Tick Tock: Partitions of a Clock

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Initial Question

The hands of an analog clock subdivides a clock's face into a variety of sector configurations.

Initial Question: Does there exist a time of day for which the three hands subdivide the clock into three equal sectors?

Modeling the clock

Imagine the face of a circular clock with center O and unit radius along with the ends of the hour, minute, and second hands located at the points H, M, and S on the boundary circle, respectively. We locate points on the boundary circle by real numbers modulo 2π in the clock friendly but mathematically nonstandard manner in which k o'clock is located at $\frac{2\pi}{12}k$ mod 2π for $k = 1, 2 \ldots, 12$.

Let t be the number of hours past 12 o'clock. Then the locations of H, M and S are given by $h(t) = \frac{2\pi}{12}t \pmod{2\pi}$, $m(t) = 2\pi t \pmod{2\pi}$, and $s(t) = 120\pi t \pmod{2\pi}$, respectively.

Notation. $x \equiv y \pmod{z}$ with $x, y, z \in \mathbb{R}$ and $z > 0$ means that $x = y + kz$ for some $k \in \mathbb{Z}$.

Modeling the clock

For t hours past 12 o'clock, we defined

$$
\begin{aligned}
h(t) &= \frac{2\pi}{12}t \pmod{2\pi} \\
m(t) &= 2\pi t \pmod{2\pi} \\
s(t) &= 120\pi t \pmod{2\pi}
\end{aligned}
$$

Notation. $x \equiv y \pmod{z}$ with $x, y, z \in \mathbb{R}$ and $z > 0$ means that $x = y + kz$ for some $k \in \mathbb{Z}$.

Example.

Time of Day	t	$h(t)$	$m(t)$	$s(t)$
$12:00:00$	0	$\frac{2\pi}{12}$	0	0
$6:00:00$	6	π	0	0
$12:01:00$	$1/60$	$\frac{2\pi}{720}$	$\frac{2\pi}{60}$	0
$12:00:30$	$1/120$	$\frac{2\pi}{1440}$	$\frac{2\pi}{120}$	π

Clock partitions

Depending on whether or not the hands of the clock coincide, the segments OH, OM, and OS divide the face of the clock into one, two, or three sectors. Using clock symmetry if necessary, we may replace t with $-t$ and assume that the points appear in the sequence H, M, and then S when moving clockwise.

Let $\triangledown_1$ be the sector with sides OH and OM, $\triangledown_2$ the sector with sides OM and OS, and $\triangledown_3$ the sector with sides OS and OH. Suppose a, b, and c are nonnegative real numbers with $a + b + c > 0$.

General Question. For what values of $a : b : c$ are realizable by $\mathrm{area}\,(\triangledown_1) : \mathrm{area}\,(\triangledown_2) : \mathrm{area}\,(\triangledown_3)$?

Clock partitions

Solving the associated system of linear congruence equations carefully gives

Theorem (Clock Partitions)

Suppose a, b, and c are nonnegative real numbers with $a + b + c > 0$. Let $N = a + b + c$. There is a time t such that

$$\mathit{area}(\triangledown_1) : \mathit{area}(\triangledown_2) : \mathit{area}(\triangledown_3) = a : b : c$$

if and only if $719b + 708c \equiv 0 \pmod{N}$.

Example

Take $a = b = c = 1$. Then $N = 3$ and $719 \cdot 1 + 708 \cdot 1 \equiv 2 \pmod{3}$. By the theorem, there is no time t which gives the ratio $1 : 1 : 1$.

Sketch of details for clock partitions

Assume that the points appear in the sequence H, M, and then S when moving clockwise. The three sectors have angles in radians respectively of $\frac{2\pi}{N}a$, $\frac{2\pi}{N}b$, and $\frac{2\pi}{N}c$ where $N = a + b + c$. Thus, we have

$$\begin{aligned} m(t) - h(t) &\equiv \frac{2\pi}{N}a \pmod{2\pi} \\ s(t) - m(t) &\equiv \frac{2\pi}{N}b \pmod{2\pi} \\ h(t) - s(t) &\equiv \frac{2\pi}{N}c \pmod{2\pi}. \end{aligned}$$

Clock partitions

Using the assumptions made and definitions of m, h, and s, we have

$$\begin{aligned} m(t) - h(t) &\equiv \frac{11}{12} \cdot 2\pi t \pmod{2\pi} \\ s(t) - m(t) &\equiv 59 \cdot 2\pi t \pmod{2\pi} \\ h(t) - s(t) &\equiv -\frac{719}{12} \cdot 2\pi t \pmod{2\pi}. \end{aligned} \tag{1}$$

Clock partitions

Comparing the two expressions for $m(t) - h(t)$ gives $\frac{2\pi}{N}a \equiv \frac{11}{12}2\pi t \pmod{2\pi}$ or equivalently $\frac{2\pi}{N}a = \frac{11}{12}2\pi t + 2\pi k$ for $k \in \mathbb{Z}$. Thus $12(a - Nk) = 11Nt$ which can be expressed as $11Nt = 12A$ where $A \equiv a \pmod{N}$.

