Quad-Packing in the Game EvenQuads

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Dani Catalá, MIT
Timothy Goldberg, Lenoir Rhyne University
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Ren Watson*, UT Austin

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January 7, 2026

Contents

1. Introduction

Quad Packing Research Group

2024 Polymath Jr virtual REU (NSF award DMS-2218374)

- *Games and Finite Geometry research group*, mentored by Lauren Rose (Bard College) and Tim Goldberg (Lenoir Rhyne University).
  - **Quad-packing project**
    - Taiki Aiba (Georgia Tech)
    - Dani Catalá (MIT)
    - and some others

Ongoing research group, meets weekly on Zoom, with folks above and also:

- Ren Watson (UT Austin)
- Dyana Harrelson (Hope College)
- Hema Gopalakrishnan (Sacred Heart University)

We were not the first!

[1] "Maximum Number of Quads", 2024

- Student authors (grades 7-9): Byrapuram, Choi, A. Ge, S. Ge, ZiaLee, Liang, Mandal, Oki, Wu, and Yang - MIT PRIMES STEP Program
- Faculty author: Tanya Khovanova (MIT)

- Our methods are more visual and geometric.
- Had an important conjecture which we proved.

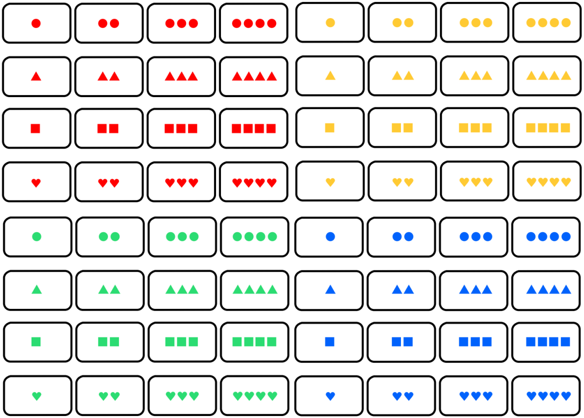
2. The Quads game

Origins of Quads

- *Quads*, initially called *SuperSET*, was introduced by Rose and Perreira in 2013.
- Published in 2021 as *EvenQuads* by the Association for Women in Mathematics in honor of its 50th anniversary. (Two different decks published so far, two more planned.)

The Quad-64 deck

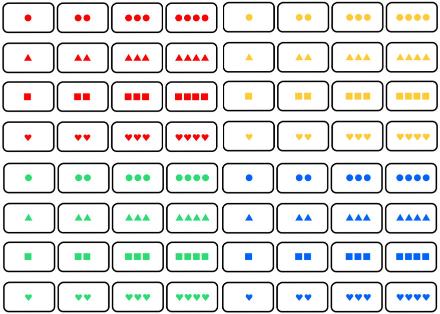
Each card has 3 attributes — color, shape, number — each with 4 possible values.



$\#(\text{cards}) = 4 \times 4 \times 4 = 64.$

Finding Quads

Goal: Find four cards that make a quad.



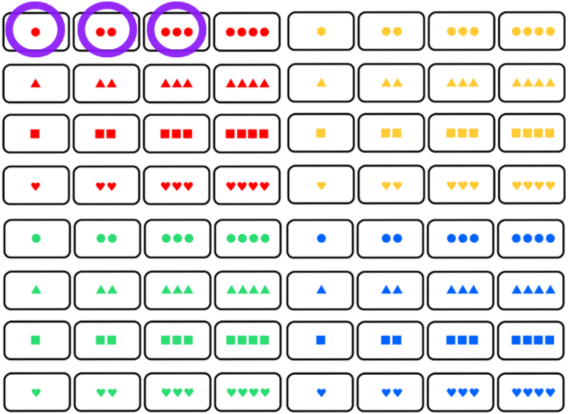
A **quad** is four cards where, for each attribute:

1. the values are all the same; or
2. the values are all different; or
3. the values are half-and-half.

