Tangle Time: Graphs and Knots in a Math Circle

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Prime Factor Math Circle
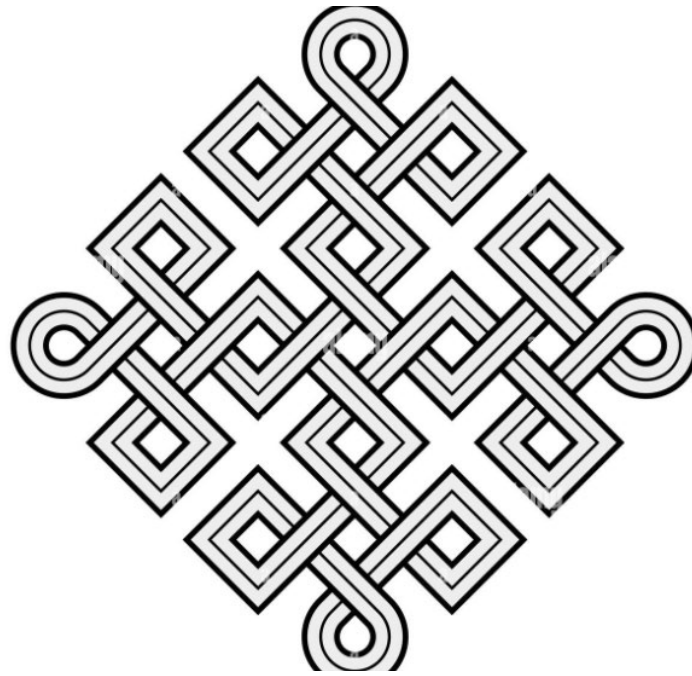
- Knots have been used by many civilizations as both tools and ornaments. We are familiar with Celtic, Nordic, and Chinese knots. Many of us have tried to draw these intricate designs, only to discover how challenging it can be.

- In today's lesson, we will explore a mathematical approach to knot drawing. Using planar graphs as frameworks for these drawings, we will create designs ranging from simple trefoils to complex Celtic patterns.

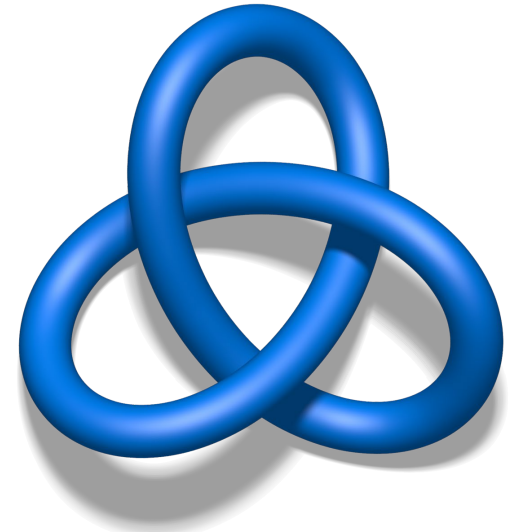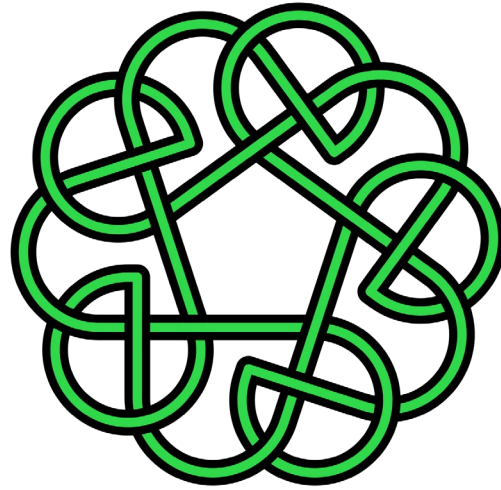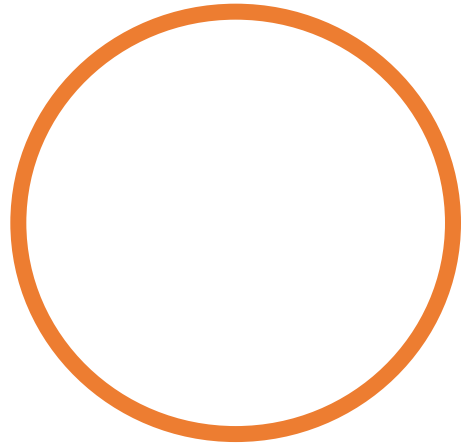Which knots will untie when you pull the rope ends in different directions?
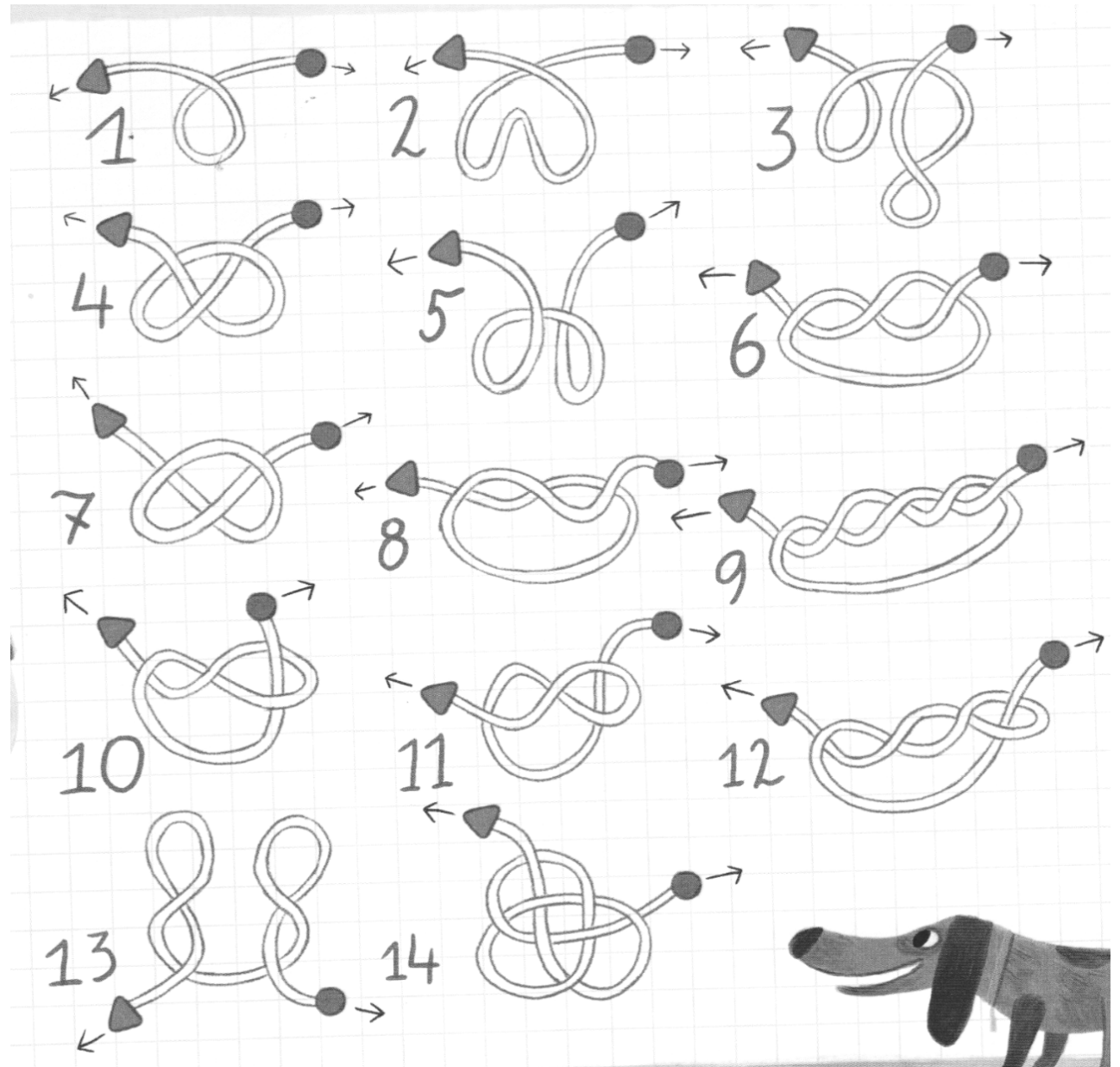
"Untie the Knots." Image Source: Kvantik Magazine, https://kvantik.com/

