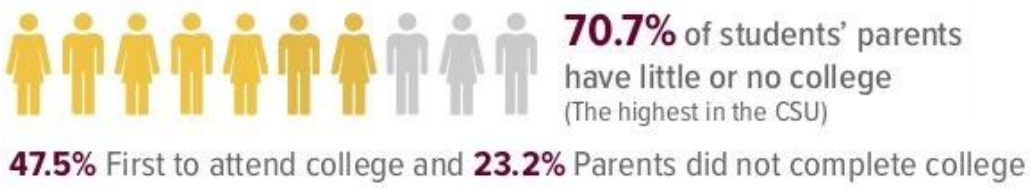
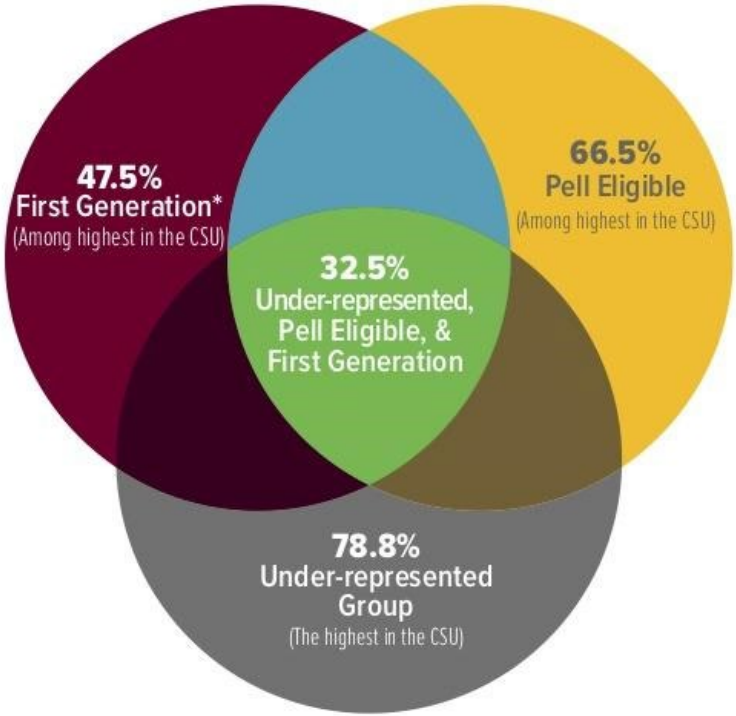
Cauldrons and Hexes

The alchemy of turning math circles into authentic research experiences for undergraduates

About CSUDH

- ~15,000 Undergraduate Students
- Urban Setting (Carson, CA)
- Demographics:
  - Hispanic/Latino 65.5%
  - Black/African American 10.8%
  - Asian 7.6%
  - White 6.2%
- Most students commute





Philosophy & Motivation

- Math research is for everyone!
  - Students who don’t have specific content knowledge or have taken specific classes
  - Students who cannot afford to devote 40 hours/week in a traditional REU
  - Students who don’t see themselves as math researchers
  - Students who don’t know what research in mathematics looks like
- We were looking for problems that
  - All students can access and engage in but that still provide opportunities for meaningful mathematics (low floor, high ceiling)
  - Allow us to give students control over all phases of research
  - Will be of interest to students of varied backgrounds
- Many math circle problems fit our needs!

Circles @ CSUDH

Math Teachers' Circle

Founded in 2016

Meets monthly during the academic year and for a week-long summer institute

Archive of current and former MTC sessions: https://www.csudh.edu/mtc/calendar/

Circles @ CSUDH

HOMES (Hands On Mathematical Experiences for Students)

- Started in Fall 2020
- For undergraduates in any major
- Meets 3-4 times per semester
- Use selected Math circle problems

Summer Research Experience Structure

Three or four students…

- Were paid hourly
- Set their own schedules
- Met regularly with each other and faculty
- Had dedicated office space in the math department

Three faculty members…

- Recruited students with an open interest form and Lunch & Learn events
- Shared responsibilities and covered for each other
- Introduced technological tools for work and collaboration (Google Drive, Co-Calc)
- Were paid occasionally :)

Bubbling Cauldrons

- First used in our Math Teachers' Circle and HOMES
- Sessions were based on a mathcircles.org activity (*thank you, AIM !*): https://mathcircles.org/activity/bubbling-cauldrons/
- Students were given three problems to explore:
  - Bubbling Cauldrons
  - Number Bracelets
  - Hidato
- Students **chose** Bubbling Cauldrons

WELCOME TO PIGWARTS!
PROFESSOR SNIPE'S POTION EXAM

Professor snipe's exam:

- You are to place as many ingredients into the cauldrons without them bubbling over!
- You can place your ingredients (numbers) in either of the two cauldrons.

