From Play to Proof: Exploring Red Ball Puzzles and Beyond

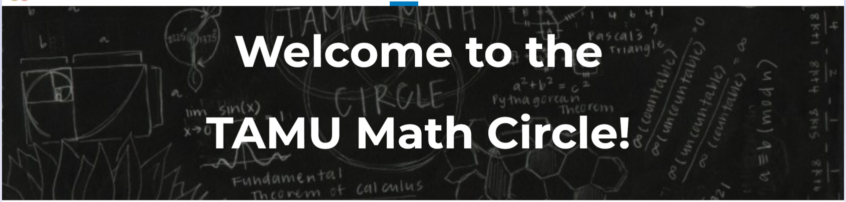
Kun Wang

Texas A&M University

January 4, 2024

Outreach Program—Math Circle

TAMU Math Circle   Home   About   Meetings   Registration   Competitions   Manuals   More



Welcome to the TAMU Math Circle!

The Texas A&M Math Circle is an outreach activity that brings 5th - 12th grade students into direct contact with mathematical professionals in an informal setting to work on fun and interesting topics in mathematics. The goal is to help students become passionate about math. Our instructors range from professors and graduate students at Texas A&M to experts from industry, all of whom are eager to share with students their passion for mathematics and its applications.

This year's meetings:

For the Fall 2023 semester, we will have Saturday meetings on 9/23, 9/30, 10/14, 10/28, 11/11, 11/18, 12/2, and 12/9.

There are two types of sessions being offered:

- 1:00 - 3:00 Discovery Activity
- 3:00 - 4:00 Problem Solving

Each is offered at three levels:

- Beginner (for those on Pre-Algebra and below)
- Intermediate (for those on Algebra 1 and Geometry)
- Advanced (for those on Algebra 2 and above)

Outreach Program—PReMa

PReMa Home Program in 2024 Organizers Mentors Volunteers Current program News Programs in previous years

PReMa

Program for Research in Mathematics

PReMa is a free research program in mathematics for talented high school students. This program is designed to provide an excellent opportunity for students to work on individual or group research projects under the guidance of academic mentors from the Texas A&M department of mathematics. Our goal is to expose students to the beauty and intricacy of advanced mathematics, as well as to allow them to get hands-on experience with all aspects of mathematical research. Moreover, by bringing students to this program to learn and work together, we will foster friendships and collaborations between students with a common love and passion for mathematics.

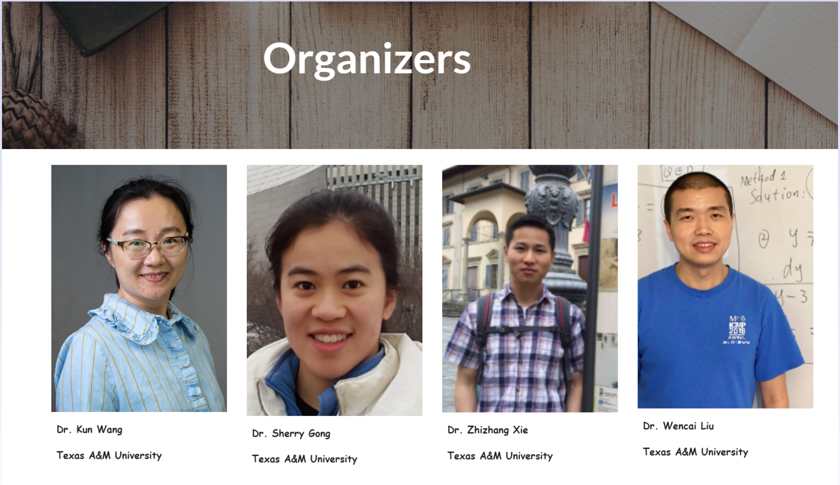
Registration Information

To register, please fill out the google form at
https://docs.google.com/forms/d/12kWV5iXvuwMz4tTlZNZC7QOEoxarcMn3APzufVP8Eh0/prefill

Contact Information

If you have any questions, feel free to contact Dr. Wang at kwang@tamu.edu.