If we pull the string ends in opposite directions, will the ring fall off the string?

"String and Ring." Image Source: Квантiк Magazine, https://kvantik.com/
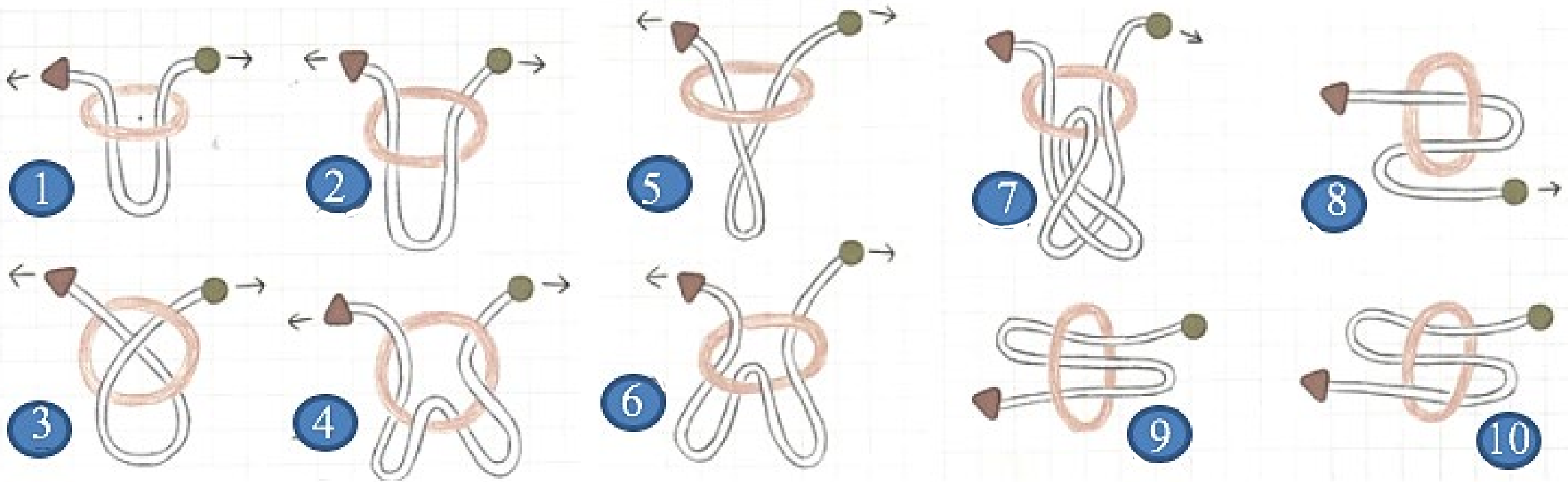
Mathematical Knots

- Unlike everyday knots, a *mathematical knot* has no ends, meaning it cannot be untied or undone. Imagine tying a knot in a piece of rope and then joining the ends together so there's no way to untie it.

- *A knot* is made of one string with closed ends.

- *A link* consists of several pieces of string with closed ends (in other words, several knots, usually linked together)

Knot diagrams

You can draw a knot on paper by showing how the loop crosses over and under itself. These diagrams are like "blueprints" for studying knots.
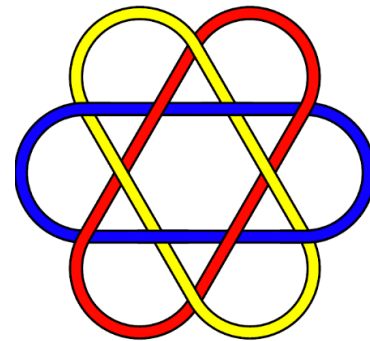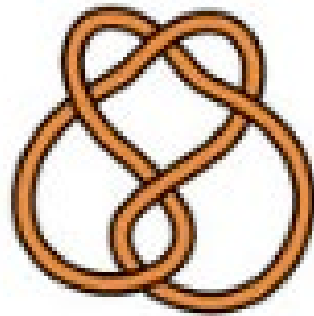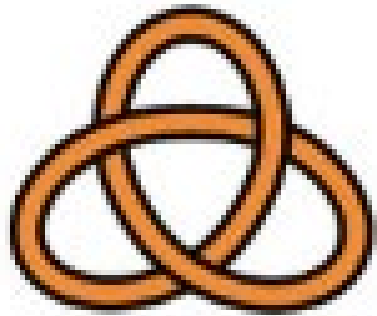
Big questions of Knot Theory

- **One big question in knot theory** is whether a knot can be untangled into the **unknot** (a simple circle).

- **Another big question** is how can we tell if two knots are the same or different?

- For example, these two knots below are the same. Pulling and twisting will convert the first one into the second one.

Knots and Graphs

Knot theory shows that knots can be represented as **planar graphs** with signed edges (**Tait graphs**).
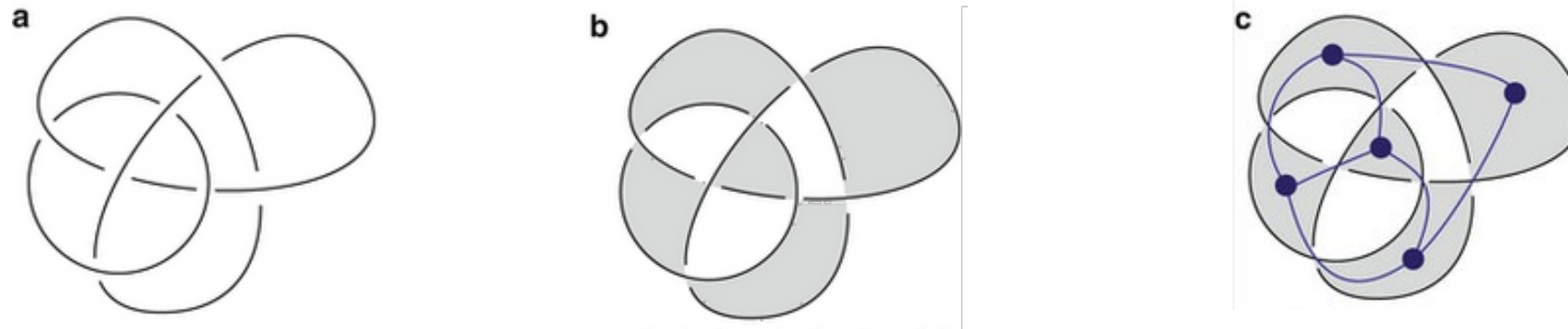

Today, we will focus on the inverse problem:

creating knots from graphs.

