Games with Special Moves

Katherine Nogin
Michelle Nogin

Clovis North High School
Fresno, California

Maria Nogin

California State University
Fresno, California

Joint Mathematics Meetings
January 4, 2024

Outline

- Fresno Math Circle and Fresno State Math Field Day
- Examples of games and winning strategies
- What are special moves?
- Questions and some answers
- More questions to investigate

- 4 groups of grades 3–4, 5–6, 7–8, 9–12
- mental math, problems on various topics, hands-on activities, puzzles, 2-person strategy games, problem solving contests

Fresno State Math Field Day

- grades 6–8, 9–10, 11–12
- 3 types of contests:
  - individual, fast problem solving (2 minutes per problem)
  - teams of 2 people, 2 hours for 20 problems
  - game tournament, 2-person games
    - a list of basic games and some variations are posted on the web, but the exact game and variation are not known in advance
    - all games are pure strategy, no element of luck
    - participants are paired randomly, each pair plays twice

Examples of games: a pile of counters

(MC, grades 5-6)

- Initially, there is a pile containing 24 counters.
- Turns alternate. In each move, the player can remove 1, 2, 3, or 4 counters from the pile.
- The game ends when the pile is empty.
- The player who made the last move wins.

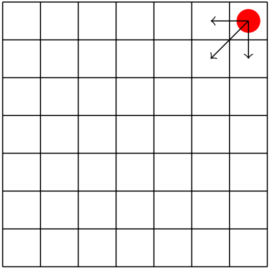
(Math Field Day)

- Choose a random number between 12 and 30 and make a pile of that many counters.
- Choose a random number N between 3 and 6. In each move, the player can remove 1, 2, ..., or N counters from the pile.
- The game ends when the pile is empty.
- The player who made the last move wins.

Examples of games: counter on a board

(MC, grades 7-8)

- A counter is placed in the upper right corner of a 7×7 board.
- Turns alternate.
- In each turn, the player can move the counter one square to the left, one square down, or one space left/down diagonally.
- The game ends when the counter is in the bottom left corner.
- The player who made the last move wins.



Variations

- The person who made the last move loses (instead of wins)
- Modified moves—can remove a different number of counters or move counters on the board in a different pattern
- Some restrictions apply, e.g. may not land on a certain position, counters are not allowed to share a square on the board or are not allowed to "jump" one over another
- Once per game, each player may make "a special move" e.g.
  - may remove a number of counters not usually permitted
  - may move a counter on the board in a way not usually permitted
  - some restrictions may apply, e.g. two special moves cannot be consecutive

Finding winning positions: a pile of counters

- Initially, there is a pile containing 24 counters.
- Turns alternate. In each move, the player can remove 1, 2, 3, or 4 counters from the pile.
- The game ends when the pile is empty.
- The player who made the last move wins.

Winning position = one we want to go to on our move

Losing position = one we do not want to go to on our move

Work backwards to classify all positions (the number of counters left in the pile):

0	1	2	3	4	5	6	7	8	9	10
W	L	L	L	L	W	L	L	L	L	W

Finding winning positions: a pile of counters

- Initially, there is a pile containing 24 counters.
- Turns alternate. In each move, the player can remove 1, 2, 3, or 4 counters from the pile.
- The game ends when the pile is empty.
- The player who made the last move ~~wins.~~ loses.

Winning position = one we want to go to on our move

Losing position = one we do not want to go to on our move

Work backwards to classify all positions (the number of counters left in the pile):

0	1	2	3	4	5	6	7	8	9	10
W	L	L	L	L	W	L	L	L	L	W
L	W	L	L	L	L	W	L	L	L	L

Finding winning positions: counter on board

- A counter is placed in the upper right corner of a 7×7 board.
- Turns alternate.
- In each turn, the player can move the counter one square to the left, one square down, or one space left/down diagonally.
- The game ends when the counter is in the bottom left corner.
- The player who made the last move wins.

W	L	W	L	W	L	
L	L	L	L	L	L	L
W	L	W	L	W	L	W
L	L	L	L	L	L	L
W	L	W	L	W	L	W
L	L	L	L	L	L	L
W	L	W	L	W	L	W

Finding winning positions: counter on board

- A counter is placed in the upper right corner of a 7×7 board.
- Turns alternate.
- In each turn, the player can move the counter one square to the left, one square down, or one space left/down diagonally.
- The game ends when the counter is in the bottom left corner.
- The player who made the last move ~~wins.~~ loses.