Fundamental Theorem of Quads

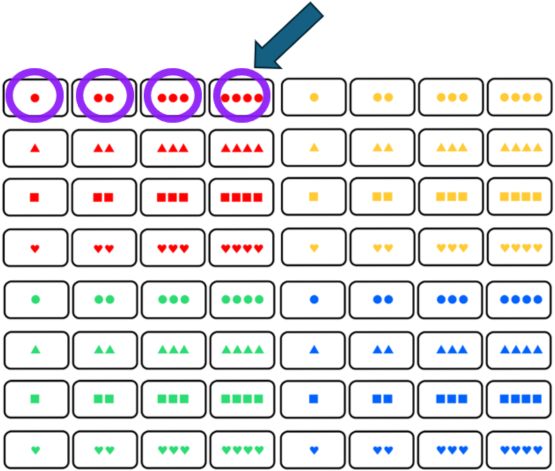
For any three distinct Quads cards, there is exactly one other card that forms a quad with them.

Some Examples of Quads



- **different numbers, same color and shape**

Some Examples of Quads



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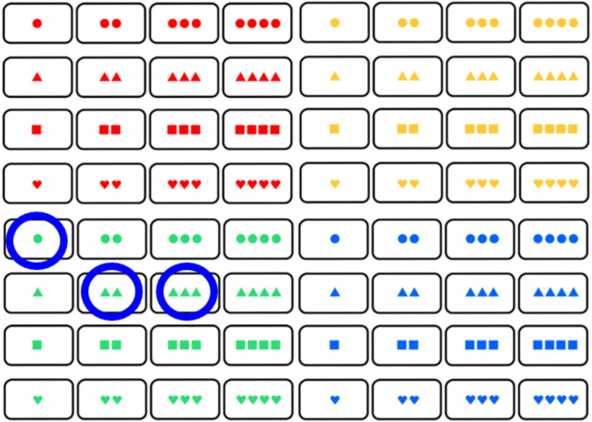

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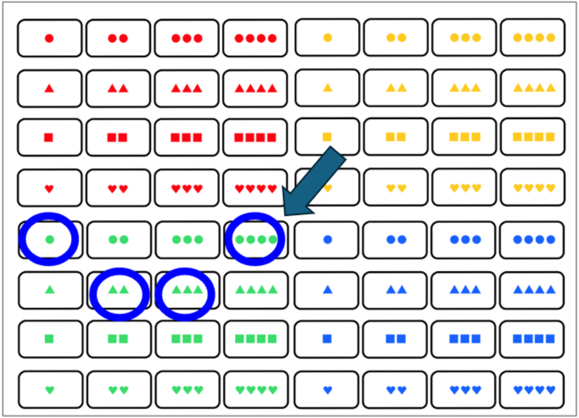

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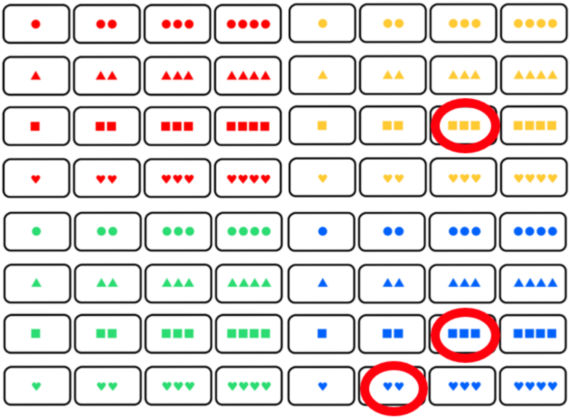
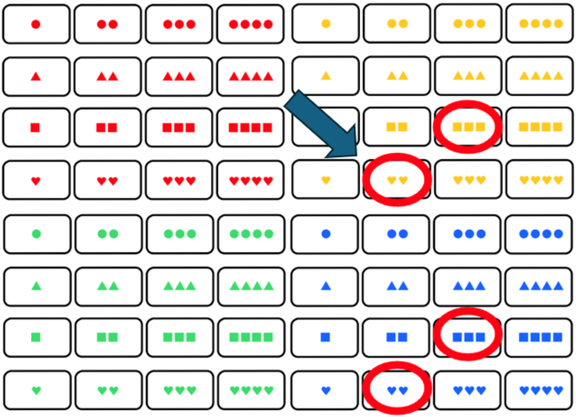
Some Examples of Quads