It's a highly rewarding process and the main topic of our activity.

A knot and its Tait graph, https://www.icts.res.in/sites/default/files/13-12-13_Abhijit_Champanerkar_1_Knots,%20Graphs%20%26%20Khovanov%20Homology-I.pdf

We can learn to draw knots!

- Start with a planar graph, a *skeleton* of the knot. You can think of it as a wire carcass on which we weave the knot.
- Put a crossing mark in the middle of each edge. Indicate clearly which stroke is above and which is under.
- Start drawing at a crossing, moving along the stroke. Continue until you reach the next crossing. There, follow the stroke that aligns with your path.
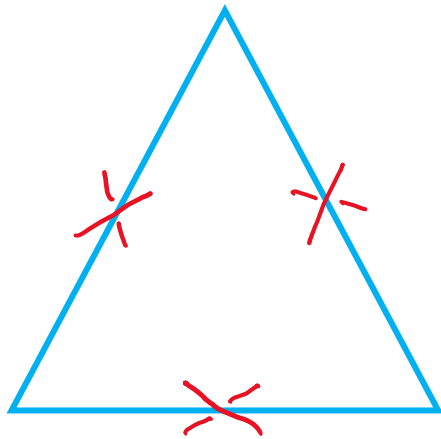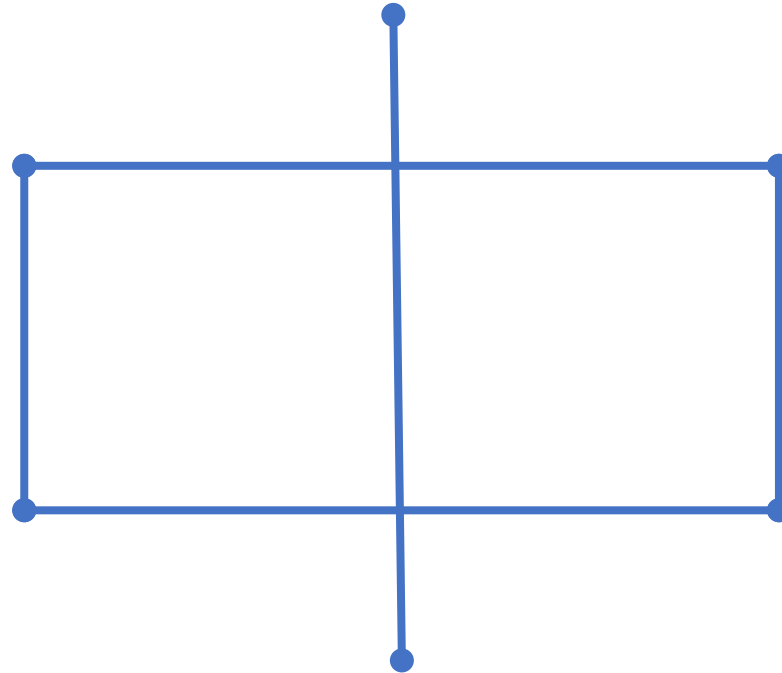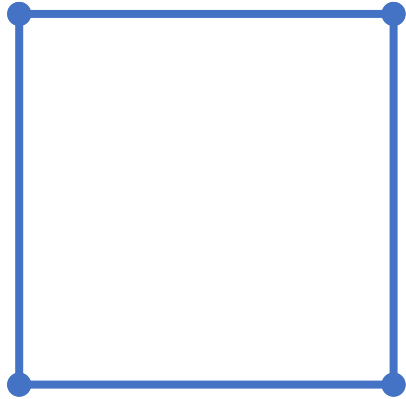
Trefoil knot

How to draw the crossings
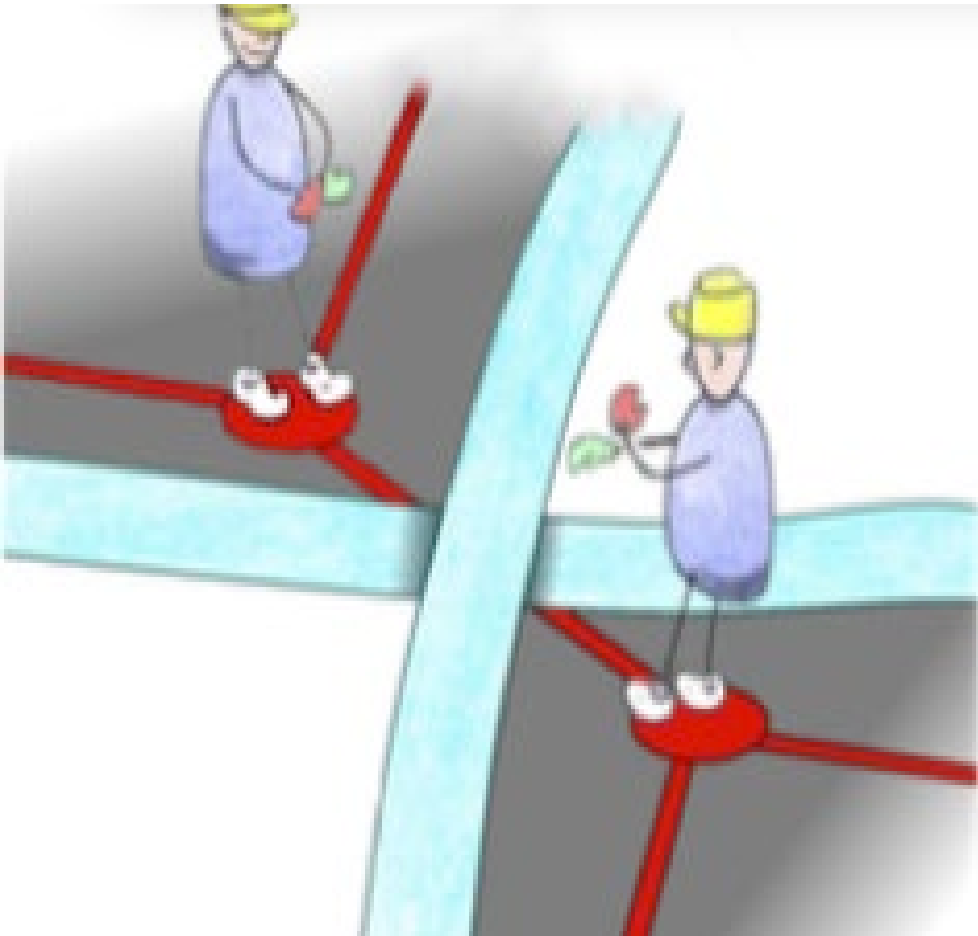
- All crossings should follow the same pattern.
- Choose an edge. Imagine that you stand on its endpoint looking straight down at it.
- Cross your arms with the left over the right one. Draw the crossing with the strokes replicating the position of your arms (the left over the right).
- Note that it doesn't matter on which end you are standing; the crossing will end up looking the same.
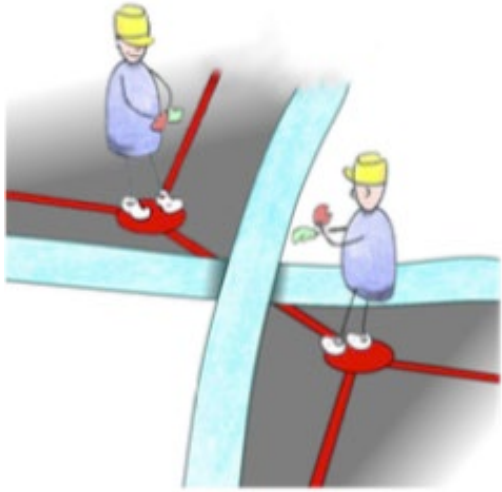
How to draw the crossings

