

Classifying Caps in Quad- 2^n via Venn Diagrams

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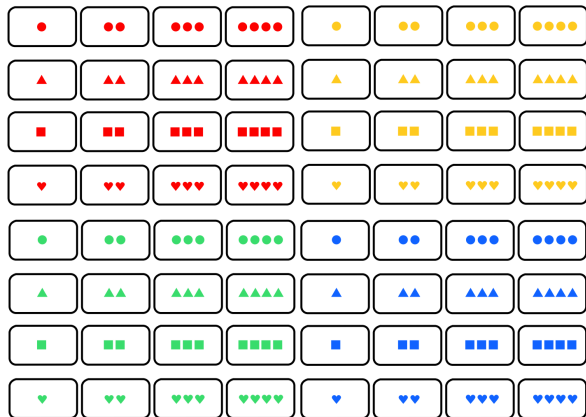
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1. Background and Motivating Questions

Quad-64



Three **attributes**, four **possible values for each**: number, shape, color.

Quads

Four cards form a **quad** if in each attribute they are:

- all the same
- all different
- half and half

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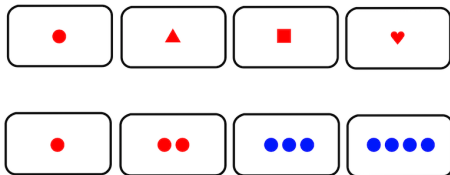
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Quads

Four cards form a **quad** if in each attribute they are:

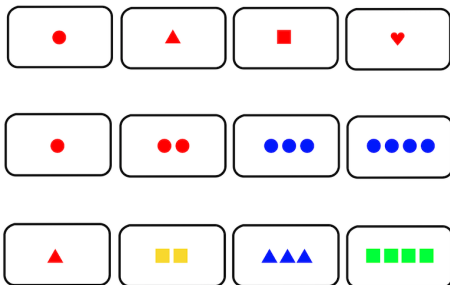
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Quads

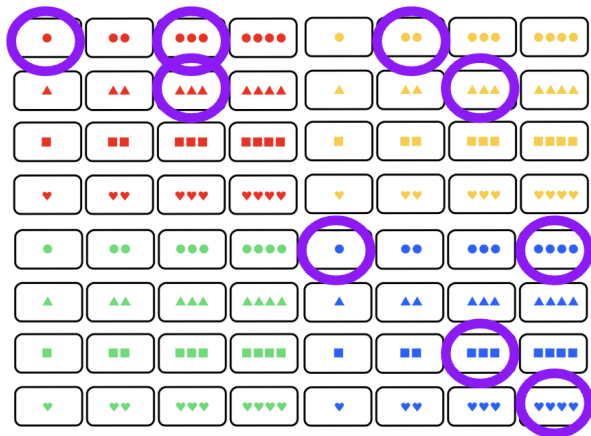
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Caps

A **cap** is a quad-free set of cards. A **k-cap** is a cap of size k .



A 9-cap in Quad-64.

Bigger Quads decks

Quad-64 can be generalized to a similar game on a deck of 2^n cards by adding attributes with two or four possible values.

Example

Quad-128 is played with two decks of different colors:



Quads and Affine Geometry

Quad- 2^n is a model of the **finite affine geometry** $AG(n, 2)$:

- To each card assign a vector in \mathbb{F}_2^n .
- Four cards form a quad iff their vectors sum to $\vec{0}$.
- Quads are exactly the **planes** in \mathbb{F}_2^n .
- A cap in Quad- 2^n corresponds to a set of points in $AG(n, 2)$ no four of which are coplanar.

Overarching Goal

Characterize caps (up to affine equivalence) in $AG(n, 2)$.

Recent progress:

- Accomplished for $n \leq 6$ in *How many cards should you lay out in a game of EvenQuads?: A detailed study of 2-caps in $AG(n, 2)$* by Crager, Flores, Goldberg, Rose, Rose-Levine, Thornburgh, and Walker
- Calta, Goldberg, Harrelson, and Rose classified all caps in $AG(7, 2)$ (coming soon)

Equivalence classes of caps

d = dimension of cap, k = size of cap

$d \backslash k$	1	2	3	4	5	6	7	8	9	10	11	12	13
0	1	0	0	0	0	0	0	0	0	0	0	0	0
1	0	1	0	0	0	0	0	0	0	0	0	0	0
2	0	0	1	0	0	0	0	0	0	0	0	0	0
3	0	0	0	1	0	0	0	0	0	0	0	0	0
4	0	0	0	0	1	1	0	0	0	0	0	0	0
5	0	0	0	0	0	1	1	0	0	0	0	0	0
6	0	0	0	0	0	0	1	2	1	0	0	0	0
7	0	0	0	0	0	0	0	1	2	2	1	1	0

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4	0	0	0	0	1	1	0	0	0	0	0	0	0
5	0	0	0	0	0	1	1	0	0	0	0	0	0
6	0	0	0	0	0	0	1	2	1	0	0	0	0
7	0	0	0	0	0	0	0	1	2	2	1	1	0

- **Blue diagonal pattern** continues with 1s off the table.

