Symmetry, Surfaces, and Knots: Geometry Activities for Middle School Students in Math Circles

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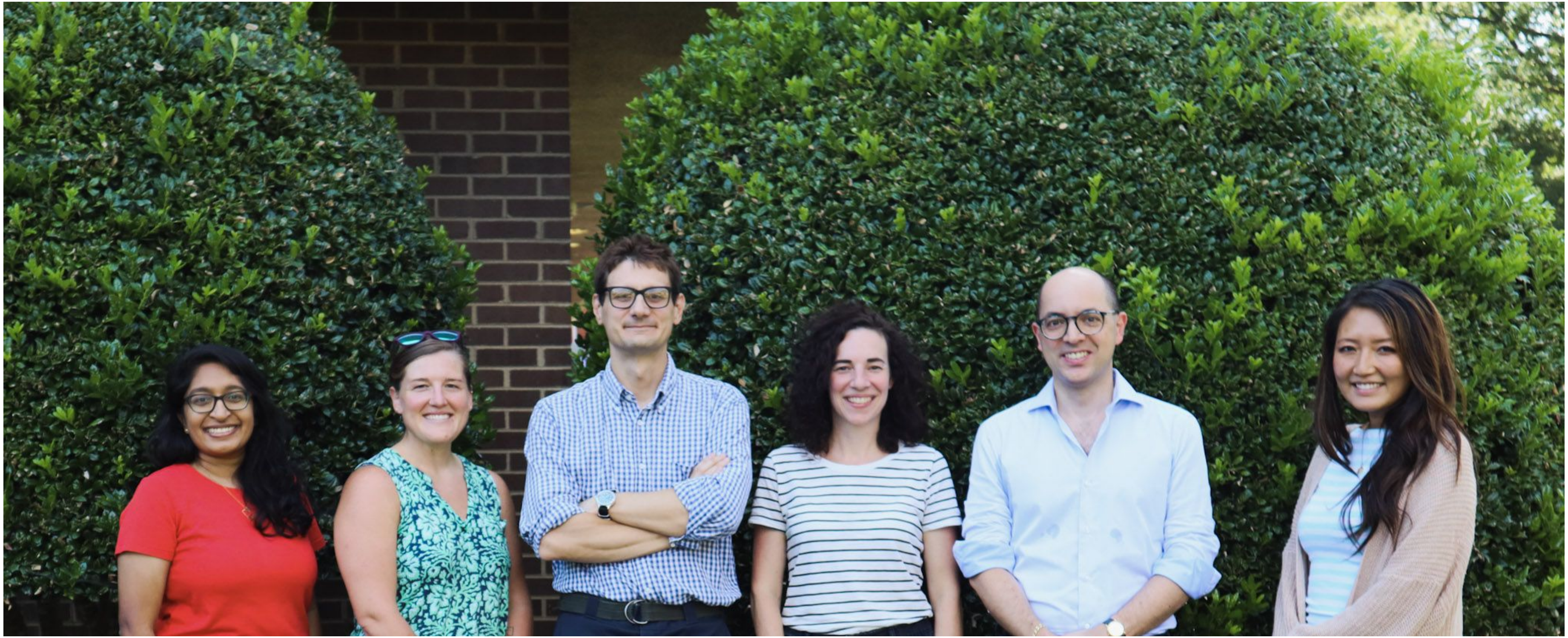
About Us: Team of Mathematics & Education Researchers



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Context: Geometry Summer Camp