- Cross your arms with the left over the right one. Draw the crossing with the strokes replicating the position of your arms (the left over the right).

- Think of the edges as walls. Follow the wall, turn a corner, follow the wall, until you meet another crossing. Connect all crossings.

What happens if we draw incorrect crossings?

- We will still get some kind of knot. However, it will be different.

- The pictures below show "incorrect" crossings in red.
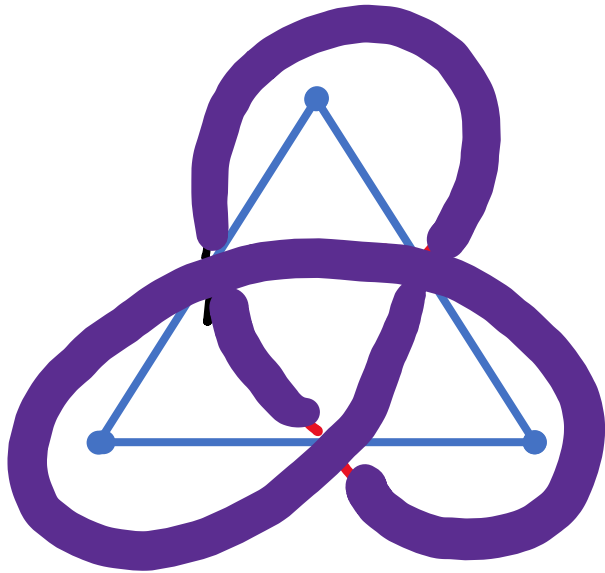
Instead of a trefoil, we got the unknot!
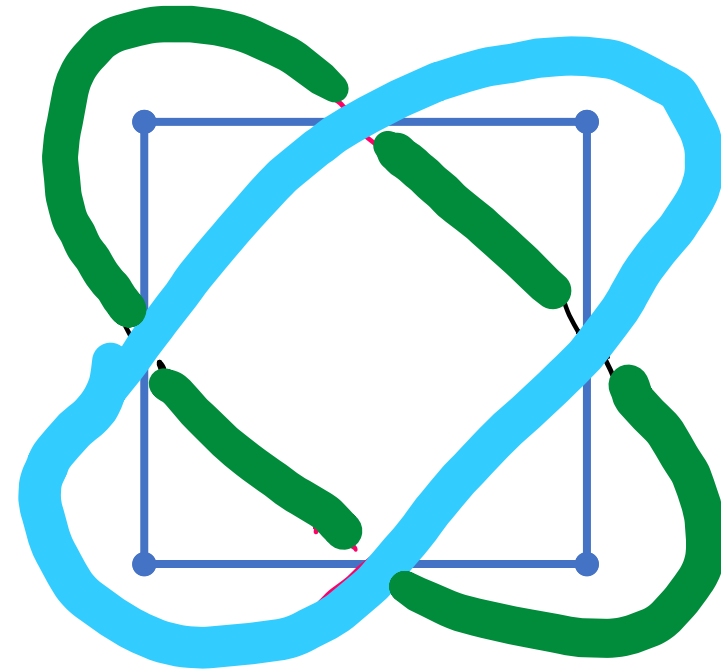
Here, we got two links not connected with each other!

Now you try!

Choose a template in handout and draw a knot.