Equivalence classes of caps

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4	0	0	0	0	1	1	0	0	0	0	0	0	0
5	0	0	0	0	0	1	1	0	0	0	0	0	0
6	0	0	0	0	0	0	1	2	1	0	0	0	0
7	0	0	0	0	0	0	0	1	2	2	1	1	0

- **Blue diagonal pattern** continues with 1s off the table.
- **Red diagonal pattern** continues with 1, 1, 2, 2, 3, 3, ... off the table.

2. Basic Definitions

Affine Geometry Basics

Let V be a vector space over a field F .

Definition

- An *affine combination* of $v_1, \dots, v_k \in V$ is a sum

$$c_1 v_1 + \dots + c_k v_k$$

where $\sum_{i=1}^k c_i = 1$.

- v_1, \dots, v_k are *affinely independent* if no v_i is an affine combination of the others.

Affine span, basis defined similarly.

Dependence relations and zero-sum sets

To any dependence relation

$$v = v_1 + \cdots + v_k$$

associate a **zero-sum set**

$$X_v = \{v, v_1, \dots, v_k\}$$

and vice versa.

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Definition

Let $S \subset \mathbb{F}_2^n$.

- S is an **even zero-sum set** if $|S|$ is even and the elements of S sum to 0.
- $E(S)$ is the set of even-zero-sum subsets of S .

The vector space of zero-sum sets

Addition of dependence relations is equivalent to **symmetric difference** of zero-sum sets.

Example:

$$v = v_1 + v_2 + v_3$$

$$w = v_1 + v_3 + v_4$$

$$v + w = v_2 + v_4$$

$$X_v = \{v, v_1, v_2, v_3\}$$

$$X_w = \{w, v_1, v_3, v_4\}$$

$$X_v \triangle X_w = \{v, w, v_2, v_4\}$$

Theorem

$E(S)$ is an \mathbb{F}_2 -vector space under \triangle .

(It is a subspace of the entire power set of S .)

Remark: $E(S)$ algebraically encodes information about dependence relations in S .

Affine Geometry Basics

Definition

Two subsets S_1 and S_2 are **affinely equivalent** if there is an affine automorphism $T : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ such that $T(S_1) = S_2$.

Affine Geometry Basics

Definition

Two subsets S_1 and S_2 are **affinely equivalent** if there is an affine automorphism $T : \mathbb{F}_2^n \rightarrow \mathbb{F}_2^n$ such that $T(S_1) = S_2$.

Theorem

Let $S, T \subset \mathbb{F}_2^n$. Then a bijection $f : S \rightarrow T$ is an affine equivalence iff f preserves even zero-sum sets and induces a **linear isomorphism** $E(S) \rightarrow E(T)$.

Cap terminology

Definition

Let $C \subset F_2^n$ be a k -cap. A **basis** for C is a basis B for $\text{Aff}(C)$ with $B \subseteq C$. The **dimension** of C is $\dim(C) = |B| - 1$.

For a k -cap C of dimension d , we write:

$$C = B \sqcup D$$

where

$$B = \{a_1, \dots, a_{d+1}\}$$

is a **basis** for C

$$D = \{x_1, \dots, x_r\}$$

is the **dependent set**.

3. Caps and Venn diagrams

Caps and Venn Diagrams

Example (A 10-cap of type 5-5-(2))

Let C be a 10-cap with basis $B = \{a_1, \dots, a_8\}$ and dependent set $D = \{x_1, x_2\}$ where $x_1 = a_1 + \dots + a_5$ and $x_2 = a_4 + \dots + a_8$.

Let

$$X_1 = \{x_1, a_1, \dots, a_5\} \quad \text{and} \quad X_2 = \{x_2, a_4, \dots, a_8\}.$$

Note that x_1 is a sum of 5, x_2 is a sum of 5, and they share two elements.