Outreach Program—PReMa



Motivations and Challenges for PReMa

- Motivations:
  Every year, we (professors in our department) receive many emails from high school students asking for research opportunities. Unfortunately, there are very limited programs available for them. So we decided to initiate a new program to address this gap.

Motivations and Challenges for PReMa

- Motivations:
  Every year, we (professors in our department) receive many emails from high school students asking for research opportunities. Unfortunately, there are very limited programs available for them. So we decided to initiate a new program to address this gap.
- Challenges:
  A primary challenge is the selection of research problems that are both suitable and engaging for our students. These topics should ideally align with their level of understanding and stimulate their interest in scientific exploration.

A math circle problem

- Among nine golden coins, one is fake. Identify the fake coin (suppose it is lighter than the others) using a balance scale. What's the least number of weighings needed to find it?

A math circle problem

- Among nine golden coins, one is fake. Identify the fake coin (suppose it is lighter than the others) using a balance scale. What's the least number of weighings needed to find it?

Solution

To find the fake gold coin, we can use a process of **elimination.** Here's how we can do it in just **TWO** uses of the balance scale:

Solution

To find the fake gold coin, we can use a process of **elimination.** Here's how we can do it in just **TWO** uses of the balance scale:

First Use of the Balance Scale:

- Divide the 9 coins into 3 groups of 3 coins each (let's call them Group A, Group B, and Group C).

Solution

To find the fake gold coin, we can use a process of **elimination.** Here's how we can do it in just **TWO** uses of the balance scale:

First Use of the Balance Scale:

- Divide the 9 coins into 3 groups of 3 coins each (let's call them Group A, Group B, and Group C).
- Compare Group A and Group B on the balance scale.

Solution

To find the fake gold coin, we can use a process of **elimination.** Here's how we can do it in just **TWO** uses of the balance scale:

First Use of the Balance Scale:

- Divide the 9 coins into 3 groups of 3 coins each (let's call them Group A, Group B, and Group C).
- Compare Group A and Group B on the balance scale.
  - Outcome 1: If the scale balances, then all coins in Group A and Group B are real, and the fake coin is in Group C.
  - Outcome 2: If the scale does not balance, then the fake coin is in the lighter group (either Group A or Group B).

Solution

Second Use of the Balance Scale:

- Take the group identified as containing the fake coin in the first use.

Solution

Second Use of the Balance Scale:

- Take the group identified as containing the fake coin in the first use.
- From this group, put one coin on each side of the scale, leaving the third coin aside.

Solution

Second Use of the Balance Scale:

- Take the group identified as containing the fake coin in the first use.
- From this group, put one coin on each side of the scale, leaving the third coin aside.
  - Outcome 1: If the scale balances, then the coin not on the scale is the fake.
  - Outcome 2: If the scale does not balance, the fake coin is the lighter one on the scale.

Solution

Second Use of the Balance Scale:

- Take the group identified as containing the fake coin in the first use.
- From this group, put one coin on each side of the scale, leaving the third coin aside.
  - Outcome 1: If the scale balances, then the coin not on the scale is the fake.
  - Outcome 2: If the scale does not balance, the fake coin is the lighter one on the scale.

So, the minimal number of times you need to use the balance scale is two.

More problems

Now we can ask them to solve more problems:
What is the most efficient method to locate the fake coin among larger sets of 20, 50, 100, or 1000 coins using a balance scale with the fewest number of weighings?”

More problems

Now we can ask them to solve more problems:
What is the most efficient method to locate the fake coin among larger sets of 20, 50, 100, or 1000 coins using a balance scale with the fewest number of weighings?"
Based on the above method, we can realize that the **KEY** is to divide the coins into **three** groups and compare their weights. (Perhaps, we need to point this for the students!)

More solutions

Here is a solution for the set of 100 coins:

More solutions

Here is a solution for the set of 100 coins:
First Weighing:

- Divide the 100 coins into three groups, but not equally. A good division could be 34, 33, and 33 coins.

More solutions

Here is a solution for the set of 100 coins:
First Weighing:

- Divide the 100 coins into three groups, but not equally. A good division could be 34, 33, and 33 coins.
- Weigh the two groups of 33 coins against each other.