- **half-and-half numbers, colors, and shapes**

Some Examples of Quads



- **half-and-half numbers, colors, and shapes**

3. Geometry of Quads

Coordinates for Quads Cards

Assigning coordinates to cards

- For each attribute, assign each value one of the elements of $\mathbb{Z}_2 \times \mathbb{Z}_2 = \{00, 01, 10, 11\}$.
- Then each Quads card is represented by an element of $(\mathbb{Z}_2 \times \mathbb{Z}_2)^3 \cong (\mathbb{Z}_2)^6$.

A possible assignment of coordinates:

Attribute	**Values**	**Coordinates (respectively)**
Color	red, green, yellow, blue	$00, 01, 10, 11$
Shape	circle, triangle, square, heart	$00, 01, 10, 11$
Number	one, two, three, four	$00, 01, 10, 11$

For example:

00 01 00	10 10 01	11 01 10	01 10 11

Possible Binary coordinates for Quad-64

000000	000001	000010	000011	100000	100001	100010	100011
000100	000101	000110	000111	100100	100101	100110	100111
001000	001001	001010	001011	101000	101001	101010	101011
001100	001101	001110	001111	101100	101101	101110	101111
010000	010001	010010	010011	110000	110001	110010	110011
010100	010101	010110	010111	110100	110101	110110	110111
011000	011001	011010	011011	111000	111001	111010	111011
011100	011101	011110	011111	111100	111101	111110	111111

Quads and Linear algebra

- The Quad-64 cards and game can be represented by the vector space $(\mathbb{Z}_2)^6$.
- For any positive integer n, there is a game **Quad**-2^n. Its cards correspond to elements of $(\mathbb{Z}_2)^n$.

Theorem

Four cards form a quad if and only if the sum of their vectors is $\vec{0}$.

$$000100 + 101001 + 110110 + 011011 = 000000$$

Quads and Affine Geometry

Quad-2^n cards are points of the affine space associated with $(\mathbb{Z}_2)^n$.

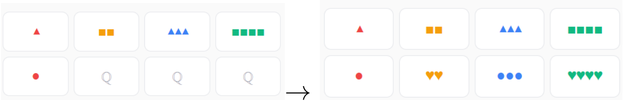
Theorem

A set of points is a flat if and only if it is **quad-closed**, *i.e. closed under triple sums of distinct points.*

- In general, an n-flat contains 2^n points.
- Lines (1-flats) are sets of two points.

- Planes (2-flats) are exactly the quads.

- The 3-flats (8 cards) are determined by any plane and one more point.



4. Counting Quads

Maximum Number of Quads in ℓ Cards

Quad-Packing: What is the largest possible number, $M(\ell)$, of quads that ℓ cards can contain?

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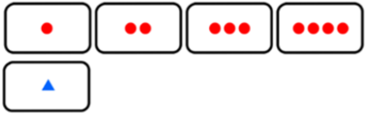
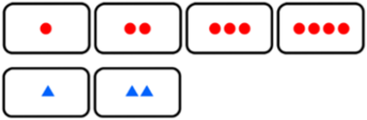
- $\ell = 4$:  $M(4) = 1$
- $\ell = 5$:  $M(5) = 1$

Maximum Number of Quads, $M(\ell)$, in ℓ Cards

- $\ell = 6$:  $M(6) = 3$

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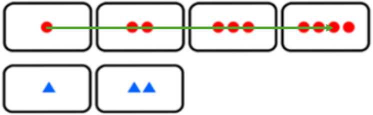
- $\ell = 6$:  $M(6) = 3$