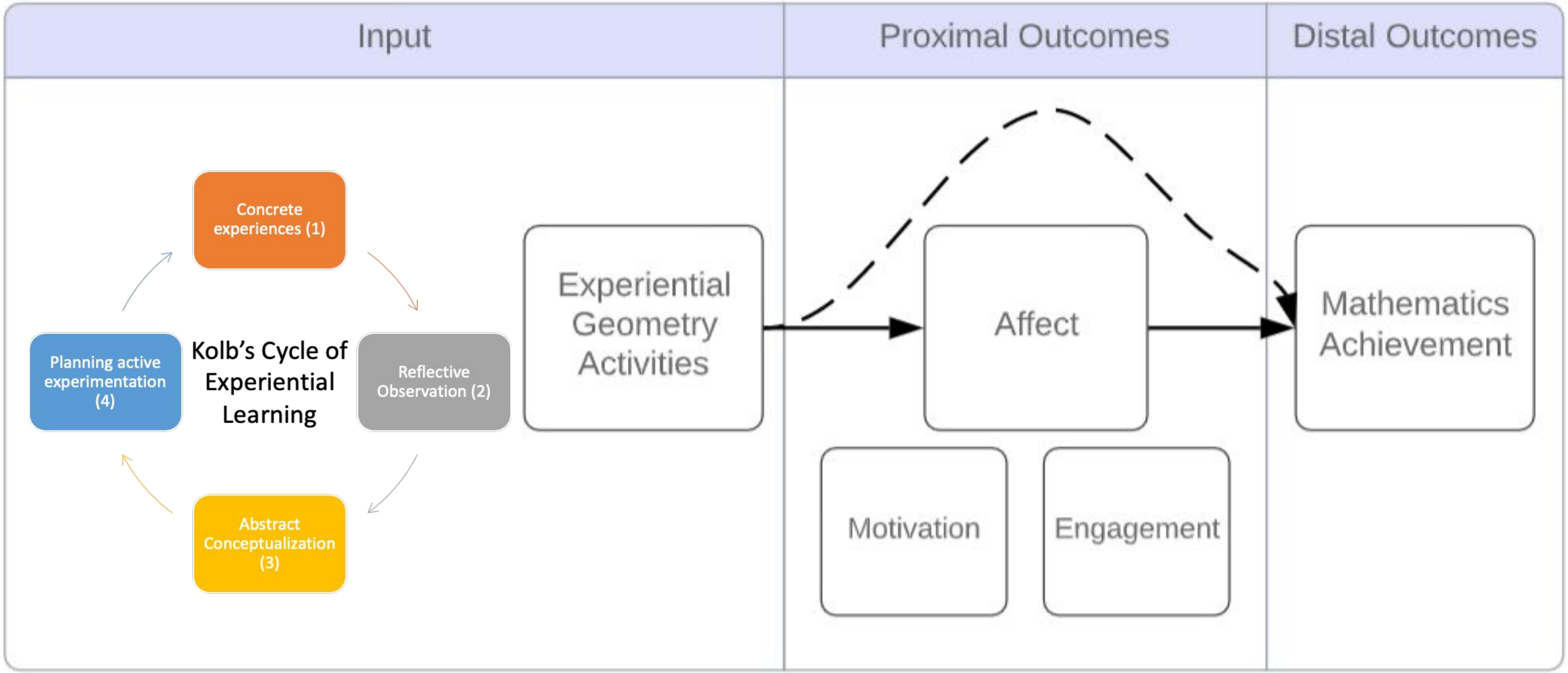
Project: Design and run a one week-long geometry summer camp

Updated geometry content, aimed to increase students' engagement in mathematics: symmetry, curvature, knotting and linking, manifolds, orientability, etc.

Target age group: Middle school students

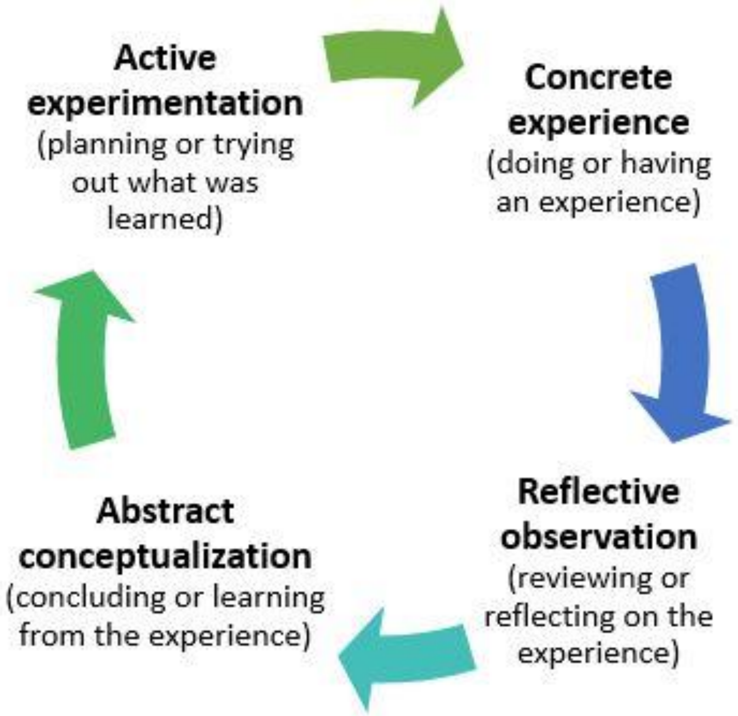
Location: Richmond, VA

Funded by an internal grant, VCU Breakthroughs



Input
Proximal Outcomes
Distal Outcomes
Concrete experiences (1)
Planning active experimentation (4)
Kolb's Cycle of Experiential Learning
Reflective Observation (2)
Abstract Conceptualization (3)
Experiential Geometry Activities
Affect
Motivation
Engagement
Mathematics Achievement