- You must first start by placing the number 1 into a cauldron, then 2, then 3, and so on… you cannot skip any numbers!
- But if any two numbers in a cauldron add up to another number in their cauldron, **they explode!** The cauldron empties, and you have to start over!

POTION'S EXAM: PROBLEM 1

Following professor snipe's rules, what's the largest number you can place in two cauldrons without them exploding?

POTION'S EXAM: PROBLEM 2

Following professor snipe's rules, what's the largest number you can place in three cauldrons without them exploding?

POTION'S EXAM: PROBLEM 3 & 4

Going back to 2 cauldrons and assuming the same rules, what is the largest number you can place in the cauldrons without them exploding if you start with the number 2? The number 3? 4? 5?

Bubbling Cauldrons - Student Questions

Students explored multiple **variations** on the theme “What is the largest number that can be successfully placed?” by:

- Increasing the number of sets (cauldrons)
- Modifying the “explosion criterion” from sums to product (i.e., product-free sets vs. sum-free sets)
- Placing consecutive terms in some arithmetic progression

Their eventual approach was to study the outcome of specific number-placing strategies and algorithms: e.g., first available, greedy, etc

Bubbling Cauldrons - Student Work

Students…

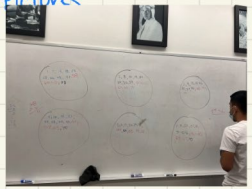
- Did A LOT of **exploration** (including both sum- and product-free sets)
- Needed help choosing and refining a question to explore in more depth
- Used a computer to generate and sort examples (eventually…this took some convincing & help with coding)
- Came up with several conjectures that they were able to prove

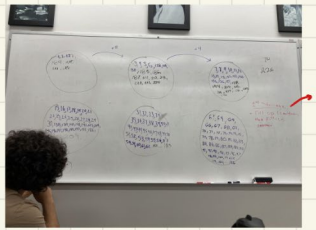
Bubbling Cauldrons - Student Work

Today's Goals

- Continue w/ Friday's strategy for 6 cauldrons, 2^+ rule problem
- Continue w/ the 2^x rule problem to include 40

Pictures





2nd strategy: Fill-up 1 cauldron, then Fill-up another

Today's Recap

What We Did

- We employed strategy #2 for the other cauldrons (2, 3, 4, 5)
- We visited Nataly's work on 2 cauldrons, 2^x
- We used another strategy for 2^x problem

What We Noticed

- The accuracy for strategy #2 on the 6 cauldrons decreased
  - Meaning the strategy may not work for 6 cauldrons
- No patterns were observed for the 2^x problem
  - The highest # we reached was 83
- Math is hard!
- Working backwards is hard!

What WDWTK

- What patterns exist for the 2^x problem?

Looking Ahead

- Revisiting 2^x problem; trying to get higher than 83
- Write proofs for 2^+ problems, starting at 2 cauldrons and ending at 5 cauldrons

Conjecture 1: Range (1, a, b)

- The range going from 1 to a with increments of b where $b \neq 1$, fills up the cauldron infinitely.
- Proof idea: Things in cauldron are congruent to 1 mod b but when added they do not equal 1 mod b.
- Congruent integers a & b are congruent mod b if their difference is a multiple of n

Proof:

Consider the set $C1 = \{1 + nb \mid b \in N, b \neq 1, n >= 0\}$.

WWTS C1 is infinite. Assume c is a finite subset of C1 with $x \in C1$. Since $x \in C1$, then $x \notin c$.

Therefore, C1 is infinite.

https://mathstats.uncg.edu/sites/pauli/112/HTML/section-40.html#:~:text=Let%20B%20be%20a%20set,natural%20numbers%20N%20is%20infinite.