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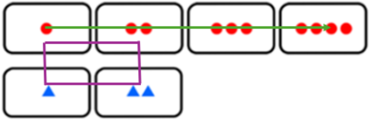
- $\ell = 6$:  $M(6) = 3$

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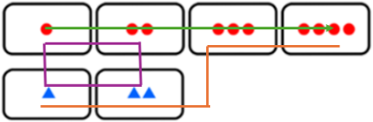
- $\ell = 6$:  $M(6) = 3$

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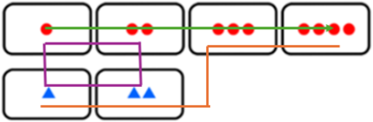
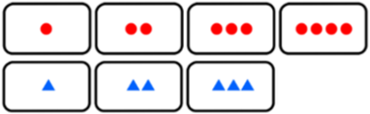
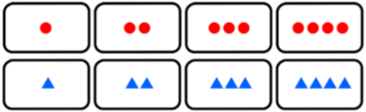
- $\ell = 6$:  $M(6) = 3$
- $\ell = 7$:  $M(7) = 7$
- $\ell = 8$:  $M(8) = 14 = \frac{\binom{8}{3}}{4}$

Flat-Packing

Suppose we want to pack $k = 93$ cards.

$$\begin{aligned} 93 &= 64 + 16 + 8 + 4 + 1 \\ &= 2^6 + 2^4 + 2^3 + 2^2 + 2^0 \end{aligned}$$

Flat-Packing

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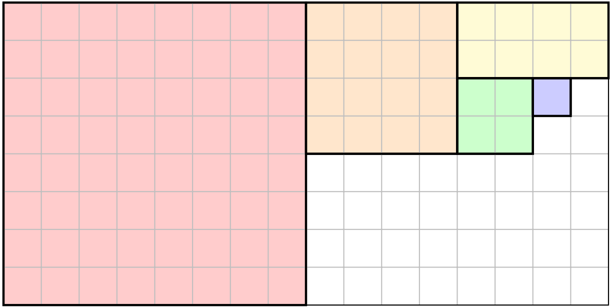


93 cards chosen to form disjoint flats of dimensions $6, 4, 3, 2, 0$, all inside a 7-flat. We say that the cards are **flat-packed** with signature $(6, 4, 3, 2, 0)$.

A Formula for Counting Quads

Theorem

Let $\Lambda \subset \mathbb{Z}_2^n$ be flat-packed with signature $\vec{r} = (r_1, r_2, \ldots, r_m)$. Then Λ contains precisely $M(\ell)$ quads where,

$$M(\ell) = \sum_{1 \leq i \leq m} \frac{1}{4}\binom{2^{r_i}}{3} + \sum_{1 \leq i < j \leq m} 2^{r_i + r_j - 2}\left(2^{r_i} - 1\right) + \sum_{1 \leq i < j < k \leq m} 2^{r_i + r_j + r_k - 1}$$

ℓ	$M(\ell)$
3	0
4	1
5	1
6	3
7	7
8	14
9	14

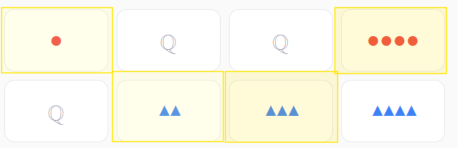
ℓ	$M(\ell)$
10	18
11	26
12	39
13	55
14	77
15	105
16	140

ℓ	$M(\ell)$
17	140
18	148
19	164
20	189
21	221
22	263
23	315

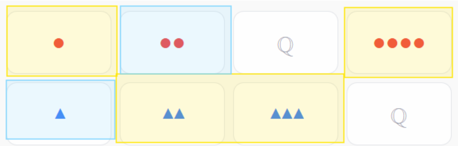
ℓ	$M(\ell)$
24	378
25	442
26	518
27	606
28	707
29	819
30	945

ℓ	$M(\ell)$
31	1085
32	1240
33	1240
34	1256
35	1288
36	1337
37	1401

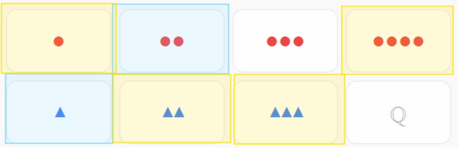
Subsets of a 3-flat with at least 5 cards are flat packed.