Caps and Venn Diagrams

Example (A 10-cap of type 5-5-(2))

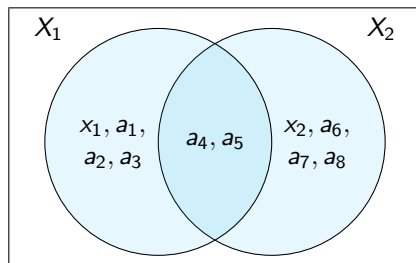
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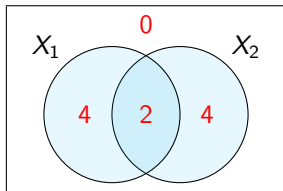
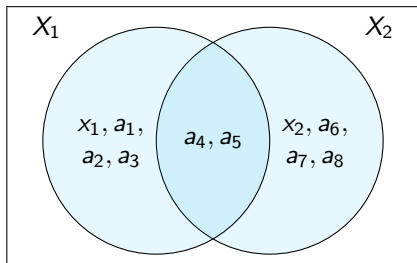
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Note that x_1 is a sum of 5, x_2 is a sum of 5, and they share two elements.

Alternatively, we can visualize the structure of this cap with respect to this basis in a [Venn diagram](#):





Partition of C into Venn regions:

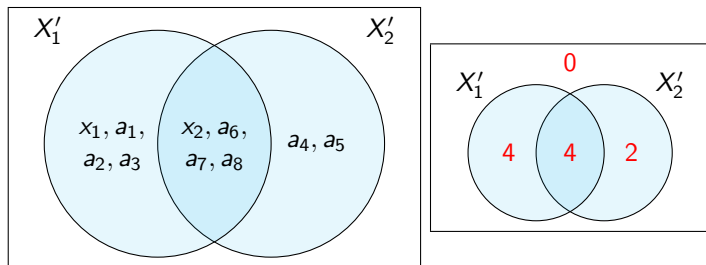
$$C = \{x_1, a_1, a_2, a_3\} \sqcup \{a_4, a_5\} \sqcup \{x_2, a_6, a_7, a_8\}$$

Change of cap basis

Note: Changing the basis (hence dependent set):

$$\{a_1, \dots, a_8\} \sqcup \{x_1, x_2\} \rightarrow \{x_2, a_1, \dots, \hat{a}_4, \dots, a_8\} \sqcup \{x_1, a_4\}$$

may change the Venn diagram:



but not the partition of C by Venn regions:

$$C = \{x_1, a_1, a_2, a_3\} \sqcup \{a_4, a_5, a_6, a_7\} \sqcup \{x_2, a_8\}$$

Affine equivalence preserves Venn regions

Theorem

If $f : C \rightarrow C'$ is an affine equivalence, then f preserves Venn regions.

Corollary

If two caps are affinely equivalent, they must have the same Venn region cardinalities.

Warning: The converse is false. Examples too large to show!

When are caps (in)equivalent?

Quick Checks:

- Do the caps have the same size?
- Do the caps have the same dimension?
- Do the caps have the same Venn region cardinalities?

If not, then the caps are inequivalent!

When are caps (in)equivalent?

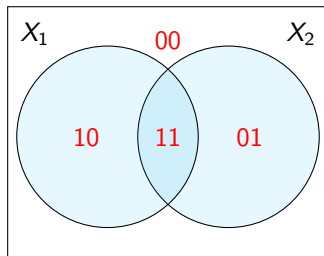
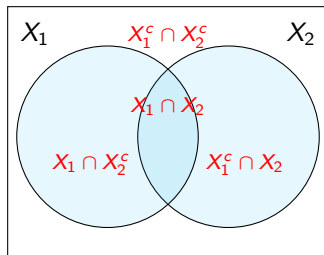
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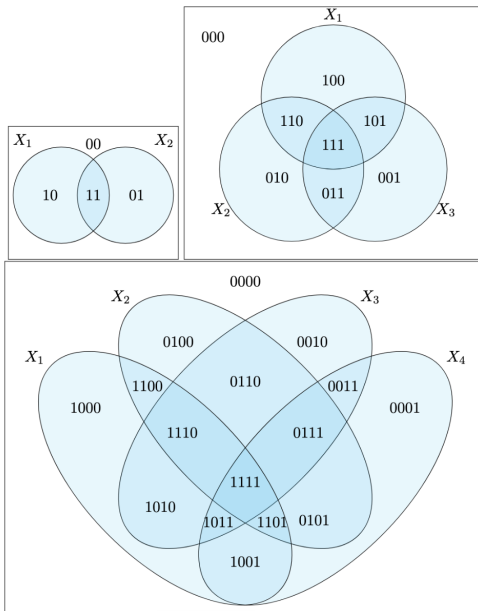
If not, then the caps are inequivalent!