Kolb's (2014) experiential learning cycle



Piloting Activities through VCU Math Circle

Weekend meetings: 1.5 hours long

4-6 meetings over a semester, each led by a different mathematician

Recruitment & attendance is variable, ranges from 4-15

Teachers providing their students extra credit and word of mouth has been effective

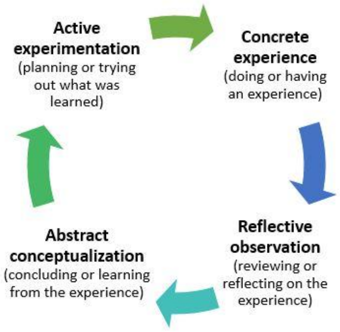
We pilot tested two of our lessons at VCU Math Circle

Activity: Knotty Knots

We'll first share one of our activities about knot theory, stepping through its experiential learning phases

Lesson Plan

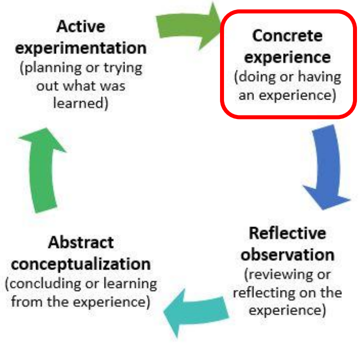
Components:



Lesson: November Pilot – Knotty Knots		**Time:** ~75 mins of activities, anticipate breaks, things taking longer, etc.
Learning Goal(s): 1. Mathematical objects can be dynamic 2. Basic concepts of knot theory: knots, diagrams, invariants, Reidemeister moves 3. Different topological knot types can be distinguished with tricoloring and with experiment		
Lesson Stage/Activity & Steps	**Guiding Questions/Goals**	**Needed Supplies**
Intro Activity **(~10 mins)** Invent and draw some knots	1. What is a knot? 2. What is a knot diagram?	• Worksheet • Blank Paper • Erasers • Pencils • Knot necklaces
Mathematical Knot Theory **(~50 mins)** 1. Equivalence via isotopy as realized by Reidemeister moves a. Which knots are equivalent? b. Planar isotopy activity c. Reidemeister move exercises d. Mirror knot exercises 2. Show that knots are inequivalent using knot invariants	1. What makes two knots equivalent? 2. How to determine two knots are different? 3. What is a knot invariant? Other concepts: - Regions of a knot diagram - Alternating diagrams - Mirroring	• Worksheet • Paper • Erasers • Pencils • Highlighters • Markers • Knot necklaces

Intro Activity: Concrete Experience

Silkies necklaces



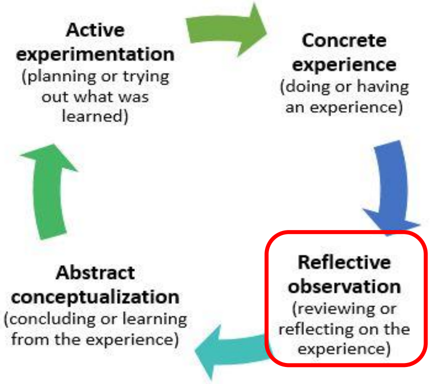
1 Knots and diagrams

A **knot** is a circle that has been placed into 3D space in an interesting or possibly complicated way. Informally, we can think of a knot as being a knotted cord or string where the ends have been sealed.

Invent some knots and draw pictures of them here.