- 5 cards, always 1 quad



- 6 cards,always 3 quads



- 7 cards, always 7 quads

5. Flat-Packing is Maximal Packing

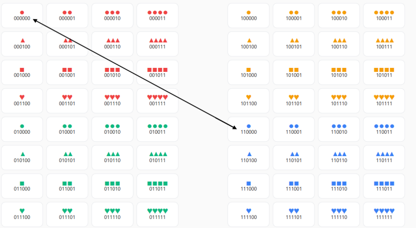
Partitioning and Reflecting

Definition (Reflection Map r)

$\mathbb{Z}_2^n$ is partitioned into two parallel $(n-1)$-flats, A and B. The function $r_{a_0,b_0} : \mathbb{Z}_2^n \to \mathbb{Z}_2^n$ defined by $x \mapsto x + a_0 + b_0$ is called a **reflection of $\mathbb{Z}_2^n$ with respect to a_0 and b_0**.

Example

A is L.H.S with $a_0 = 000000$ and B is R.H.S. with $b_0 = 110000$



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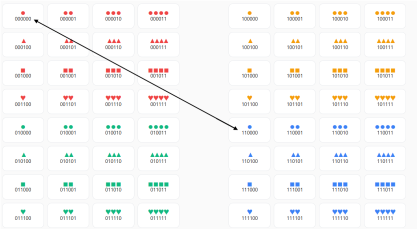
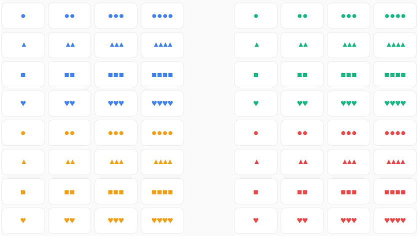


$\rightarrow$



The φ Map

$\Lambda \subset \mathbb{Z}_2^n$ represents a set of cards. $\Lambda_A = \Lambda \cap A$ and $\Lambda_B = \Lambda \cap B$

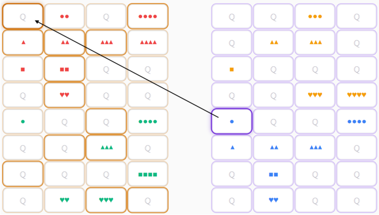
Definition

Define $\varphi \colon \Lambda \to \mathbb{Z}_2^n$ by

$$\varphi(x) = \begin{cases} r_{a_0,b_0}(x) & \text{if } x \in \Lambda_B \text{ and } r_{a_0,b_0}(x) \notin \Lambda_A, \\ x & \text{otherwise} \end{cases}$$

Example

A is gold (L.H.S.) with $a_0 = 000000$ and B is purple (R.H.S.) with $b_0 = 110000$



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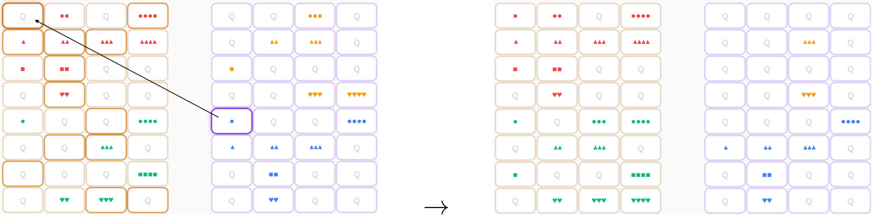