If so, the caps may still not be equivalent, but how can we tell??

Assigning bitstrings to Venn regions



Assigning bitstrings to Venn regions



Cap equivalence and bitstrings

Note: A map between caps that preserves Venn regions lifts to a map of bitstrings:

$$f : C \rightarrow C' \rightsquigarrow \tilde{f} : \mathbb{F}_2^r \rightarrow \mathbb{F}_2^r$$

Big Theorem

A bijection $f : C \rightarrow C'$ is an affine equivalence iff

- 1 it preserves Venn regions and
- 2 it lifts to a linear isomorphism $\tilde{f} : \mathbb{F}_2^r \rightarrow \mathbb{F}_2^r$.

(In our examples here, $r = 2$.)

A map on bitstrings

Let f be the map from the 5-5-(2) cap to the 7-5-(4) cap in the previous examples. The induced map on bitstrings (via Venn regions) is:

$$\begin{array}{llll} 00 & \mapsto & \emptyset & \mapsto & 00 \\ 10 & \mapsto & \{x_1, a_1, a_2, a_3\} & \mapsto & 10 \\ 11 & \mapsto & \{x_2, a_8\} & \mapsto & 01 \\ 01 & \mapsto & \{a_4, a_5, a_6, a_7\} & \mapsto & 11 \end{array}$$

Note: This map is linear with matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

Examples: 10-caps in dimension 7

There are three types of 10-caps of dimension 7.

Let $\{a_1, \dots, a_8\}$ be a basis.

basis extended type	dependent elements
7-5-(4)	$x_1 = a_1 + \dots + a_7$ $x_2 = a_4 + \dots + a_8$
5-5-(2)	$x_1 = a_1 + \dots + a_5$ $x_2 = a_4 + \dots + a_8$
5-5-(3)	$x_1 = a_1 + \dots + a_5$ $x_2 = a_3 + \dots + a_7$

Venn cardinality diagrams for 10-caps

These are their corresponding Venn cardinality diagrams:

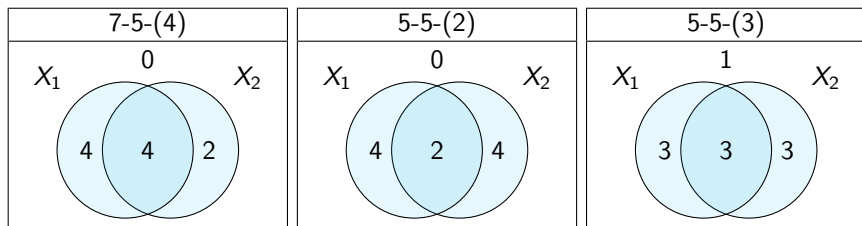


Figure: The Venn cardinality diagrams for three caps of size 10 and dimension 7.

Observe that a 7-5-(4) cap cannot be equivalent to a 5-5-(3) cap because they have different Venn region cardinalities.

But what about 7-5-(4) and 5-5-(2)? It turns out that those are equivalent, also, using our big theorem.

4. Applications

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- A relatively straightforward computational method for:
 1. checking whether two caps are affinely equivalent;
 2. describing equivalence classes of caps.

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Applications

- A relatively straightforward computational method for:
 1. checking whether two caps are affinely equivalent;
 2. describing equivalence classes of caps.
- Complete characterization of k -caps of dimension $d = k - 3$.
- Complete characterization of k -caps of dimension 7.
- Work continues on caps of higher dimensions.

Visualizing Caps- the Qap Finder

- Each square is an element of \mathbb{Z}_2^7 (a card in Quad-128).
- Green diamonds = cap points
- Red squares = points that form quads with select cap elements.
- Number = # triples of cap points that form a quad with this point

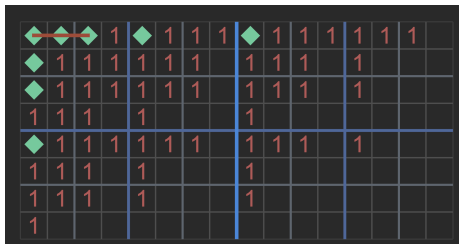


Figure: A cap with 8 points in AG(7,2)