A *knot diagram* or a *knot projection* is how mathematicians draw knots in the plane. Over and under-crossings are an important part of the picture. We don't allow more than two strands to intersect in a knot diagram. ***Can you think of any reasons why?***

Intro Activity: Reflective Observation



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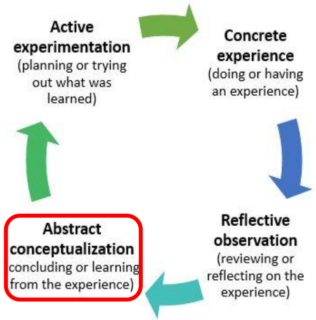
Knotty Knots: Abstract Conceptualization

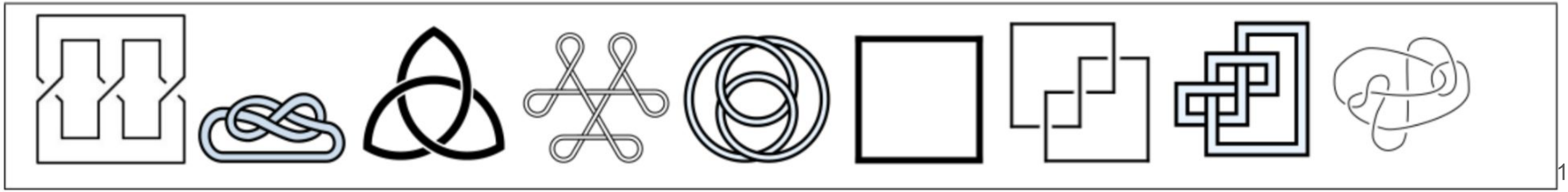


2 Equivalent knots

If one knot can be smoothly transformed into another knot (no cutting or breaking allowed!) then those knots are equivalent.

- The act of stretching, shrinking and moving around in space is called doing an *isotopy*.
- Equivalent knots are called *isotopic* and notated with a squiggly equals sign: $K_1 \simeq K_2$

Which knots are equivalent to each other? Highlight equivalent knots. ← Task, to segue into next phase

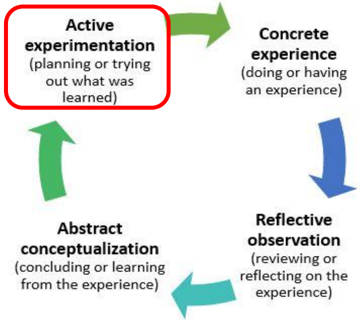


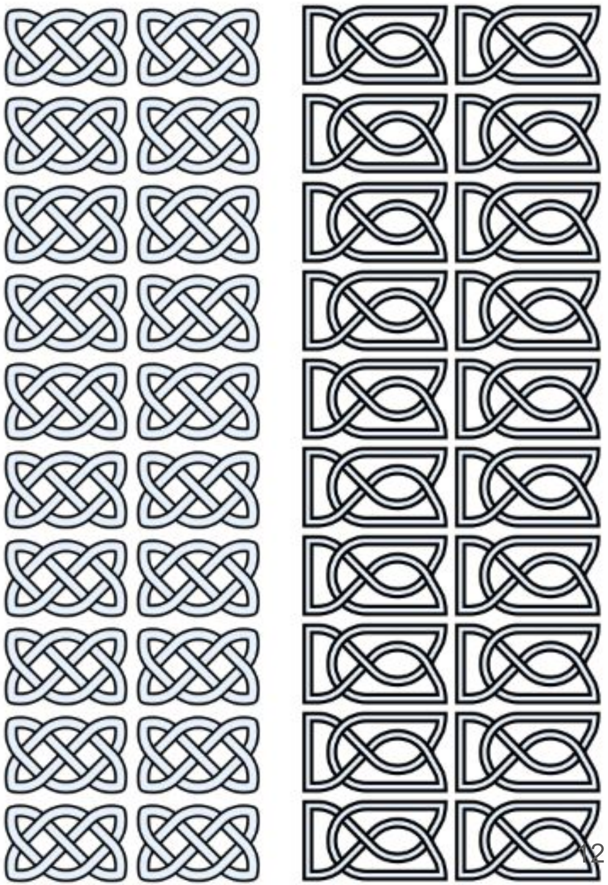
Knotty Knots: Active Experimentation

Tricoloring exercise 1: Are either of these knots tricolorable?

Figure 4: The knot on the left is 7_4. The knot on the right is 4_1, the Figure 8 knot.