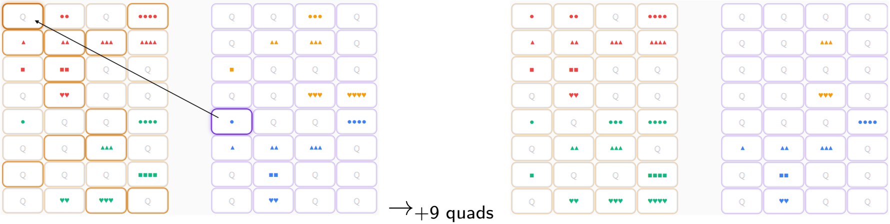
The φ Map Maintains Or Increases the Quad Count

Theorem

$\varphi(\Lambda)$ *has at least as many quads as* Λ.

Example

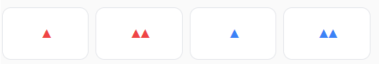
A is gold (L.H.S.) with $a_0 = 000000$ and B is purple (R.H.S.) with $b_0 = 110000$
Λ contains 338 quads while $\varphi(\Lambda)$ contains 347 quads.



Quads that are symmetric under r.

Example

Consider a quad $Q \subset \Lambda$. If $r(Q) = Q$, then Q is fixed under φ.



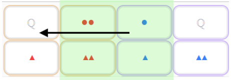
- Each of these 4 cards is in Λ.
- The φ map used is based on the reflection that pairs blue and red.
- This Λ-quad, Q, reflects onto itself.
- We call Q a symmetric quad

Quads that are asymmetric under r

Example

Consider an asymmetric Λ-quad, Q.

- $Q \cup r(Q)$ is a 3-flat
- There are the same number of quads before and after the φ map.
- This example shows 3 quads in the same 3-flat
  - 1 symmetric on previous slide
  - 2 asymmetric quads shown here: Q_1 and Q_2

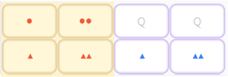


Λ quad Q_1

$\rightarrow$

$\varphi(Q_1)$ not a quad

but



asymmetric Λ' quad.

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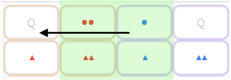

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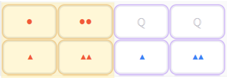
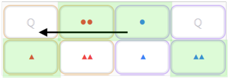
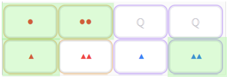


asymmetric Λ' quad.



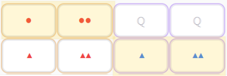
Λ quad Q_2

$\rightarrow$



$\varphi(Q_2)$ not a quad

but



asymmetric Λ' quad.

Outline of proof that φ maintains or increases quads.

Lemma

Let $\Lambda' := \varphi(\Lambda)$. There is an injective function $f : quads(\Lambda) \to quads(\Lambda')$.

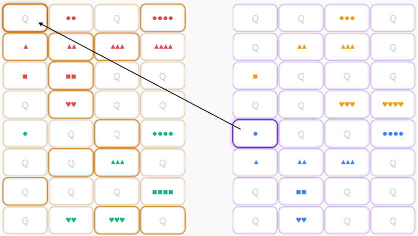
Proof.

. Define $f : quads(\Lambda) \to quads(\Lambda')$ piecewise.

- For $Q \in \Lambda$ s.t. $r(Q) = Q$, then let $f(Q) = Q$.
- For $Q \in \Lambda$ s.t. $r(Q) \neq Q$, then consider the 3 flat given by $Q3 = Q \cup r(Q)$.
  - $\varphi(Q3 \cap \Lambda) = Q3 \cap \Lambda'$
  - $Q3 \cap \Lambda$ and $Q3 \cap \Lambda'$ both contain the same number of cards and at least one quad.
  - The number of quads is determined by the number of cards. Therefore, $Q3 \cap \Lambda$ and $Q3 \cap \Lambda'$ contain the same number of quads.
  - This allows one to define f for quads in $Q3 \cap \Lambda$

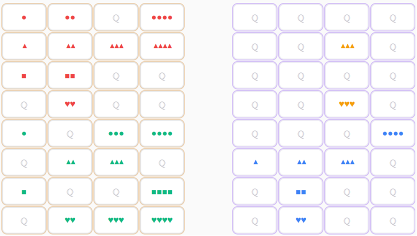
□

Iterate φ map until a set of cards is flat-packed.



