Exploring Prime Numbers and The *abc* Conjecture

Joint Mathematics Meetings

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Art of Problem Solving
aops.com

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Prime numbers are fundamental to the universe.

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- A great low-floor high-ceiling activity with lots of side explorations
- In the news! Mathematics is a living subject
- Exploration of primes can give participants a sense of what research mathematicians do

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The New York Times

A Possible Breakthrough in Explaining a Mathematical Riddle

Give this article

By Kenneth Chang

Sept. 17, 2012

3 MIN READ

Numbers, addition, multiplication — the basic stuff of grade-school arithmetic — are suddenly the excited talk of cutting-edge mathematicians.

On Aug. 30, with no fanfare, Shinichi Mochizuki, a mathematician at Kyoto University in Japan, dropped onto the Internet four papers.

The papers, encompassing 500 pages and four years of effort, claim to solve an important problem in number theory known as the abc

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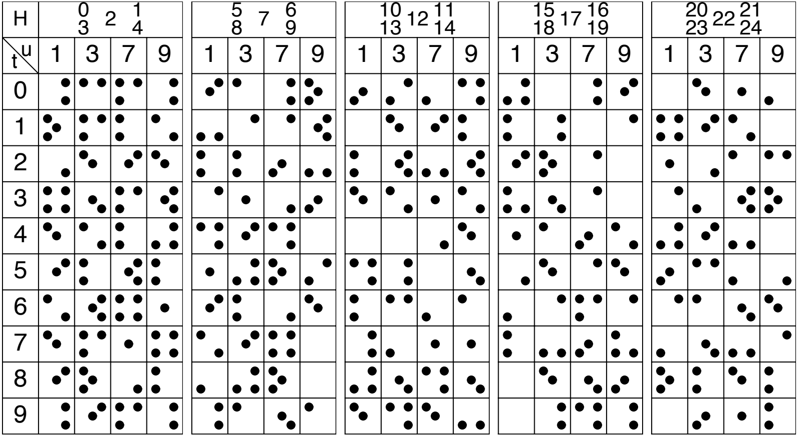
Prime Number Chart

Prime Numbers to 2500 (+2,5)



Paul Zeitz, MTC Network

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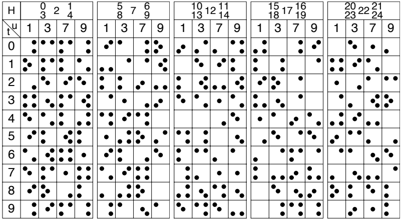

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Explore the Chart

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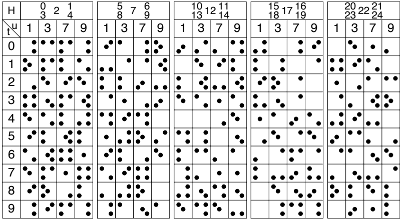

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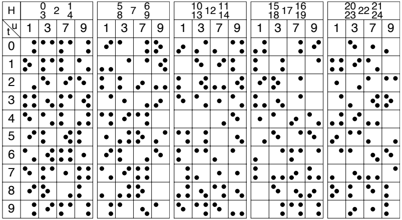

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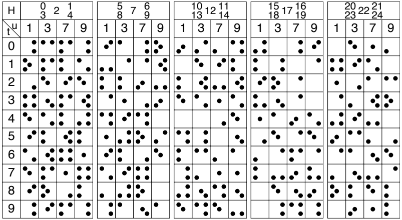

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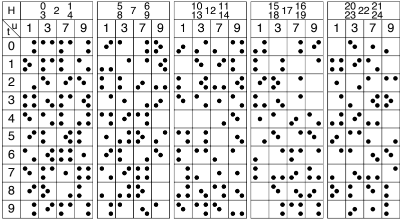

- How do we read it?
- What patterns do you notice?
  Can you prove them?
- The four units columns 1,3,7,9 have roughly an equal number of dots? Is this a coincidence or a general pattern?
- As the numbers get larger, there appear to be fewer dots. Is this a coincidence or a general pattern?

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- Probability that $\gcd(m, n) = 1$ for randomly chosen m and n
  - Can think about this experimentally (i.e. compute for $m, n \leq 10$)
  - Can "prove" that this equals $\frac{6}{\pi^2}$ (you probably have to cheat somewhat)
  - Connection to the Riemann Hypothesis

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abc Conjecture

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This still seems profoundly unexciting.

abc Conjecture

Definition

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$\text{rad}(abc) = 2 \cdot 5 = 10$

Notice that in all these examples, $\text{rad}(abc) > c$.

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The **radical** of a number n, denoted rad(n), is the product of all the prime factors of n.

So here's the game:

Are there solutions in relatively prime positive integers to

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for which $\text{rad}(abc) \leq c$?

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Only $1 + 1 = 2$. (There are no primes left for a or b if $c = \text{rad}(abc)$.)

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Are there solutions in relatively prime positive integers to

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$1 + 8 = 9$ is the smallest
$\text{rad}(72) = 2 \cdot 3 = 6$.

$5 + 27 = 32$ is the smallest with all numbers greater than 1
$\text{rad}(5 \cdot 27 \cdot 32) = 5 \cdot 3 \cdot 2 = 30$.

Are there solutions in relatively prime positive integers to

$$a + b = c$$

for which $\text{rad}(abc) < c$?

Four others with $c < 100$:
$1 + 48 = 49$ (radical is 42)
$1 + 63 = 64$ (radical is 42)
$1 + 80 = 81$ (radical is 30)

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Are there infinitely many solutions?

Computer search has found over 23 million solutions!

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So $\text{rad}(abc) \leq \frac{2b}{3} < b < c$.
Examples:
$1 + 63 = 64$, $\text{rad}(63 \cdot 64) = 42$
$1 + 4095 = 4096$, $\text{rad}(4095 \cdot 4096) = 3 \cdot 5 \cdot 7 \cdot 13 \cdot 2 = 2730$

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abc Conjecture

So we modify the question a little. . .

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Are there finitely many solutions in relatively prime positive integers to

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abc Conjecture

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The value q such that $c = \text{rad}(abc)^q$ is called the **quality** of the triple (a, b, c).
So the question is: if we fix a baseline $Q > 1$ for quality, are there finitely many solutions that have a higher quality (that is, that have $q > Q$)?

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. . . but it's unknown whether this is true or not!

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Are there finitely many solutions in relatively prime positive integers to

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Highest known quality:

$$2 + 6436341 = 6436343$$

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$$15042^{1.62991168...} = 6436343$$

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There are 239 known triples with quality $q \geq 1.4$. The largest one is:

$$2^{37}3^{12}9109^3 + 5^{13}13^{15}2939^1 = 7^{23}11^1793345871^1.$$

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The right-hand side of this

$$238841709663649705652770167283,$$

a 30-digit number.

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9/17/12 New York Times

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The mathematical community is divided as to whether the proof is correct or not.

Thanks!

A version of this activity (without the *abc* conjecture) is on the Math Circles website at:

https://mathcircles.org/activity/primes/

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aops.com
beastacademy.com