WWTS C1 is sum free. Let n,k be arbitrary $\in N$ such that 1+nb and 1+kb. Then, (1+nb) + (1+kb) = 2 + nb + kb = 2+ b(n+k) = 2 + bc.

Let (n+k) = c and c is $\in N$.

Then $2 + bc \equiv 1 \bmod b \rightarrow 1 + bc \bmod b$, since 1 + bc isn't a multiple of b,

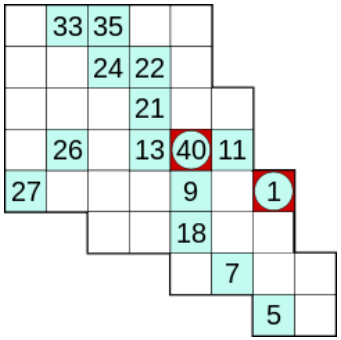
then C1 is sumfree

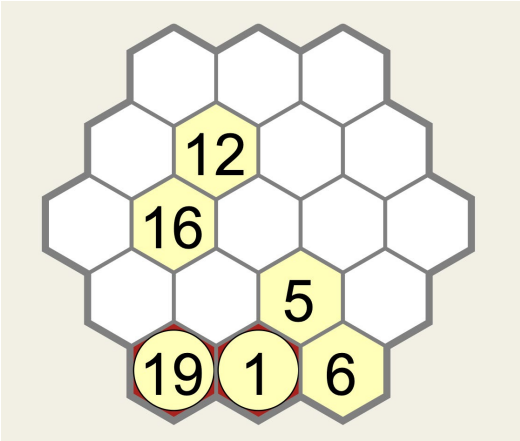
Students presented their work at our campus Student Research Day…and won first prize in their division!

Hidato

- First used in our Math Teachers' Circle and HOMES as part of a session about Number Puzzles (Hidato, Kakuro, Minesweeper)
- Hidato was used for a (successful!) funding proposal
- Students were introduced to the puzzle and given time to solve puzzles and pose questions.