338 quads

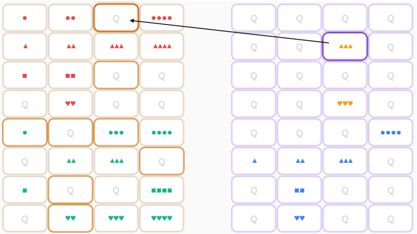
$\rightarrow$



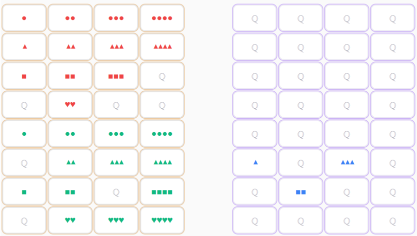
347 quads

$+9$ quads

Iterate φ map until a set of cards is flat-packed.



$\rightarrow$

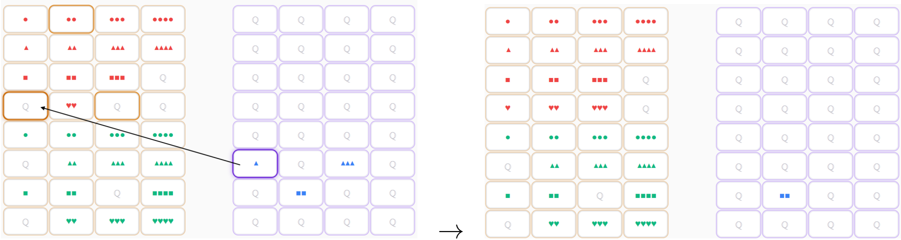


347 quads

463 quads

+116 quads

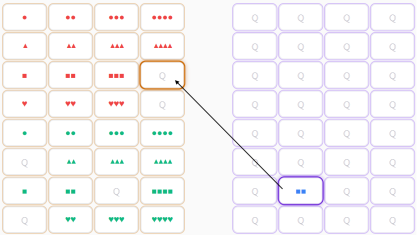
Iterate φ map until a set of cards is flat-packed.



463 quads $\rightarrow$ 605 quads

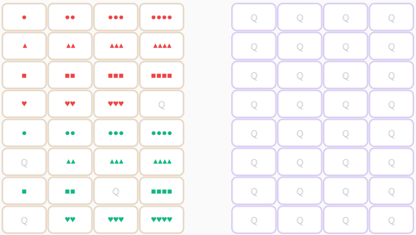
+142 quads

Iterate φ map until a set of cards is flat-packed.



605 quads

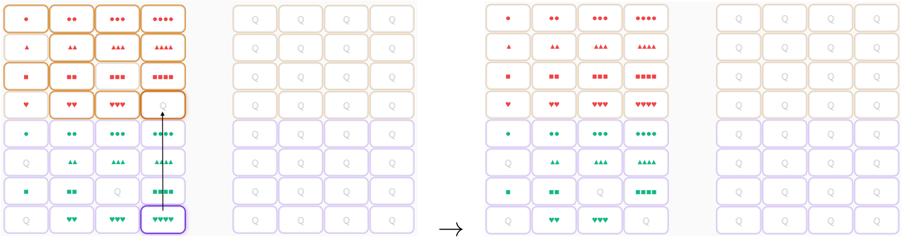
$\rightarrow$



706 quads

$+101$ quads

Iterate φ map until a set of cards is flat-packed.



706 quads $\rightarrow$ 706 quads

$+0$ quads

Iterate φ map until a set of cards is flat-packed.