More solutions

Here is a solution for the set of 100 coins:
First Weighing:

- Divide the 100 coins into three groups, but not equally. A good division could be 34, 33, and 33 coins.
- Weigh the two groups of 33 coins against each other.
  - If they balance, the fake coin is in the group of 34.
  - If they don't balance, the fake coin is in the lighter.

More solutions

Here is a solution for the set of 100 coins:
First Weighing:

- Divide the 100 coins into three groups, but not equally. A good division could be 34, 33, and 33 coins.
- Weigh the two groups of 33 coins against each other.
    - If they balance, the fake coin is in the group of 34.
    - If they don't balance, the fake coin is in the lighter.

Subsequent Weighings:
Continue dividing the group suspected to contain the fake coin. The key is to divide the remaining coins into three groups as evenly as possible and weigh two of them against each other. With each weighing, you significantly reduce the number of potential fake coins.

More solutions

Here is a solution for the set of 100 coins:
First Weighing:

- Divide the 100 coins into three groups, but not equally. A good division could be 34, 33, and 33 coins.
- Weigh the two groups of 33 coins against each other.
  - If they balance, the fake coin is in the group of 34.
  - If they don't balance, the fake coin is in the lighter.

Subsequent Weighings:
Continue dividing the group suspected to contain the fake coin. The key is to divide the remaining coins into three groups as evenly as possible and weigh two of them against each other. With each weighing, you significantly reduce the number of potential fake coins.
In total, we need 5 times weighings for 100 coins.

Complete the table

Next, we can encourage students to fill out the following table:

# of coins	Num. of weighings	# of coins	Num. of weighings
2	1	16	
3		17	
4		18	
5		19	
6		20	
7		21	
8		22	
9		23	
10		24	
11		25	
12		26	
13		27	
14		28	
15		29	

Complete the table

By this process, the students can also learn the idea of reduction method.

# of coins	Num. of weighings	# of coins	Num. of weightings
2	1	16	3
3	1	17	3
4	2	18	3
5	2	19	3
6	2	20	3
7	2	21	3
8	2	22	3
9	2	23	3
10	3	24	3
11	3	25	3
12	3	26	3
13	3	27	3
14	3	28	4

Completed Solution

Now, let's encourage students to formulate a general rule. Their objective is to point out the specific quantities of coins at which the required number of weighings increases.

Completed Solution

Now, let's encourage students to formulate a general rule. Their objective is to point out the specific quantities of coins at which the required number of weighings increases.

- The number of weighings increases after each power of 3!!!

Completed Solution

Now, let's encourage students to formulate a general rule. Their objective is to point out the specific quantities of coins at which the required number of weighings increases.

- The number of weighings increases after each power of 3!!!

Theorem (Original problem)

The optimal weighing times for identify the fake coin among n coins is to determined by identifying the highest integer k for which

$$3^k < n.$$

In this case, $k+1$ times weighings are necessary.

Completed Solution

Now, let's encourage students to formulate a general rule. Their objective is to point out the specific quantities of coins at which the required number of weighings increases.

- The number of weighings increases after each power of 3!!!

Theorem (Original problem)

The optimal weighing times for identify the fake coin among n coins is to determined by identifying the highest integer k for which

$$3^k < n.$$

In this case, $k+1$ times weighings are necessary.

Remark

In other words, dividing coins into three piles as evenly as possible, allow us, with k weighings, to find the fake coin among up to 3^k coins.

Modified Coin puzzle problems

More famous coin puzzle appeared around the same time as the previous puzzle:

Modified Coin puzzle problems

More famous coin puzzle appeared around the same time as the previous puzzle:

• There are 12 coins: one of them is fake. All real coins weigh the same. The fake coin is either lighter or heavier than the real coins. Find the fake coin using a balance scale with 3 weighings.

Modified Coin puzzle problems

More famous coin puzzle appeared around the same time as the previous puzzle:

• There are 12 coins: one of them is fake. All real coins weigh the same. The fake coin is either lighter or heavier than the real coins. Find the fake coin using a balance scale with 3 weighings.

Theorem (Modified case 1)

For this case, the maximum number of coins for which the problem can be solved in k weighings is $\frac{3^k-3}{2}$.

