Rational Approximations to Irrational Square Roots

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Famous rational sequences that converge to $\sqrt{2}$

- Newton's Method (Herron's method, Babylonian method) gives:

$$1, \frac{3}{2}, \frac{17}{12}, \frac{577}{408}, \frac{665857}{470832}, \ldots$$

$$1, 1.5, 1.4167, 1.414213,$$

- Continued Fraction *convergents*: $\sqrt{2} = 1 + \cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\ddots}}}$. These give

$$1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \frac{99}{70}, \ldots$$

A simple Fibonacci-type construction

- $g_n = 2g_{n-1} + g_{n-2}$ for $n \geq 2$ and initial conditions g_0 and g_1
- The general solution is $g_n = c_1\lambda_1^n + c_2\lambda_2^n$
- λ_1 and λ_2 are the roots of the characteristic equation $\lambda^2 = 2\lambda + 1$ and c_1 and c_2 are determined from the initial conditions
- In this case we get $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$. Hence,

$$g_n = c_1\left(1 + \sqrt{2}\right)^n + c_2\left(1 - \sqrt{2}\right)^n$$

- Note $\boldsymbol{\lambda_2 < 0}$ and $|\boldsymbol{\lambda_1}| \boldsymbol{>} |\boldsymbol{\lambda_2}|$

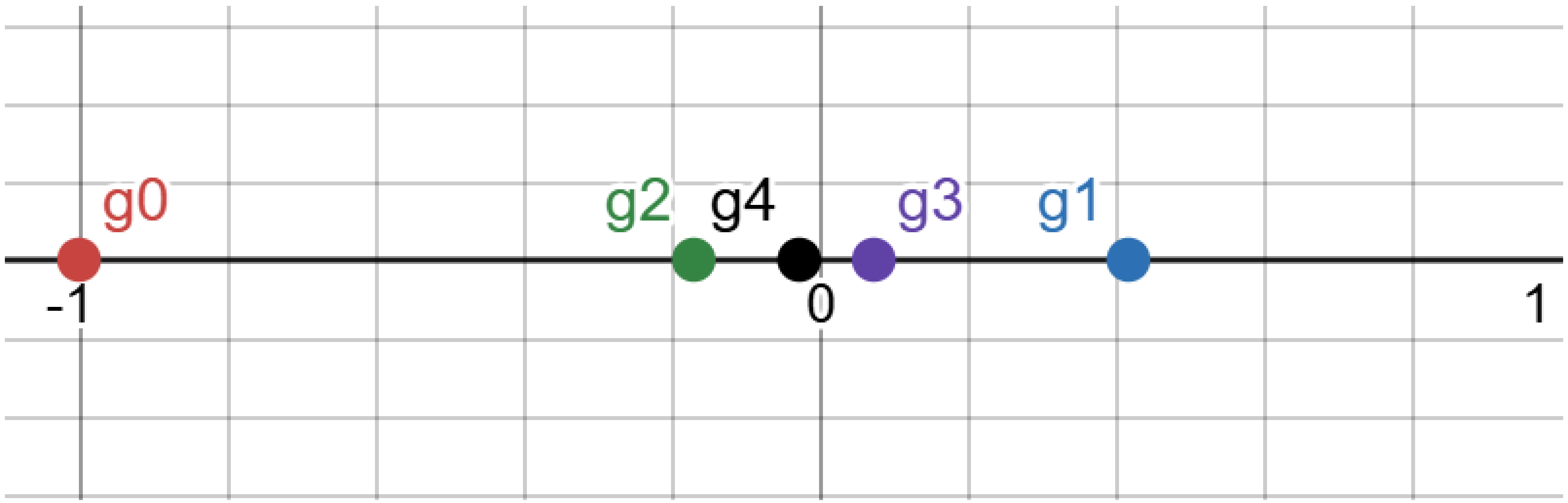
The Motivating Question

- Assume we set $\boldsymbol{g_0 = -1}$ and $\boldsymbol{g_1 = x}$
- How do we choose $\boldsymbol{x}$ so that the terms **alternate sign**?
- **Answer:** Recall the general solution to the recurrence relation is $\boldsymbol{g_n = c_1\lambda_1^n + c_2\lambda_2^n}$ and the coefficients c_1 and c_2 are determined by solving

$$\begin{aligned} \boldsymbol{c_1 + c_2} &= \boldsymbol{-1} \\ \boldsymbol{c_1\lambda_1 + c_2\lambda_2} &= \boldsymbol{x} \end{aligned}$$

- We must have $\boldsymbol{c_1 = 0}$ because $|\boldsymbol{\lambda_1}| > |\boldsymbol{\lambda_2}|$ and otherwise the positive root would eventually dominate the behavior.
- Hence $\boldsymbol{c_1 = 0}$, so $\boldsymbol{c_2 = -1}$ giving $\boldsymbol{x = -\lambda_2 = \sqrt{2} - 1;}$ thus, $\boldsymbol{g_n = -(\lambda_2)^n}$

The First Five Terms show Oscillating Behavior



- Setting $g_1 = -1$ and $g_2 = x = -\lambda_2 = \sqrt{2} - 1$, makes the recursive sequence alternate sign.

- But what if we didn't know that? How else could we determine x?