Student worksheets had 20 of each knot to allow for trial-and-error in determining tricolorability

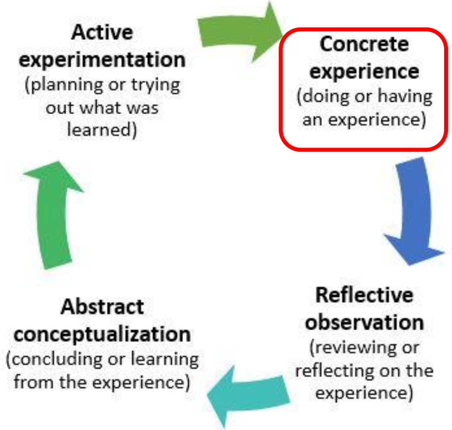




Knotty Knots: Concrete Experience

Back to a concrete experience: “Try to make a knot that’s going to sink faster”

Solution in each container: 2 ½ cups water to ½ cup sugar



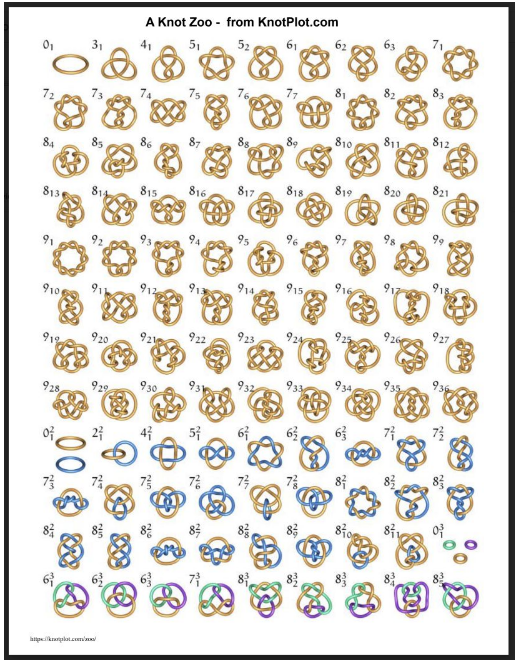
Knotty Knots: Concrete Experience (cont'd)

Discuss whether we expect knots to descend in the fluid at different rates Hypothesize which knots will descend the fastest. Tie candidates into necklaces. Conduct experiments (knot races) Analyze Results Record the winners, losers. Describe their knot-theoretic properties.	1. Do mathematically different knots have different physical properties? 2. What other physical properties of knots would be worth investigating?	• Sugar-water solution • Graduated buckets • Knot necklaces • Chopsticks • Kitchen Timer • Wet-wipes • Tea kettle • Towels

Learning Goal(s):

1. Mathematical objects can be dynamic
2. Basic concepts of knot theory: knots, diagrams, invariants, Reidemeister moves
3. Different topological knot types can be distinguished with tricoloring and with experiment

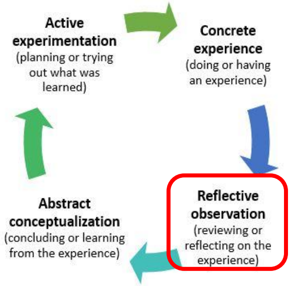
Knotty Knots: Reflective Observation



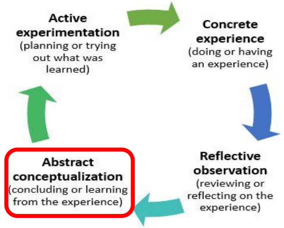
"What we're observing is that the denser knots seem to sink faster. That probably doesn't seem surprising"

Reflective Prompt: "Look at the table, do you think there's any correlation between their placement in the table and how fast they would move through fluid?"

Students hypothesize out loud why: number of crossings, etc.

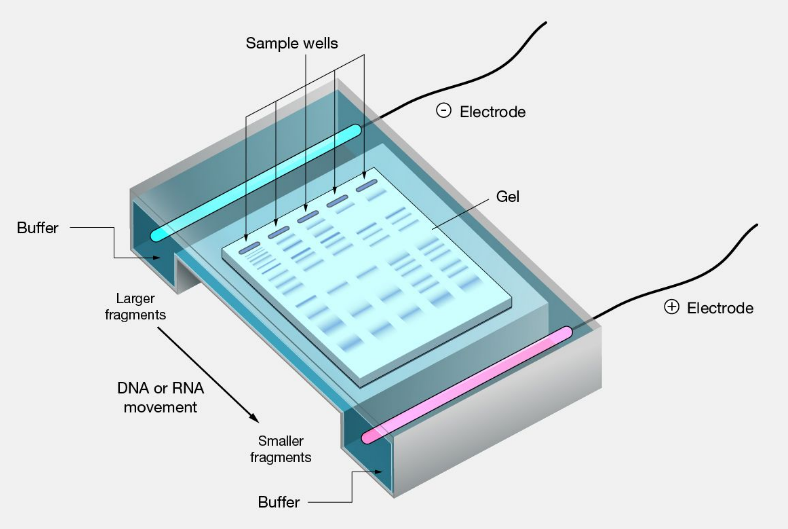


Knotty Knots: Abstract Conceptualization



We played a video that demonstrates/explains a gel electrophoresis experiment.