Modified Coin puzzle problems

Theorem (Modified case 2)

If in a set S of coins one coin is a different weight than the rest and each coin is labelled "possibly heavy" (p.h.) or "possibly light" (p.l.), the least number of weighings on a beam balance in which the odd coin can be found is the unique k satisfying $3^{k-1} < |S| \leq 3^k$.

Modified Coin puzzle problems

Theorem (Modified case 2)

If in a set S of coins one coin is a different weight than the rest and each coin is labelled "possibly heavy" (p.h.) or "possibly light" (p.l.), the least number of weighings on a beam balance in which the odd coin can be found is the unique k satisfying $3^{k-1} < |S| \leq 3^k$.

Theorem (Modified case 3)

If we are given a set S of coins, plus a standard coin, and one coin in S is a different weight than the rest, then the least number of weighings in which the odd coin can be found is the unique k satisfying $(3^{k-1} - 1)/2 < |S| \leq (3^k - 1)/2$.

Proof of Modified case 3

Let us denote by $M(k)$ the maximum number of coins for which the odd coin problem can be solved in k weighings if a standard coin is provided. The lemma claims that $M(k) = (3^k - 1)/2$. It is easy to see that this is correct for $k = 1$ and 2.

Proof of Modified case 3

Let us denote by $M(k)$ the maximum number of coins for which the odd coin problem can be solved in k weighings if a standard coin is provided. The lemma claims that $M(k) = (3^k - 1)/2$. It is easy to see that this is correct for $k = 1$ and 2.

Suppose now we are given a set S of coins from which we are to find the odd coin in k weighings. On our first weighing, we must place equal sets of coins S_1 and S_2 on the scale, leaving off a set S_3. If the beam balances, we are left with S_3, so we must require $|S_3| \leq M(k-1)$.

Proof of Modified case 3

Let us denote by $M(k)$ the maximum number of coins for which the odd coin problem can be solved in k weighings if a standard coin is provided. The lemma claims that $M(k) = (3^k - 1)/2$. It is easy to see that this is correct for $k = 1$ and 2.

Suppose now we are given a set S of coins from which we are to find the odd coin in k weighings. On our first weighing, we must place equal sets of coins S_1 and S_2 on the scale, leaving off a set S_3. If the beam balances, we are left with S_3, so we must require $|S_3| \leq M(k-1)$.

On the other hand, if the scale does not balance, we are left with S_1 and S_2, each coin labelled "possibly heavy" or "possibly light". So we must have $|S_1| + |S_2| \leq 3^{k-1}$ with each labelled.

Proof of Modified case 3

Let us denote by $M(k)$ the maximum number of coins for which the odd coin problem can be solved in k weighings if a standard coin is provided. The lemma claims that $M(k) = (3^k - 1)/2$. It is easy to see that this is correct for $k = 1$ and 2.

Suppose now we are given a set S of coins from which we are to find the odd coin in k weighings. On our first weighing, we must place equal sets of coins S_1 and S_2 on the scale, leaving off a set S_3. If the beam balances, we are left with S_3, so we must require $|S_3| \leq M(k-1)$.

On the other hand, if the scale does not balance, we are left with S_1 and S_2, each coin labelled "possibly heavy" or "possibly light". So we must have $|S_1| + |S_2| \leq 3^{k-1}$ with each labelled.

Thus we can solve the balancing problem if $|S| = (|S_1| + |S_2|) + |S_3| = 3^{k-1} + M(k-1)$. This yields $M(k) = 3^{k-1} + M(k-1)$, which leads to $M(k) = \sum_{i=0}^{k-1} 3^i$. Thus $M(k) = (3^k - 1)/2$.

Research problems for high school students

Based on the above scenario, we can ask our students explore several research questions:

Based on the above scenario, we can ask our students explore several research questions:

1. Two Counterfeit Coins Among n Coins:

• Given that two counterfeit coins are mixed among n gold coins, with each fake being lighter and of distinct weights, what is the minimum number of weighings required to identify them?

Research problems for high school students

Based on the above scenario, we can ask our students explore several research questions:

1. Two Counterfeit Coins Among n Coins:

• Given that two counterfeit coins are mixed among n gold coins, with each fake being lighter and of distinct weights, what is the minimum number of weighings required to identify them?