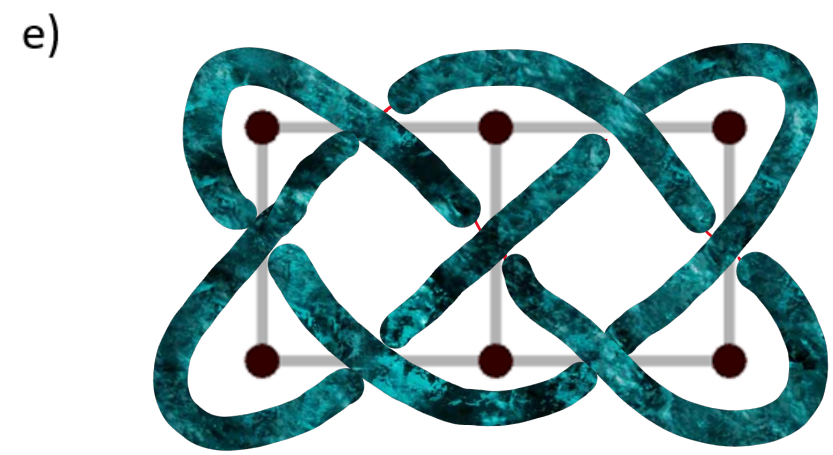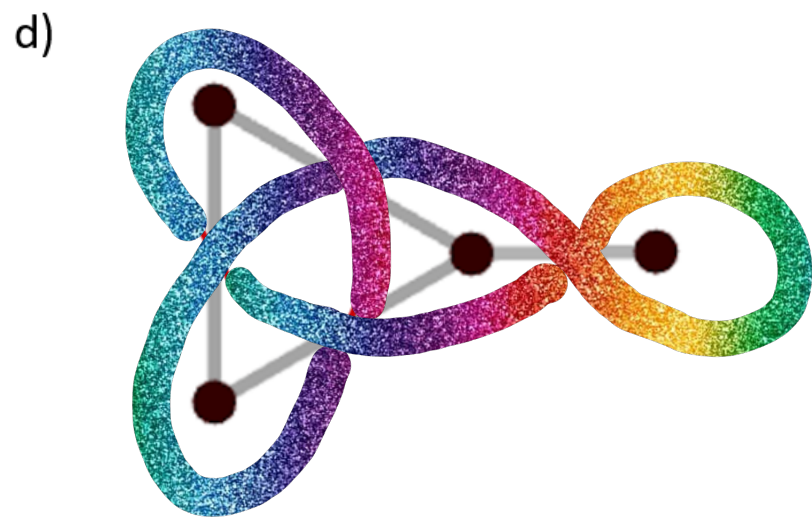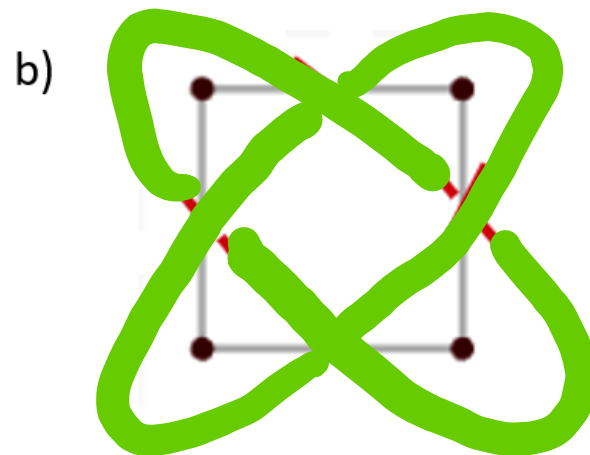**Try it**! Draw the knots for these skeletons with a pencil, then outline with a marker.
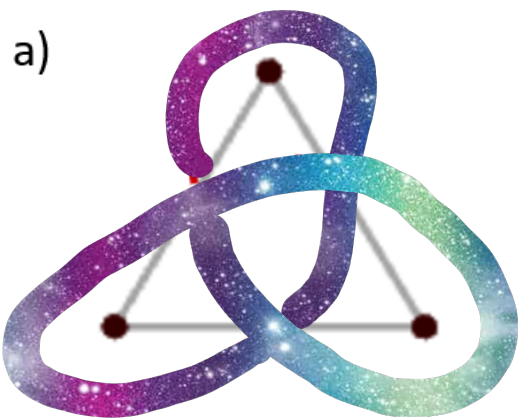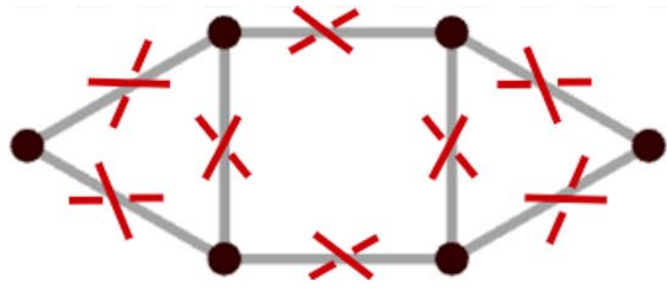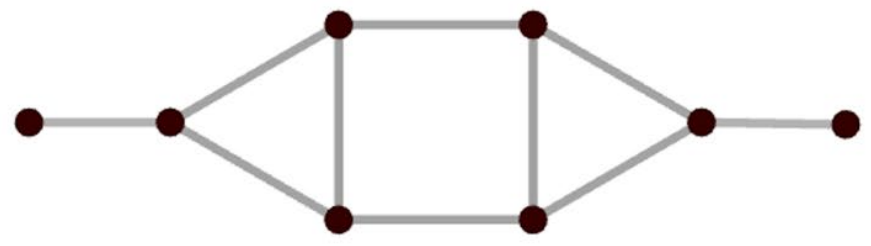
a)
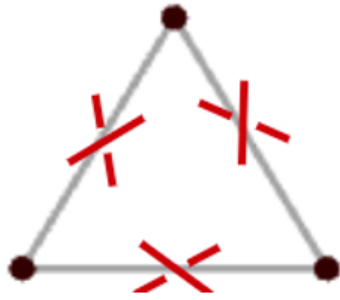
b)
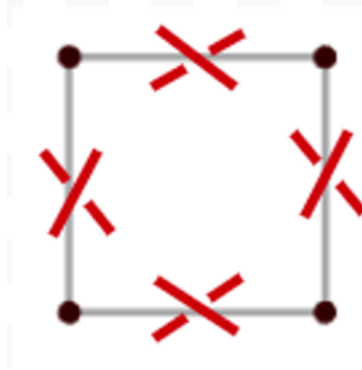
c)

d)
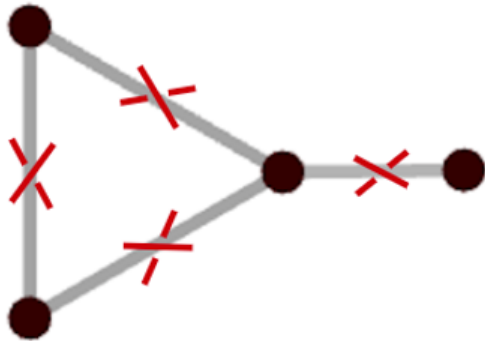
e)
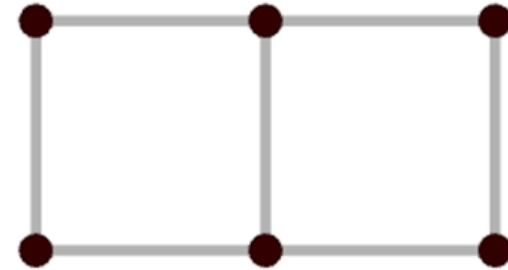
Try it! Draw the knots for these skeletons with a pencil, then outline with a marker.

f)

g)

g)

h)
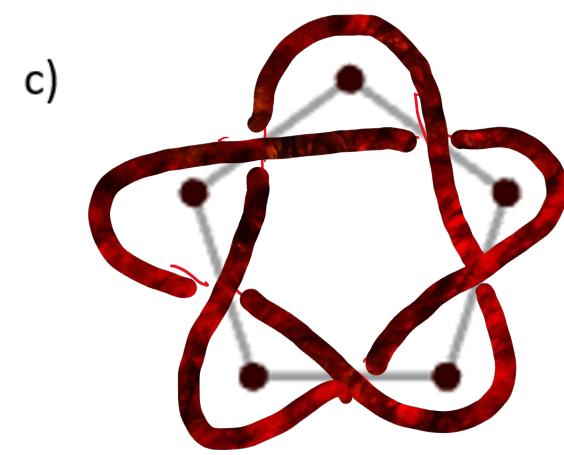
f)