www.hidato.com

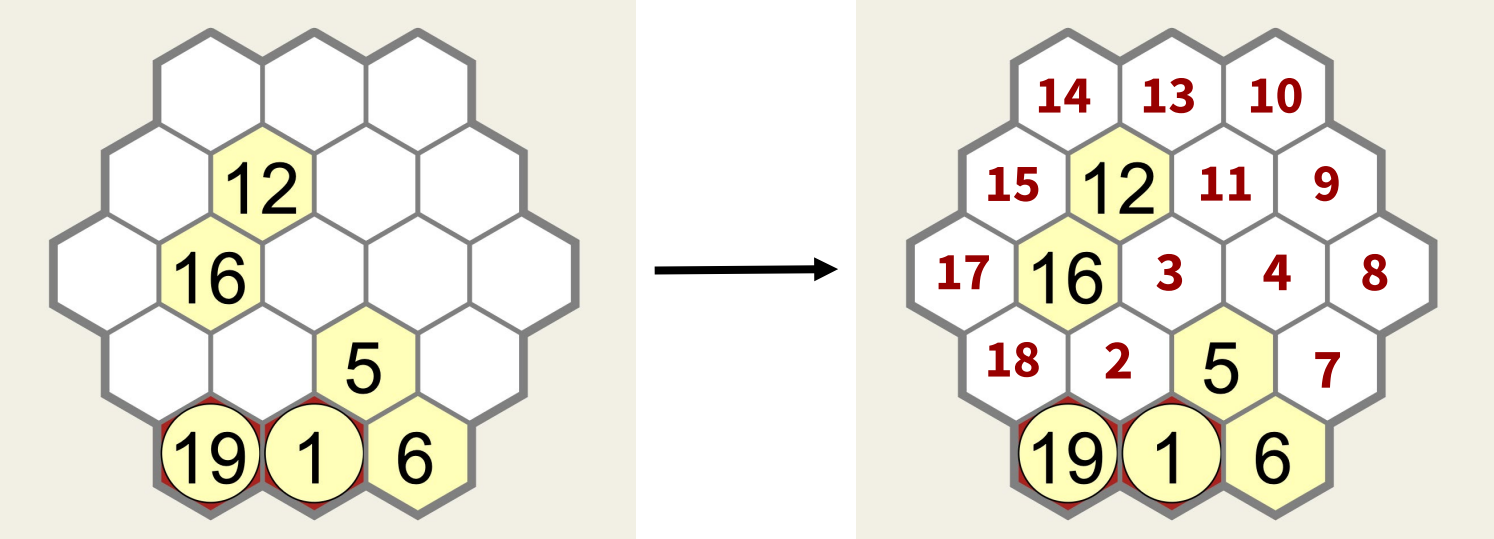




Beehive Hidato

Starts with a partially filled in grid of numbers.

Goal: Create a path from 1 to n so that consecutive numbers are in adjacent hexes.



Beehive Hidato - Student Questions

Students posed questions about **creating puzzles** as well as strategies for solving. For example:

- How many puzzles can be created?
- What are efficient ways to create a puzzle?
- How do you know if your puzzle has a unique solution?
- What kind of computer algorithms could solve a puzzle?

This led to students creating **definitions** and connecting to important mathematical themes:

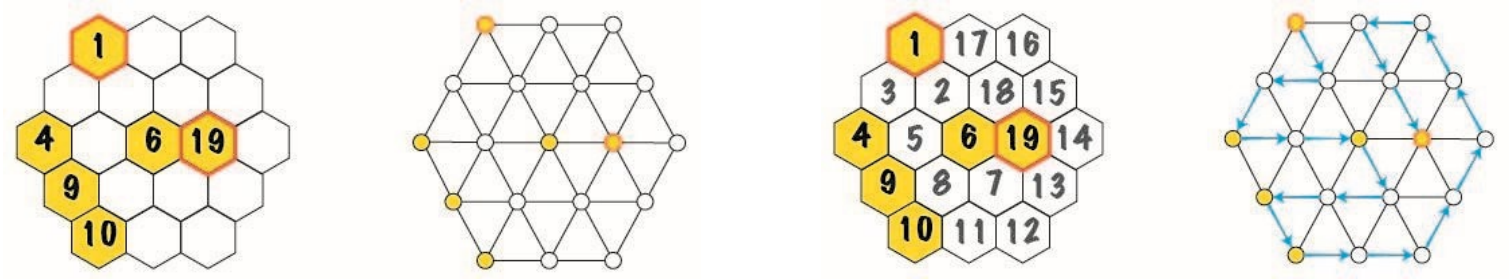
- Valid Puzzles & Solutions
- Minimal Puzzles
- Ideas of Equivalence

Beehive Hidato - Tools

Students were introduced to **graph theory** as a tool:

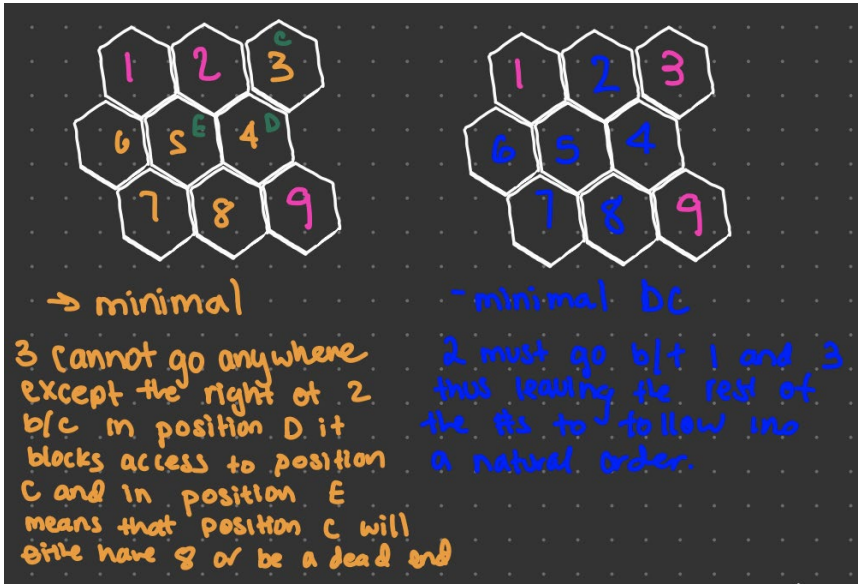
Hexes : Vertices Edges Between Adjacent Hexes