- Here is the sequence without knowing the value of x:

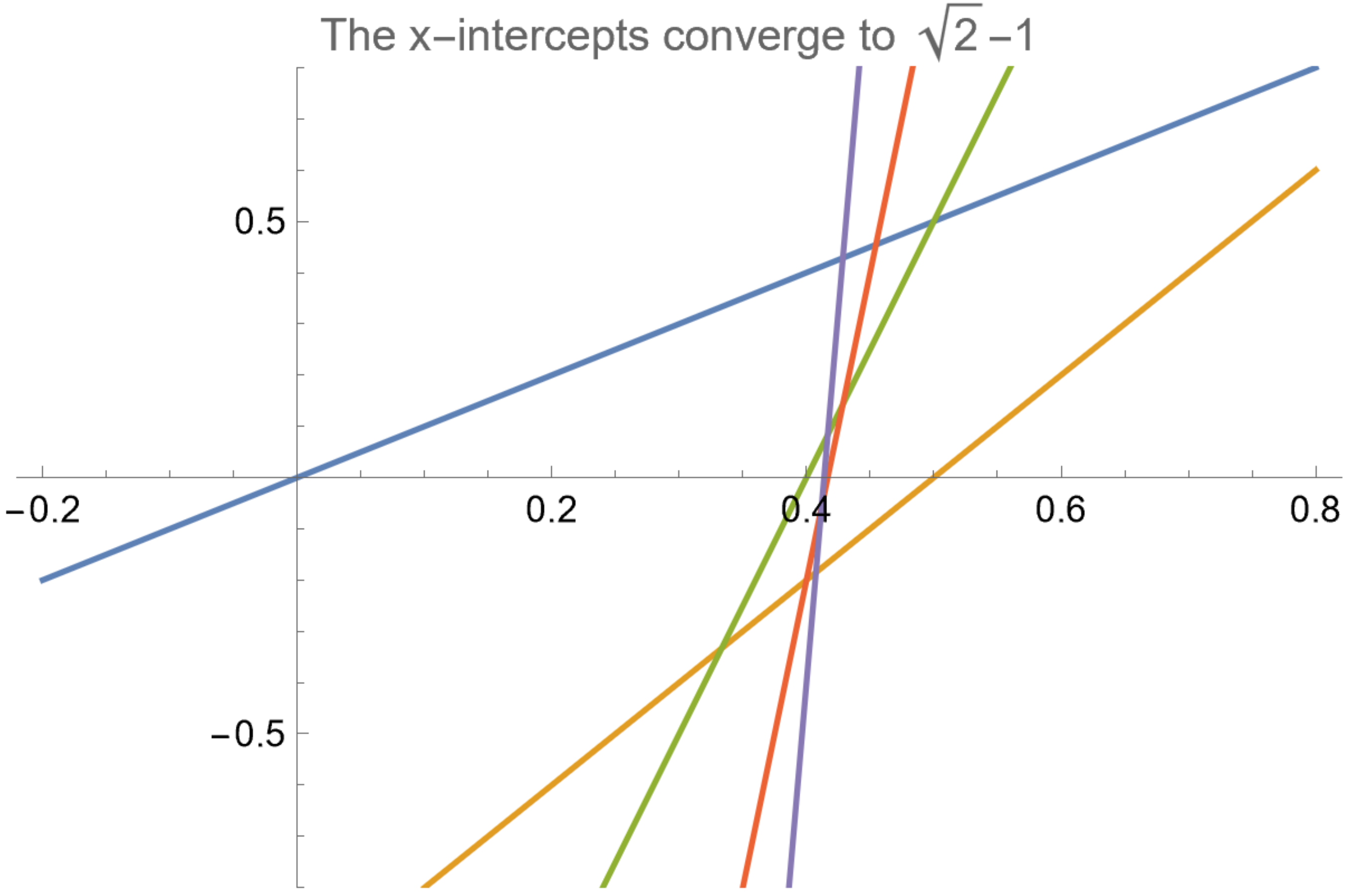
$$-1, x, 2x - 1, 5x - 2, 12x - 5, 29x - 12, \ldots$$

We want the terms to alternate signs; hence x must satisfy the infinite chain of inequalities:

$$x > 0, 2x - 1 < 0, 5x - 2 > 0, 12x - 5 < 0, \ldots$$

Solve for the end point of each inequality: $0, \frac{1}{2}, \frac{2}{5}, \frac{5}{12}, \ldots$

Each inequality involves a linear function



The intercepts converge $\sqrt{2} - 1$

$$\frac{1}{2}, \frac{2}{5}, \frac{5}{12}, \frac{12}{29}, \ldots \to \sqrt{2} - 1$$

- Now add 1 to each term to obtain a sequence converging to $\sqrt{2}$

$$\frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \frac{41}{29}, \ldots \to \sqrt{2}$$

- This is the sequence of **convergents** from the continued fraction expansion of $\sqrt{2}$

This works in general

- $g_n = ag_{n-1} + bg_{n-1}$, with $a, b > 0$; $g_0 = -1$, $g_1 = x$

- The characteristic equation is $\lambda^2 = a\lambda + b \;\Rightarrow \lambda_{1,2} = \frac{a \pm \sqrt{a^2+4b}}{2}$

- General solution: $g_n = c_1\lambda_1^n + c_2\lambda_2^n$

- Alternating signs requires $c_1 = 0$ and $x = -\lambda_2 = \frac{\sqrt{a^2+4b}-a}{2}$

- Writing the sequence in terms of x and forcing them to alternate sign gives a rational sequence that converges to $-\lambda_2 = \frac{\sqrt{a^2+4b}-a}{2}$

- Multiplying by 2 and adding a to each terms yields a rational sequence converging to $\sqrt{a^2+4b}$.

Example (Fibonacci & Lucas):

- When $a = b = 1$, this method gives $\frac{L_n}{F_n} = \frac{1}{1}, \frac{3}{1}, \frac{4}{2}, \frac{7}{3}, \frac{11}{5}, \frac{18}{8} \ldots \to \sqrt{5}$

- By choosing appropriate a and b, we can create rational sequences that converge to $\sqrt{n}$ for all $n \geq 2$.

Thank you

Questions?