706 quads

$\rightarrow$

706 quads

$+0$ quads

Iterate φ map until a set of cards is flat-packed.

706 quads

$\rightarrow$

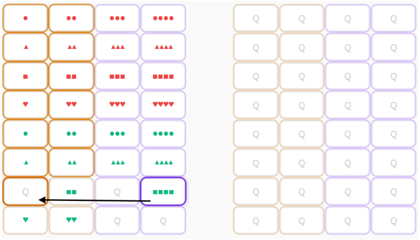
706 quads

$+0$ quads

Iterate φ map until a set of cards is flat-packed.

Theorem

A flat-packed set of ℓ cards has the maximum number of quads, $M(\ell)$.



706 quads

$\rightarrow$

+1 quad

707 quads

28 flat packed cards

6. Maximal Packing is Flat Packing

6. Maximal Packing is Flat Packing

- (1) conjectured that If $\dim(\Lambda) > \lceil \log_2(\ell) \rceil$, then the number of quads is at most $M(\ell - 1)$.
- (1) showed that if this conjecture holds, max packing implies flat packing.
- We prove

Theorem

Let $\ell = |\Lambda|$ and $Q(\Lambda)$ be the number of quads in Λ.
If $\dim(\Lambda) > \lceil \log_2(\ell) \rceil$, then the $Q(\Lambda) \leq M(\ell - 1)$.

thus finishing the proof that Maximal Packing is Flat Packing.

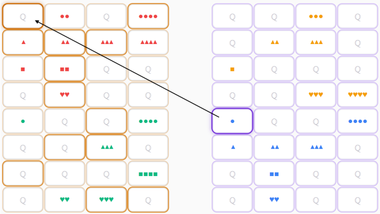
Example

Λ contains 28 cards but Λ spans 6 dimensions. Then Λ contains **at most** $M(27)$ quads.

Case 1: Iterate φ map until one card is left behind

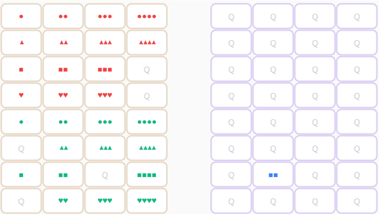
Example

Λ contain 28 cards that could fit in dimension 5, but Λ spans 6 dimensions. After 3 iterations of the φ map, there is a lone card in B. This card is in no quads. Thus Λ contains **at most** $M(27)$ quads.



338 quads

$\rightarrow_3$ φ maps



605 quads

$\leq M(27) = 606$ quads

Case 2: There is no φ map that leaves a card in B

Lemma

If there is no pair a_0, b_0 such that φ based on $r(x) = x + a_0 + b_0$ moves at least one card in Λ_B without moving all cards in Λ_B, then Λ_A is the union of flats parallel and equal in size to span(Λ_B).

Outline of Proof.

Theorem

If dim$(\Lambda) > \lceil \log_2(\ell) \rceil$*, then* $Q(\Lambda) \leq M(\ell - 1)$.

Outline of proof. Use φ map to pack until

1. **Case 1: One card is left in B.**
   - This one card in B is not in any quads. Thus
   $$Q(\Lambda) \leq M(\ell - 1)$$
2. **Case 2: The Λ_A cards are a union of flats.**
   - Found a lower bound, L, for $M(\ell) - Q(\Lambda)$.
   - Found an upper bound, U, for $M(\ell) - M(\ell - 1)$.
   - Showed $L \geq U$. Thus
   $$Q(\Lambda) \leq M(\ell - 1)$$

Results and Conjectures

- **Result 1.** An explicit formula for counting quads in a flat packed set.
- **Result 2.** Flat packing is equivalent to max packing.
- **Conjecture.** If $2^{n-k-1} < \ell \leq 2^{n-k}$, but the ℓ cards have dimension n, the the cards contain at most $M(\ell - k)$ quads.

Bibliography

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2. https://math.hope.edu/harrelson/quads/quad_packing.html

The End