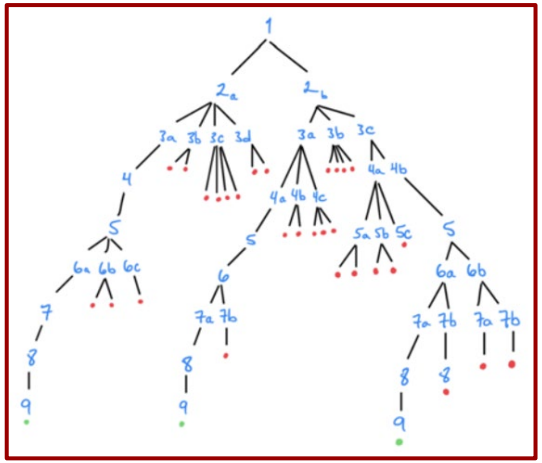
A Solution is a Hamiltonian Path

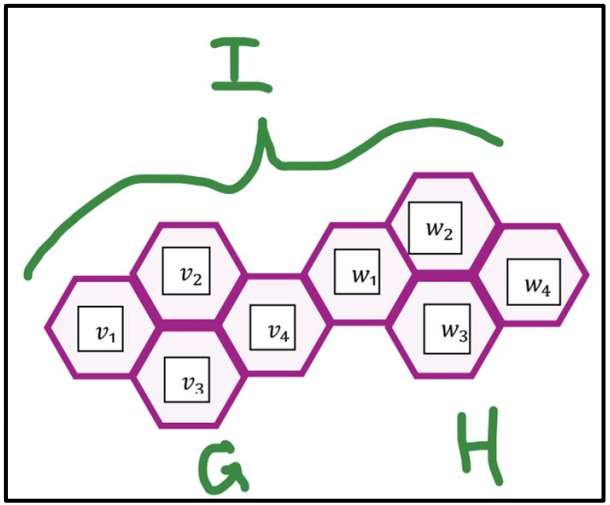


Students were also taught some **programming** techniques to generate examples and aid with enumeration.

Beehive Hidato - Student Work







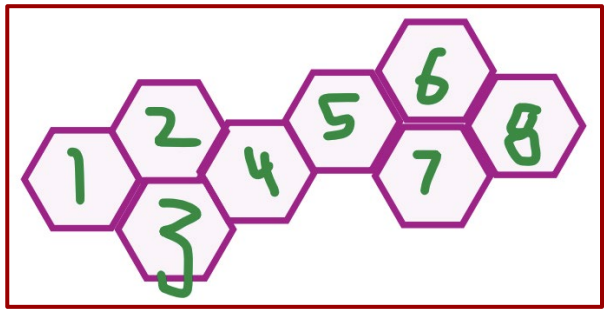
Let the first module be represented by graph G = (V, E), where $V = \{v_1, v_2, v_3, v_4\}$, and

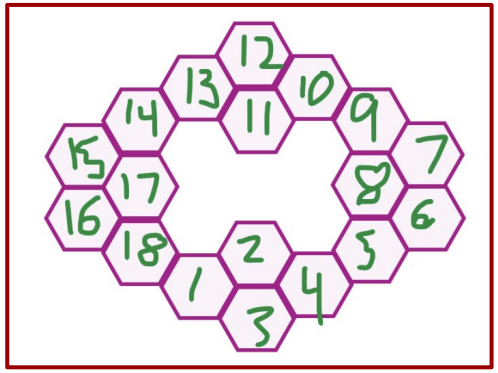
$E = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_3v_4\}$, and the second module be represented by graph H = where $W = \{w_1, w_2, w_3, w_4\}$ and $F = \{w_1w_2, w_1w_3, w_2w_3, w_2w_4, w_3w_4\}$.

Beehive Hidato - Results

Students focused on **enumerating puzzles** but soon learned this was very challenging!

They devised a strategy to enumerate paths in **smaller modules** and stitch these together to create and enumerate larger puzzles.