Electrophoresis… separate[s] DNA, RNA or protein molecules based on their size and electrical charge. An electric current is used to move the molecules through a gel or other matrix. Pores in the gel or matrix work like a sieve, **allowing smaller molecules to move faster than larger molecules**" (National Human Genome Research Institute, 2023).



https://www.genome.gov/genetics-glossary/Electrophoresis

Note: We wish we had something that fit this stage of the cycle better

What Worked

- Students were drawn to the hands-on manipulatives (Silkies necklaces, as stretchy string) to create representations
- Mix of students working with knots, discussing with each other, and looking around at what others were doing
- They were engaged but also read definitions and listened during times of direct instruction
- Knot races activity - mini-competition, for fastest and slowest
- Variety of activity types (experimentation, instruction, observations/sharing)
- In general, they were excited to show what they'd created and accomplished

What we'd do differently

- *Equipment*: Have a document camera available to project students' necklace knots or drawn knots, or reference visuals on papers
- *Setting*: Ensure students are sitting with others
- *Student support:*
    - Provide a document or explain mathematical symbols used when discussing equivalency (e.g., $\simeq$, $\leftrightarrow$)
    - Provide an example of a knot that is tricolorable and a knot that is not before letting student experiment; ensure they have a clear idea of what they're working towards

      AND/OR

      Slightly more structured support during tricolorability trial-and-error OR less exploration time (students had 15-20 minutes, might have been too long)

Suggestions for Implementation

- Use physical string where knots will be visible and not unravel
- Ensure students can clearly see examples when discussing or referencing particular knots (e.g., document camera)
- If students are working towards a particular goal (e.g., determining tricolorability), ensure they have a clear definition/example of what they're working towards
- Consider student levels and current knowledge that is relevant to knots, provide basic definitions and necessary concepts (e.g., equality symbols)
- Plan for a variety of activity types (e.g., experimentation, guided instruction)

Feedback & Thank you

We thank:

- Joe Flenner, who runs the VCU Math Circle

We welcome any feedback on how to improve our activities or make them more engaging for middle school students

VCU Breakthroughs Team:

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Activity: Technical Drawing

Lesson: November Pilot - Constructing Triangles	**Time:** ~75 mins of activities, anticipate breaks, things taking longer, etc.

Learning Goal(s):

1. Math is not just numbers; math involves drawings and understanding relationships between objects
2. How to use a compass and a straightedge
3. Compasses and straightedges can be used to create specific geometric shapes

Lesson Stage/Activity & Steps	Guiding Questions/Goals	Needed Supplies
Intro Activity **(~15 mins)** 1. Show examples of artwork created with just a compass and straightedge **(~5 mins)** 2. Distribute compasses & straightedges and allow students to "play" with them (*allow them to figure out how to use - some may know and can help others*) and explore creating their own images **(~5 mins)** a. Provide support to students struggling to use compass and straightedge	1. Compass and straightedge are helpful tools for design and understanding mathematical concepts	• Compasses & straightedges • Paper • Erasers • Colored pencils, crayons, markers • Examples of compass and straightedge artwork • Examples of Ancient Greek columns, mathematics, etc

Scaffolding of Lesson Plan: Technical Drawing

1. What are the most basic geometric shapes? How can we use the compass and straightedge to construct? *(Go through and discuss together, provide support to struggling students, encourage students to help each other, demonstrate to whole group as needed)* (~10-15 mins)
   a. Point
   b. Line
   c. Rays
   d. Angles

2. What shape would come after an angle? How can we construct a triangle using the straightedge and compass? (~10 mins)
   a. Allow students to experiment and just draw triangles (~5 mins)
   b. Discuss what they found (~5 mins), possibly only using ruler

3. How can we construct specific types of triangles? Review: What are the different types of triangles? (~20-25 mins)
   a. Separate into two groups, as STATIONS: one group explores how to create an equilateral triangle and the other looks at isosceles triangles. Once a student is done, they can move to the other station. (Provide support to both groups) (~10-15 mins)
   b. Come back as a whole group and have students demonstrate their findings to the whole class. (~10 mins)
4. How can we construct a hexagon?

5. Show pictures of constructions in art and nature, e.g. honeycomb structure