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g)
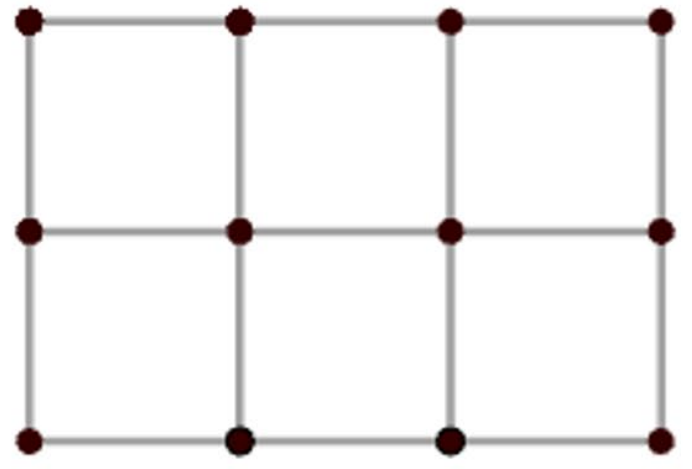
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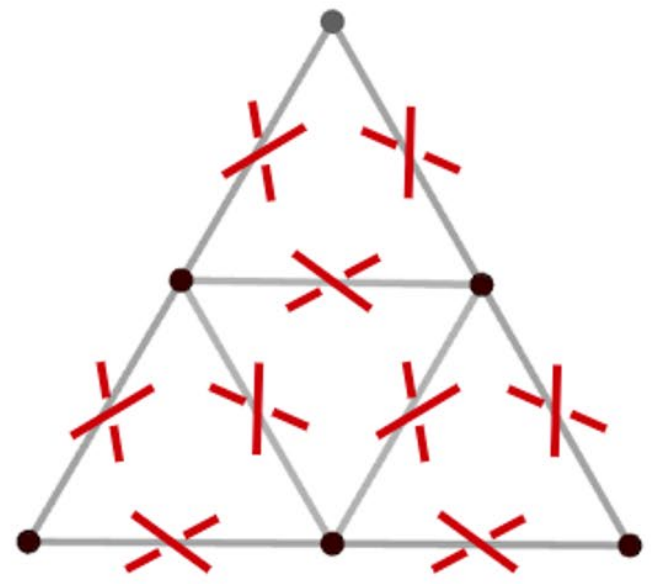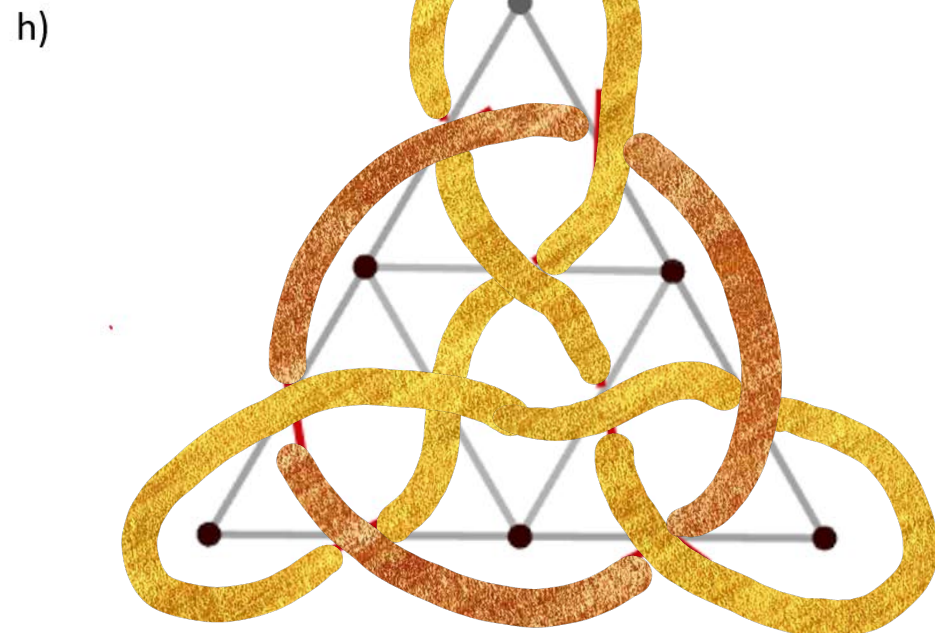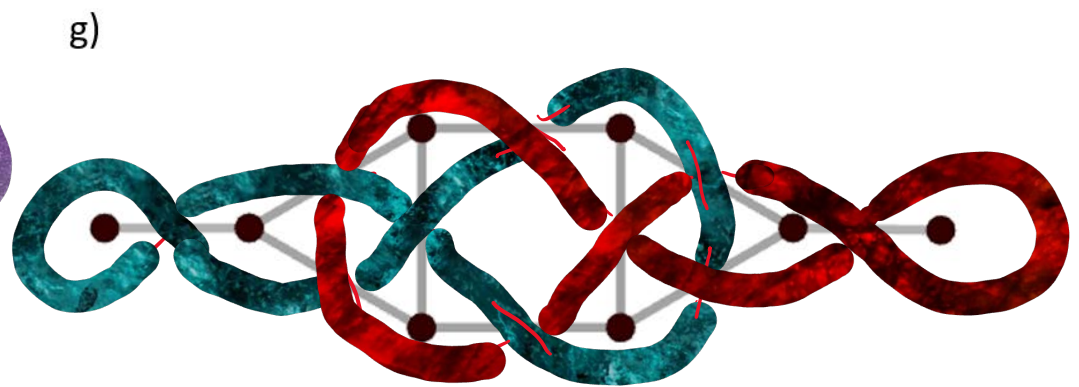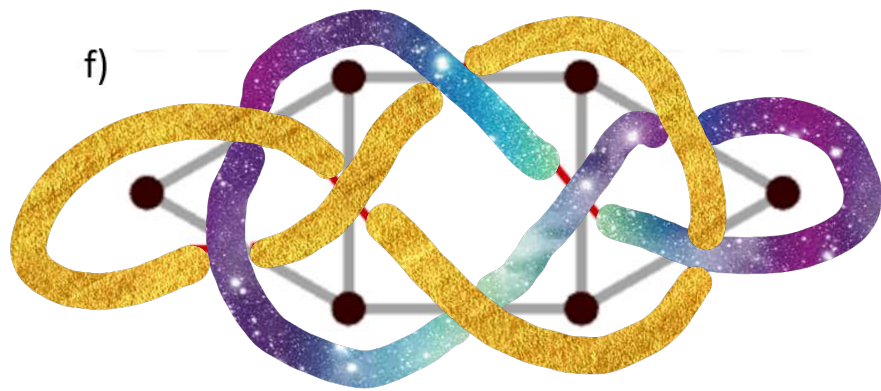
i)
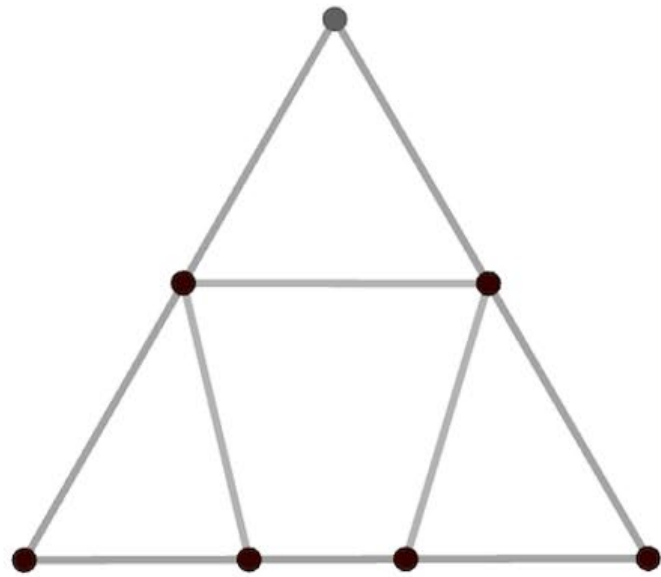
k)

i)

k)