Student Experience - What They Learned

how to initiate a project from scratch, even when ideas were scattered and uncertain

there is a diverse spectrum within the [mathematical research] field and thus you are not limited to just one type of math

This process has transformed the way I perceive problems and influenced how I articulate my research questions.

the importance of strategic planning and effective communication to consolidate diverse ideas into a unified result

the tangible impact of mathematics beyond theory

Student Experience - What They Liked

the researchers decide which path they want to take with some guidance from the professors overlooking the research

Collaborative nature of mathematical research

All the questions and ideas explored were of my own choosing, enabling me to encounter both failures and successes.

what we as a team were able to achieve in such a short time

It empowered me to chart my own unique path

It provided me a valuable opportunity to comprehend a specific problem, formulate my own questions about it, and immerse myself in the research world surrounding that issue

Takeaways - Challenges & Lessons Learned

- Students needed help nailing down a focused and answerable research question
- Needed to encourage students to use technology → scaffolding & modeling
- Challenges unique to our students: outside responsibilities, scheduling, etc
- Faculty debriefs were important to plan how to support students

Good afternoon professors,

I hope that this email finds you all doing well.

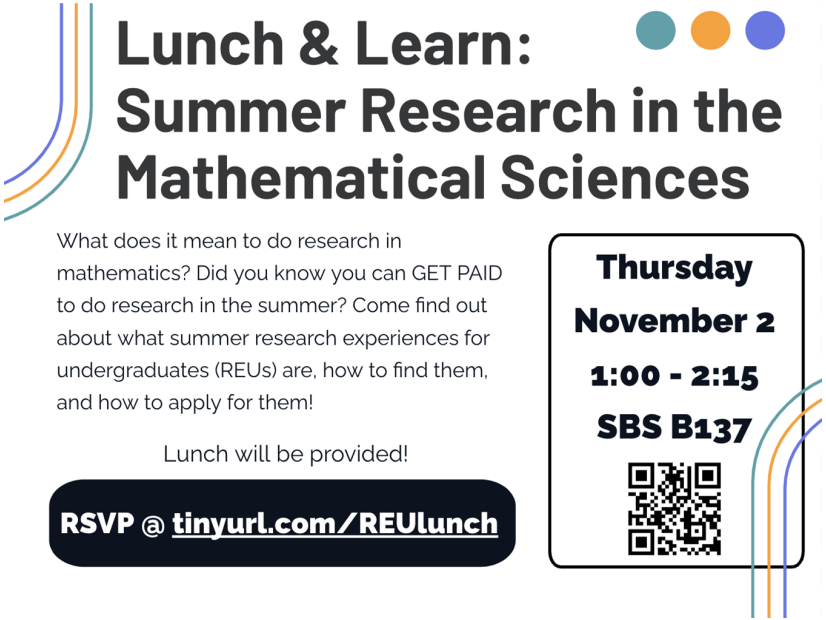
I wanted to ask if we could have a pep talk tomorrow after our meeting.
Our group could use one.

Takeaways - Highlights

- Students seemed to have a good experience (and they also learned things…)!
- At Student Research Day, students were able to articulate their work in all phases of their project (Year 1).
- Several students successfully went on to other REUs or research projects with other faculty and most are applying to graduate school (Year 1).
- Collaborating between multiple faculty made the experience more fun and practical, and made it easier to balance with other obligations (summer teaching, etc).

What's next?

- Continue using Math Circle sessions for undergraduate research!
- More Math Department events to connect students to research



Resources

CSUDH Math Circle Sessions: https://www.csudh.edu/mtc/calendar/

Bubbling Cauldrons & Sum Free Sets:

- https://mathcircles.org/activity/bubbling-cauldrons/
- P. Blanchard, F. Harary, & R. Reis, “Partitions into sum-free sets”. Integers. 6. (2006).
- F. Harary. “Sum-Free Games”, Chapter in: D. Wolfe & T. Rodgers “Puzzlers' tribute: A feast for the mind.” A.K. Peters. (2002)

Hidato:

- https://www.hidato.com/home
- https://archive.nytimes.com/wordplay.blogs.nytimes.com/2015/03/02/benedek/
- M. M. P. Silva, C. S. Magalhães. “A Genetic Algorithm for Solving Beehive Hidato Puzzles”, *ChemBioChem* (2018)

Thank you!

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