2. Three Counterfeit Coins Among n Coins:

• If three counterfeit coins are among n gold coins, and these counterfeits are lighter with differing weights, what is the least number of weighings needed for their identification?

Research problems for high school students

Based on the above scenario, we can ask our students explore several research questions:

1. Two Counterfeit Coins Among n Coins:

• Given that two counterfeit coins are mixed among n gold coins, with each fake being lighter and of distinct weights, what is the minimum number of weighings required to identify them?

2. Three Counterfeit Coins Among n Coins:

• If three counterfeit coins are among n gold coins, and these counterfeits are lighter with differing weights, what is the least number of weighings needed for their identification?

3. Uncertain Weight Difference:

• What changes in the strategy if it's unknown whether the counterfeit coins are lighter or heavier than the real ones?

Research problems for high school students

Based on the above scenario, we can ask our students explore several research questions:

1. Two Counterfeit Coins Among n Coins:

• Given that two counterfeit coins are mixed among n gold coins, with each fake being lighter and of distinct weights, what is the minimum number of weighings required to identify them?

2. Three Counterfeit Coins Among n Coins:

• If three counterfeit coins are among n gold coins, and these counterfeits are lighter with differing weights, what is the least number of weighings needed for their identification?

3. Uncertain Weight Difference:

• What changes in the strategy if it's unknown whether the counterfeit coins are lighter or heavier than the real ones?

4. Counterfeit Coins with Equal Weights:

• How does the problem alter if the counterfeit coins may have the same weight (but different from the real coins)?

Research problems for high school students

Based on the above scenario, we can ask our students explore several research questions:

1. Two Counterfeit Coins Among n Coins:

• Given that two counterfeit coins are mixed among n gold coins, with each fake being lighter and of distinct weights, what is the minimum number of weighings required to identify them?

2. Three Counterfeit Coins Among n Coins:

• If three counterfeit coins are among n gold coins, and these counterfeits are lighter with differing weights, what is the least number of weighings needed for their identification?

3. Uncertain Weight Difference:

• What changes in the strategy if it's unknown whether the counterfeit coins are lighter or heavier than the real ones?

4. Counterfeit Coins with Equal Weights:

• How does the problem alter if the counterfeit coins may have the same weight (but different from the real coins)?

5. Uncertain number of Counterfeit Coins

Methodology

- Divide and Conquer

Methodology

- Divide and Conquer
- Binary Search Method

Methodology

- Divide and Conquer
- Binary Search Method
- Information Theory

Methodology

- Divide and Conquer
- Binary Search Method
- Information Theory
- Combinatorics

Methodology

- Divide and Conquer
- Binary Search Method
- Information Theory
- Combinatorics
- Recursive Techniques

Methodology

- Divide and Conquer
- Binary Search Method
- Information Theory
- Combinatorics
- Recursive Techniques
- Programming

Methodology

- Divide and Conquer
- Binary Search Method
- Information Theory
- Combinatorics
- Recursive Techniques
- Programming
- Linear Algebra

References

- Bennet Manvel, Counterfeit Coin Problems, Mathematics Magazine, Vol. 50, No. 2 (Mar., 1977), pp. 90-92
- D. Eves, Problem E712-The extended coin problem, Amer. Math. Monthly, 53 (1946) 156.
- J. G. Mauldon, Strong solutions for the counterfeit coin problem, IBM Research Report RC 7476 (31437) (1978)
- N. Diaco, Counting Counterfeit Coins: A New Coin Weighing Problem, Mathematics, Computer Science, June 2016
- E. Purdy, Lower Bounds for Coin-Weighing Problems, ACM Transactions on Computation Theory, Volume 2, Issue 2, Article No. 3, pp 1-12.
- J. Dominguez-Montes, Solution to the Counterfeit Coin Problem and its Generalization, arXiv: 1005.1391, (2010).
- L. Halbeisen and Norbert Hungerbuhler. The general counterfeit coin problem, Discrete Mathematics, 147(1-3) (1995), 139-150.

Thank you!