Using a similar approach with $s(t) - m(t)$ and $h(t) - s(t)$, we obtain the relationships

$$\begin{aligned} 11Nt &= 12A \\ 59Nt &= B \\ -719Nt &= 12C \end{aligned}$$

where $A \equiv a \pmod{N}$, $B \equiv b \pmod{N}$, and $C \equiv c \pmod{N}$.

Clock partitions

Eliminating t from each pair of equations yields

$$\begin{aligned} 708A &= 11B \\ -719A &= 11C \\ -719B &= 708C. \end{aligned}$$

The latter system of equations is equivalent to

$$\begin{gathered} A + B + C = 0 \\ 719B + 708C = 0. \end{gathered} \tag{2}$$

The second equation implies $719b + 708c \equiv 0 \pmod{N}$.

Clock partitions

Theorem (Clock Partitions)

Suppose a, b, and c are nonegative real numbers with $a + b + c > 0$. *Let* $N = a + b + c$. *There is a time t such that*

$$\mathit{area}(\triangledown_1) : \mathit{area}(\triangledown_2) : \mathit{area}(\triangledown_3) = a : b : c$$

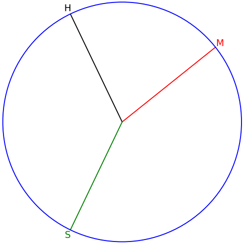
if and only if $719b + 708c \equiv 0 \pmod{N}$.

Example

Take $a = b = c = 1$. Then $N = 3$ and $719 \cdot 1 + 708 \cdot 1 \equiv 2 \pmod{3}$. By the theorem, there is no time t which gives the ratio $1 : 1 : 1$.

Example

There is a time with ratio $3:6:5$. Observe that $N = 3 + 6 + 5 = 14$ and $719 \cdot 6 + 708 \cdot 5 \equiv 0 \pmod{14}$. Working backwards through the computations and since the position of the system of three hands is periodic with period of 12 hours, we can take $t = 11 + \frac{1}{7}$ which corresponds to the time $11:08:34$ and $2/7$ seconds and is shown in the figure below.



Using clock symmetry, taking $t = -\left(11 + \frac{1}{7}\right)$ also realizes the ratio $3:6:5$ and corresponds to the time $12:51:25$ and $5/7$ seconds.

When hands coincide

Theorem

Up to reordering the sectors, there exists a time t which realizes the ratio $x : y : 0$ with $x + y > 0$ if and only if x, y are nonnegative integers with $x + y = 11$, $x + y = 708$, or $x + y = 719$.

Remark. All three hands coincide only at 12 o'clock.

Question. Is there any significance to the parameter values of 11, 708, and 719?

Physical interpretation of constants

In each 12-hour period,

- the minute hand passes the hour hand 11 times;
- the second hand passes the minute hand 708 times;
- the second hand passes the hour hand 719.

Bonus Question

Suppose $a : b : c$ is a realizable clock partition. Can any of its permutations also be realized? If so, when?

More explicitly, assume $a : b : c$ is realizable. When can we realize any of the following?

$$\begin{aligned} a &: c : b \\ b &: a : c \\ b &: c : a \\ c &: a : b \\ c &: b : a \end{aligned}$$

Transpositions

We found classes of examples which realize each of the three transpositions in a nontrivial way.

Theorem

Let a, b, c be positive integers. There exists realizable pairs of ratios

i. $a : b : c$ *and* $b : a : c$ *with* $a \neq b$;
ii. $a : b : c$ *and* $a : c : b$ *with* $b \neq c$; *and,*
iii. $a : b : c$ *and* $c : b : a$ *with* $a \neq c$.

Example

The following ratios are realizable as clock partitions.

i. $708 : 11 : 500424$ and $11 : 708 : 500424$
ii. $14256 : 1434 : 7$ and $14256 : 7 : 1434$
iii. $1427 : 514716 : 697$ and $697 : 514716 : 1427$

Cycles

Using linear algebra, we found a class of examples which are *cyclically* permutable.

Theorem

Suppose a, b, c be positive integers. The lattice points (x, y) which lie strictly within the triangle with vertices $(0, 708)$, $(719, 0)$, and $(708, 719)$ are in a one-to-one correspondences with cyclically permutable ratios $a : b : c$.

Example

The ratios $1 : 2 : 4$ and $170178 : 168065 : 170930$ are both cyclically permutable.

Thank you

Thank you to the session organizers!

Remarks

1. This paper will appear in an upcoming issue of Mathematics Magazine.
2. If you have any challenge problems at an advanced high school up to easier Putnam level, consider sending them to Math Horizons at <MHproblems@maa.org> for consideration in its problems column, The Playground.