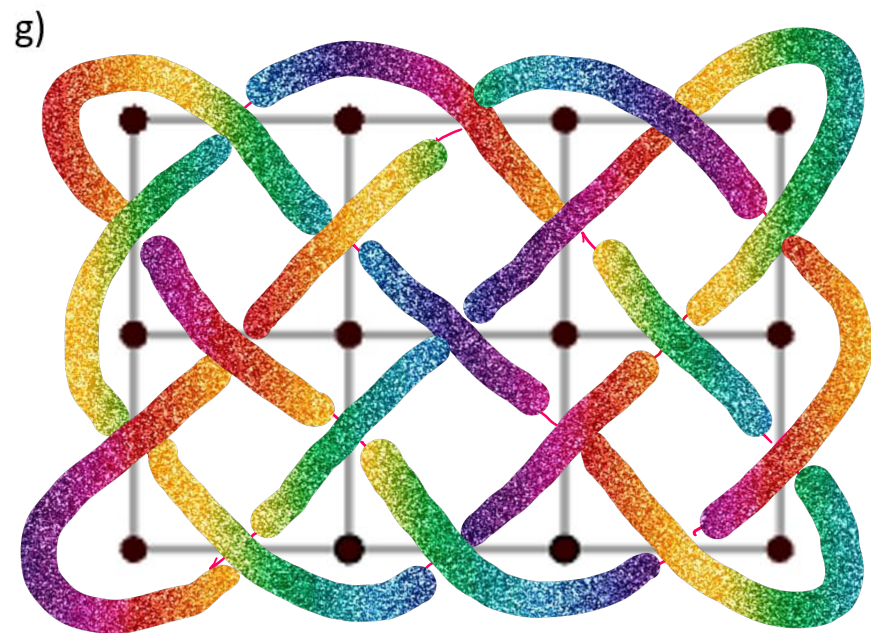# Rope trick!

