Hyperbolic Soccer Ball

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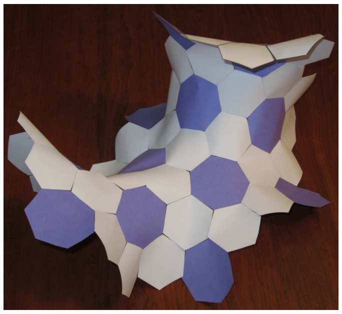
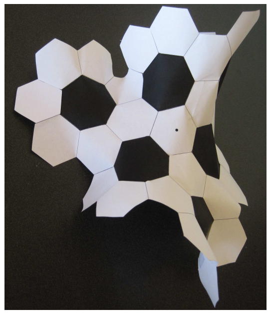
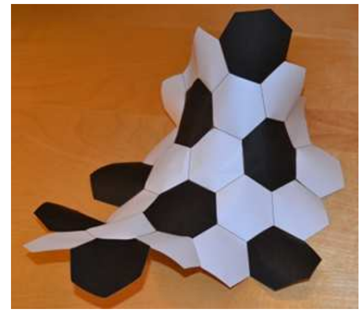
Constructing and studying the hyperbolic soccer ball is an enjoyable activity resulting in a beautiful model which can be used to investigate the hyperbolic plane and non-Euclidean geometry.

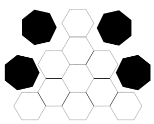
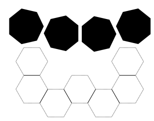





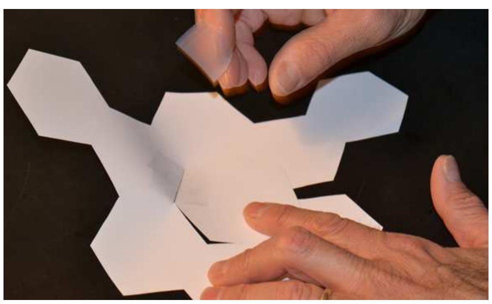
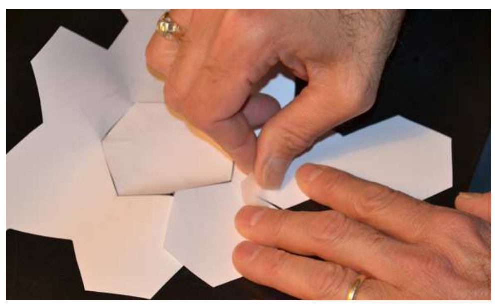
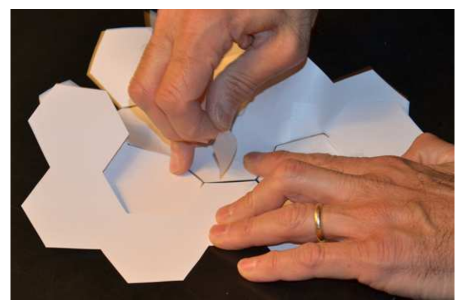
Constructing the Hyperbolic Soccer Ball.

These instructions construct the rightmost model above.

Materials:

- Printed templates. One:  One or two: 
- Scissors and Tape.

Directions: Carefully cut out *all* polygons along the dotted lines. Save the central hexagon in the first template, cutting a path to it. Tape a black heptagon edge-to-edge to the six hexagons in the hole, taping on the unprinted side. Place pieces of the tape across (*not* along) the edges, using two pieces for each edge, and taking care to *not* put tape over a vertex. When taping a second edge at a vertex, the model must be opened a bit (below left), and it will not lie flat. Tape the central hexagon into the gap which is left (below center). Tape two black heptagons

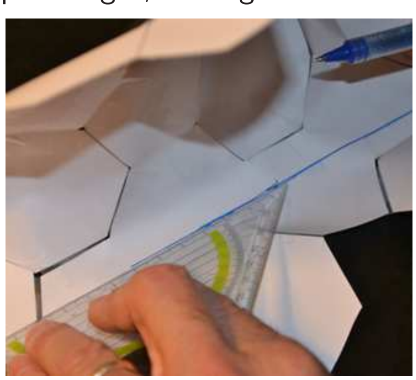
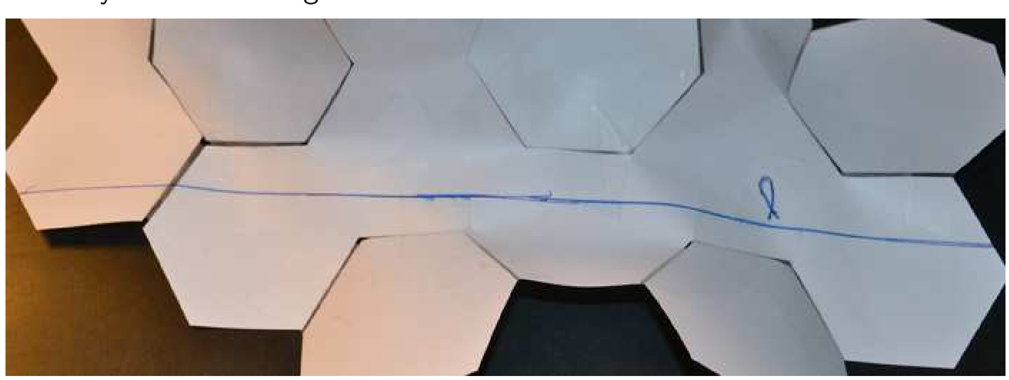




into the indentations along a side and then the W-shaped strip of hexagons from the second template to them (above right). This gives three heptagons surrounding a central hexagon, surrounded by 15 hexagons with one more breaking the symmetry. Tape the remaining five heptagons to complete this model (top right).

