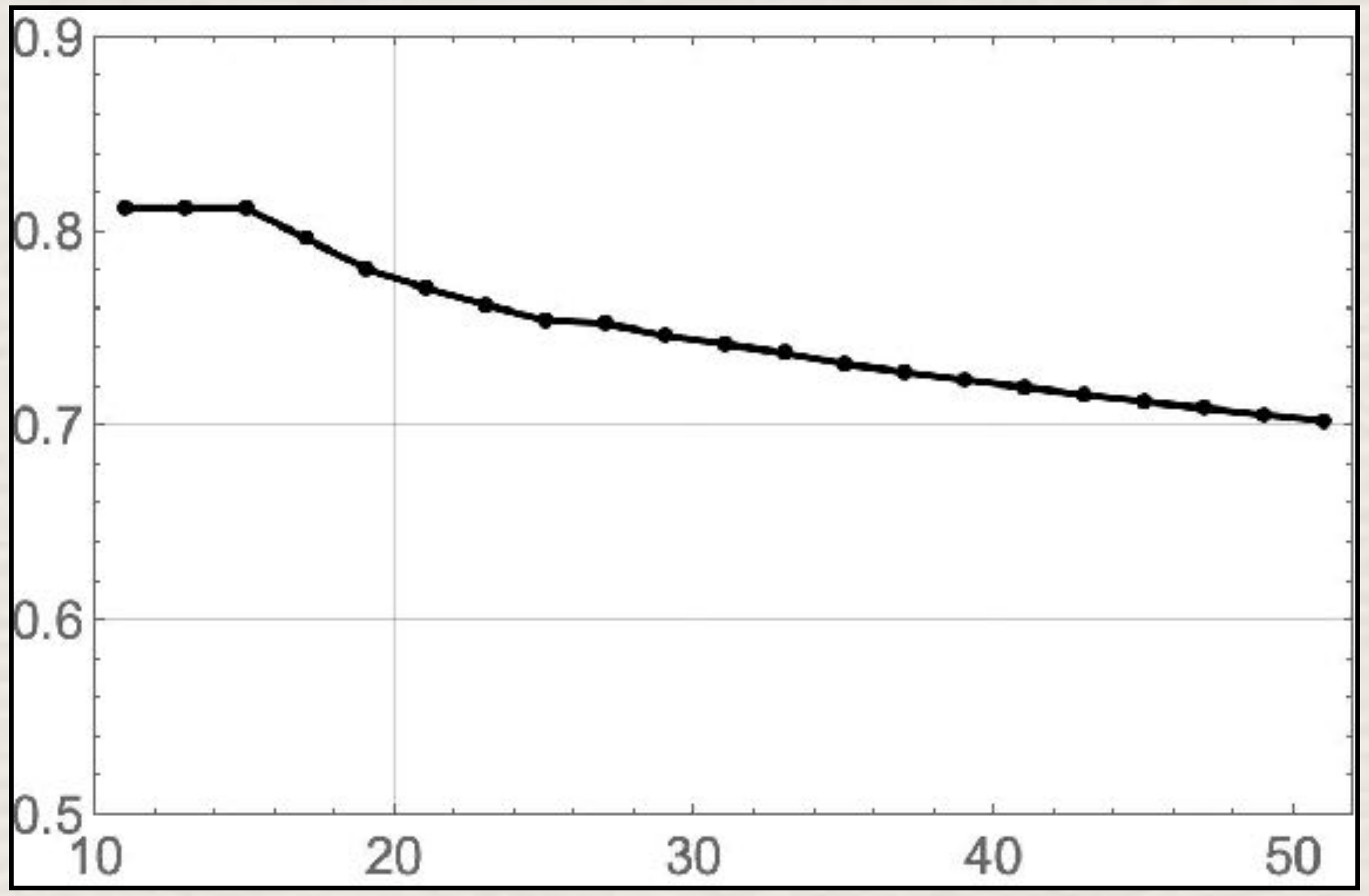
Eleven Crossings!

- $s = 117$, …BBBBBBBBBBBBBBBBBBABBABABBBABBBBBABBABAAAABBBAABBABAABABB.



Further Explorations

- Simulations check predictions (up to $4.8 \cdot 10^{15}$ games).
- Is 73 best possible for $r = 9$? Or 117 for $r = 11$?
- Can you find s and H with $r = 13$?
- Is r unbounded? Is there a lower bound on s in terms of r?
- Do almost all H exhibit the home-field paradox?



Thanks!