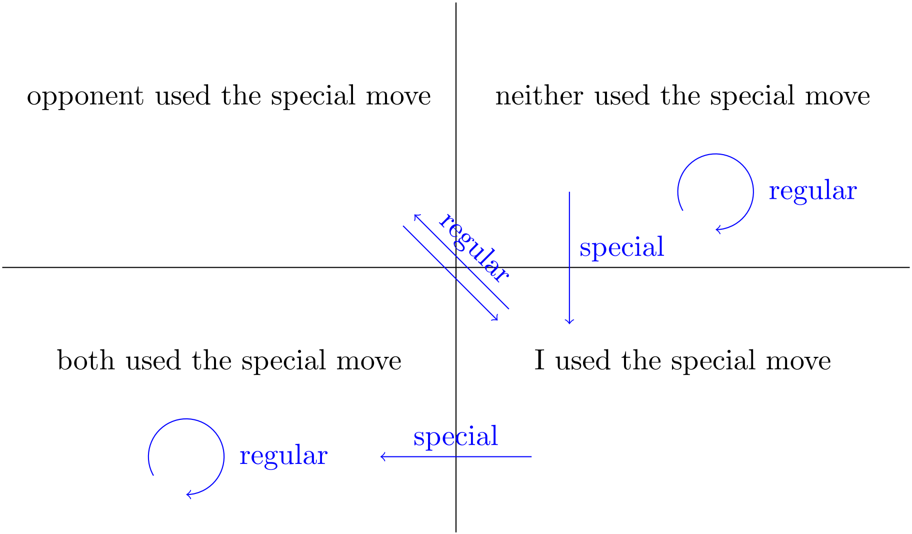
W	L	W	L	W	L	
L	L	L	L	L	L	L
W	L	W	L	W	L	W
L	L	L	L	L	L	L
W	L	W	L	W	L	W
L	L	L	L	L	L	L
W	L	W	L	W	L	W

L	L	W	L	W	L	
W	L	L	L	L	L	L
L	L	W	L	W	L	W
W	L	L	L	L	L	L
L	L	W	L	W	L	W
W	L	L	L	L	L	L
L	W	L	W	L	W	L

Questions

- How does a special move, allowed at most once per game for each player, affect the distribution of winning positions for each player?
  - Is it a "shift"?
  - Is the distribution drastically different?
- Can the percentage of initial positions for which the 1st player has a winning strategy change?
  - Can it increase?
  - Can it decrease?

Finding winning/losing positions for a game with a special move



Game investigated

Let $m < n$ and p be natural numbers.

- Initially, a pile contains some number of counters.
- Turns alternate. In each move, the player removes m, $m + 1$, $\ldots$, or n counters from the pile.
- At most once per game, each player may remove p counters.
- The game ends when the pile is empty.
- The player who made the last move wins.

Results

- For the original game (with no special move), positions 0–$(m-1)$ are winning, positions m–$(m+n-1)$ are losing, then this pattern repeats in cycles of length $m+n$.
- For $p=0$ or $m \leq p \leq n$, the game is equivalent to the original one.
- For any p, the winning/losing positions still form cycles of length $m+n$ eventually.
- For $1 \leq p < \frac{m}{2}$, the pattern is different, with fewer winning positions, so the percentage of times the first player has a winning strategy is higher.
- For $\frac{m}{2} \leq p < m$, the original pattern is shifted by $p-m$.
- For $n < p \leq m+n$, the original pattern is shifted by $p-n$.
- For $p > m+n$, let $r = p \mod (m+n)$. The pattern is shifted by m if $r=0$, shifted by $p-m$ if $0 < r < m$, shifted by $p-n$ if $r > n$, and the same as in the original game if $m \leq r \leq n$ eventually.

Example: $m = 5$, $n = 8$

Winning/losing positions for the case when neither player has used their special move:

p	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26
none, 0	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	✓
1	✓	×	✓	✓	✓	×	×	×	×	×	×	×	×	✓	×	✓	✓	✓	×	×	×	×	×	×	×	×	✓
2	✓	✓	×	×	✓	×	×	×	×	×	×	×	×	✓	✓	×	×	✓	×	×	×	×	×	×	×	×	✓
3	✓	✓	✓	×	×	×	×	×	×	×	×	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	✓	✓	✓
4	✓	✓	✓	✓	×	×	×	×	×	×	×	×	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	✓	✓
5–8	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	✓
9	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	×	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×
10	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	×	×	✓	✓	✓	✓	✓	×	×	×	×	×	×	×
11	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	×	×	×	✓	✓	✓	✓	✓	×	×	×	×	×	×
12	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	×	×	×	×	✓	✓	✓	✓	✓	×	×	×	×	×
13	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	×	×	×	×	×	✓	✓	✓	✓	✓	×	×	×	×
14	✓	✓	✓	✓	✓	×	×	×	×	×	×	×	×	✓	×	×	×	×	×	×	×	×	✓	✓	✓	✓	✓

Further questions

- What happens if we add the restriction that two special moves cannot be consecutive?
- What happens if each player can make up to two (or more) special moves per game?
  - Does the percentage of initial positions for which the first player has a winning strategy still either stay the same or increase?
- How do restrictions (e.g. cannot land in a certain position) affect the distribution/percentage of winning positions?
- How do special moves, restrictions, or other rules affect the distribution/percentage of winning positions for other games (e.g. the ones with counters on a board)?
  - Is there a game for which a special move decreases the percentage of initial positions for which the first player has a winning strategy?

Thank you!