Investigate Non-Euclidean Geometry.

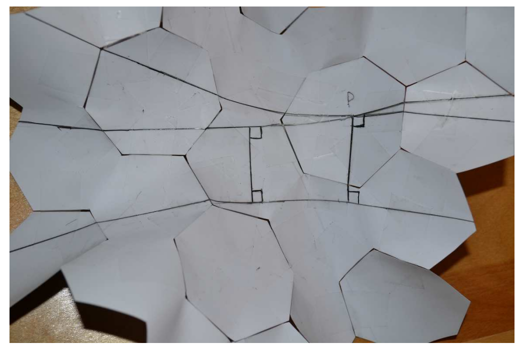
Materials: Pencil and a $3" \times 5"$ index card that will function as a straightedge.

Draw a line: With unprinted side up, flatten two adjacent polygons and draw a line segment. Extend this segment across model with straightedge (card) to a line ℓ. Grasp the model with both hands (index fingers on the lines), pull taught, and sight down the line to verify that it is straight.

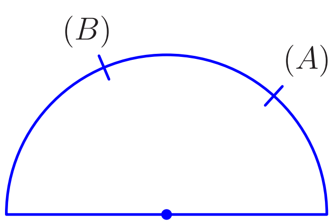



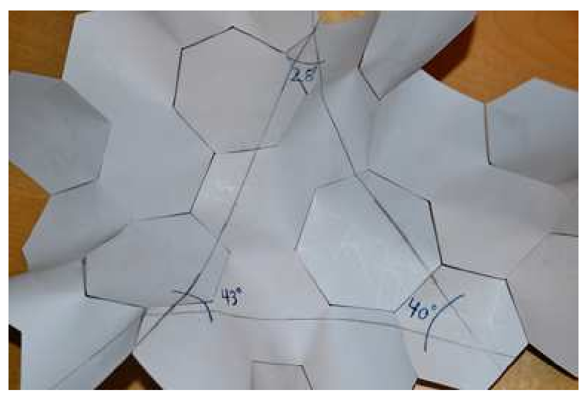
Diverging parallels. Draw a segment m perpendicular to ℓ, and then a line ℓ' perpendicular to m. The lines ℓ and ℓ' are parallel (lines perpendicular to the same line are parallel). Notice how they diverge.

Parallel Postulate. Given parallel lines, extend a new perpendicular segment from one (ℓ) meeting the second at point P. The line perpendicular to the segment at P is parallel to ℓ, giving two parallels to ℓ through $P(\notin \ell)$. Thus Euclid's parallel postulate, or rather the equivalent Playfair's axiom (*At most one line can be drawn through any point not on a given line parallel to the given line in a plane*) does not hold. (This also constructs a Lambert quadrilateral (*three right angles*), but that is another story.) Compare your figure to that of your friends.



Triangle. Draw a large triangle on your model. What are its angles? You can measure their sum as follows: Cut a small semicircle from your card, marking a centre on the straight side. Put the centre at the vertex of an angle, align one ray along the edge, and mark the other ray (A) on the arc. Put the center at the vertex of a second angle, align one ray with the mark (A), mark the other ray (B) on the arc. Repeat for the third angle to get the angle-sum. Compare your angle-sum to that of others and try to formulate a conjecture for this angle-sum.

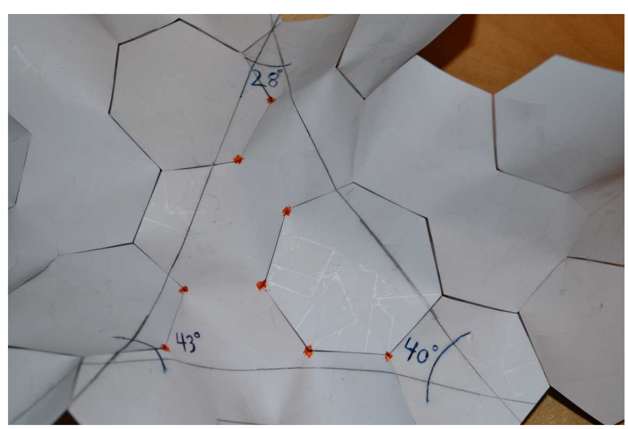




Curvature. The model is flat, except at its vertices, each of which contributes $-60/7^\circ \approx -8.5^\circ$ of curvature in a δ-function. (This is the *excess* at the vertex, the difference of 360° and the sum of the incident angles.) This gives the deviation from Euclidean geometry. For your triangle check that

$$\text{angle sum} + 8.5^\circ \cdot \#\ \text{vertices inside triangle} \ \approx\ 180^\circ\,.$$

For the triangle at right, we have $28+43+40+8\cdot 8.5 = 179.0$. Check this relation, or better compare your angle-sum to that of other students. Do this also with your Lambert quadrilaterals. This makes a very fascinating guided discovery activity.



For more information, see `www.math.tamu.edu/~sottile/research/stories/hyperbolic_football/` or google "sottile hyperbolic football". I highly recommend making a very big model.