Can you place a rope in front of you and tie a knot on it by holding its ends and never letting them go?
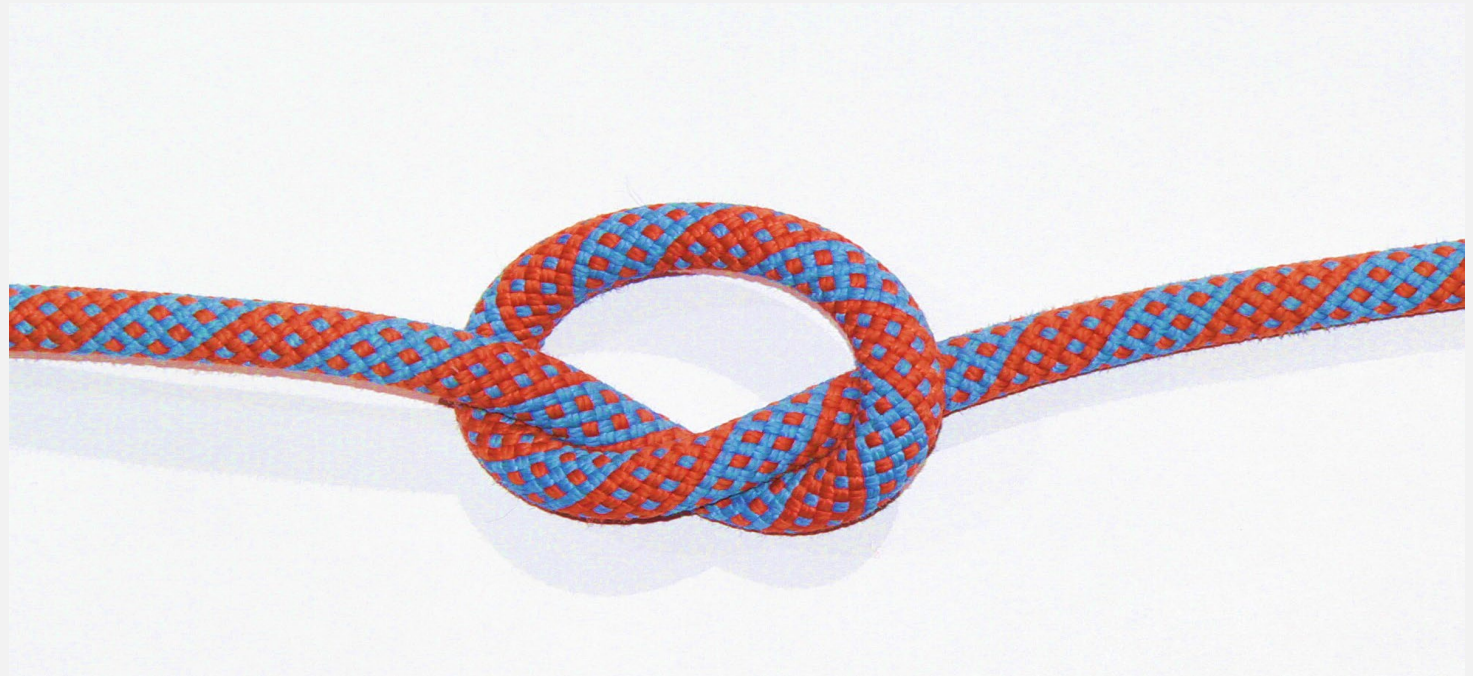
Can you make a trefoil knot from a Mobius strip?
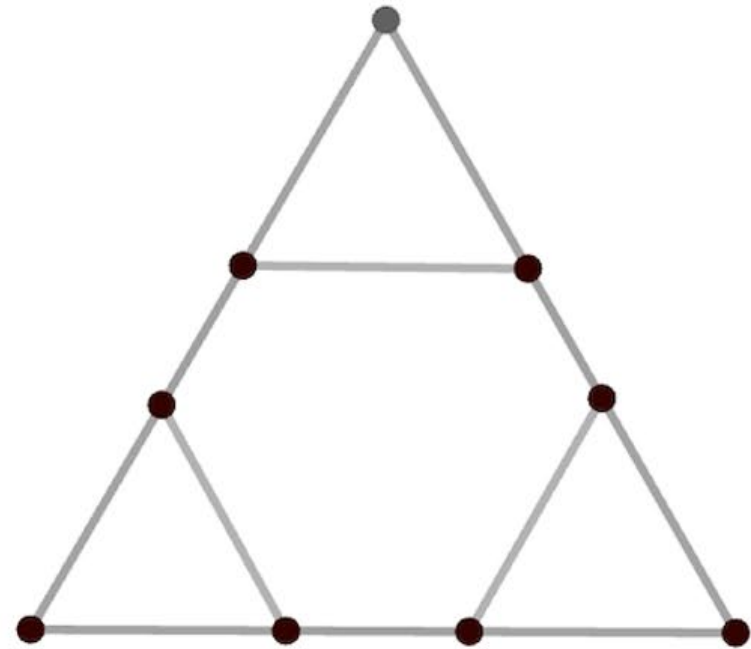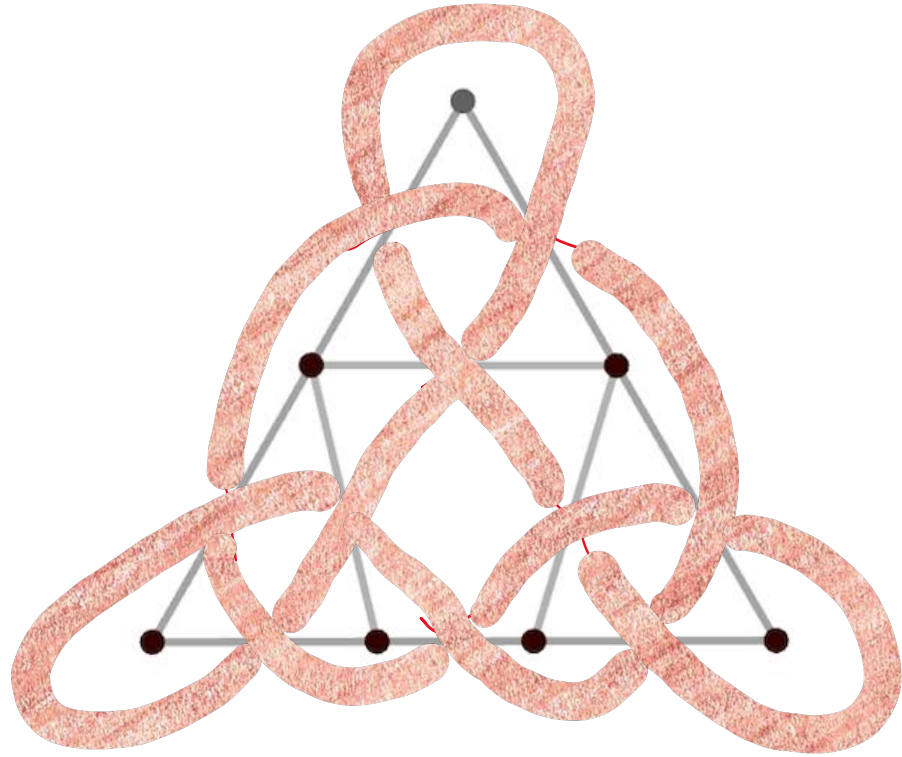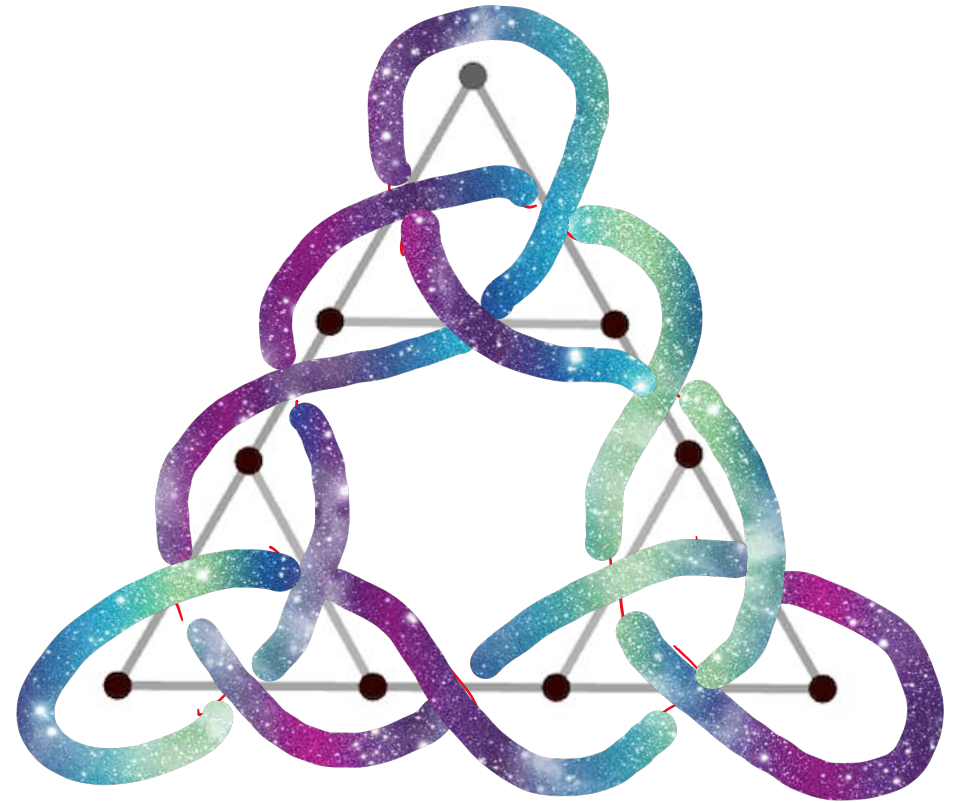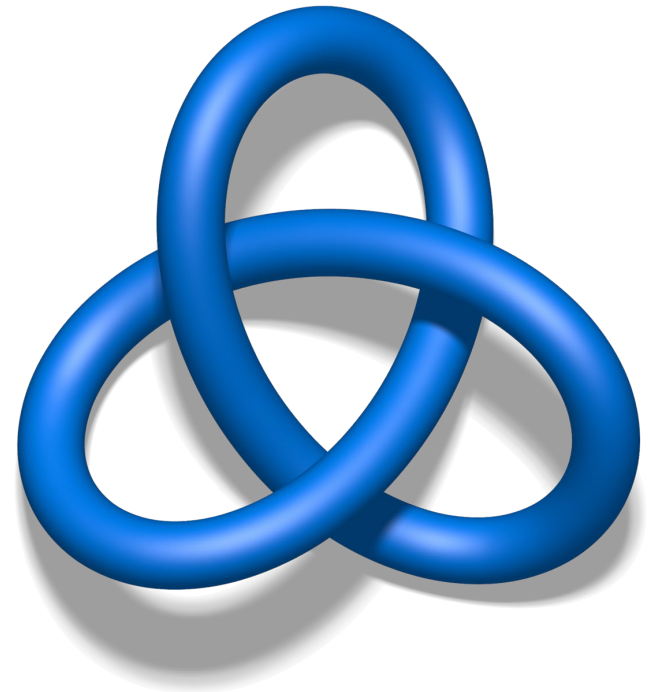
1. Create a special type of Mobius strip by half-twisting the strip of paper three times and glue the ends together.

2. You can check it is a one-sided surface by tracing the line along the center of your strip.

3. Cut the strip along the center line: you will get a trefoil knot!

Images

- " Untie the Knots." Image Source: Квантік Magazine, https://kvantik.com/

- "String and Ring." Image Source: Квантік Magazine, https://kvantik.com/

- "Celtic knot" Celtic Art: https://www.pngegg.com/en/png-nlucc

- "A knot and its Tait graph", Knots, graphs and Khovanov homology I https://www.icts.res.in/sites/default/files/13-12-13_Abhijit_Champanerkar_1_Knots,%20Graphs%20%26%20Khovanov%20Homology-I.pdf

- "Equivalent knots," Knot Theory https://knotplot.com/knot-theory/

- "Direction of a crossing." Celtic Knotwork: the Ultimate Tutorial. Christian Mercat, https://entrelacs.net/

Sources

- Celtic Knotwork: the Ultimate Tutorial. Christian Mercat, https://entrelacs.net/