Comments on the session

Wednesday, June 29, 2022 4:22 PM

The question:

- Poker players do some strange things
  - Bluff
  - Fold good hands
  - Pass with great hands (slow play)

Is this good strategy?

Are they just messing with each other?

What if you are playing against a compute

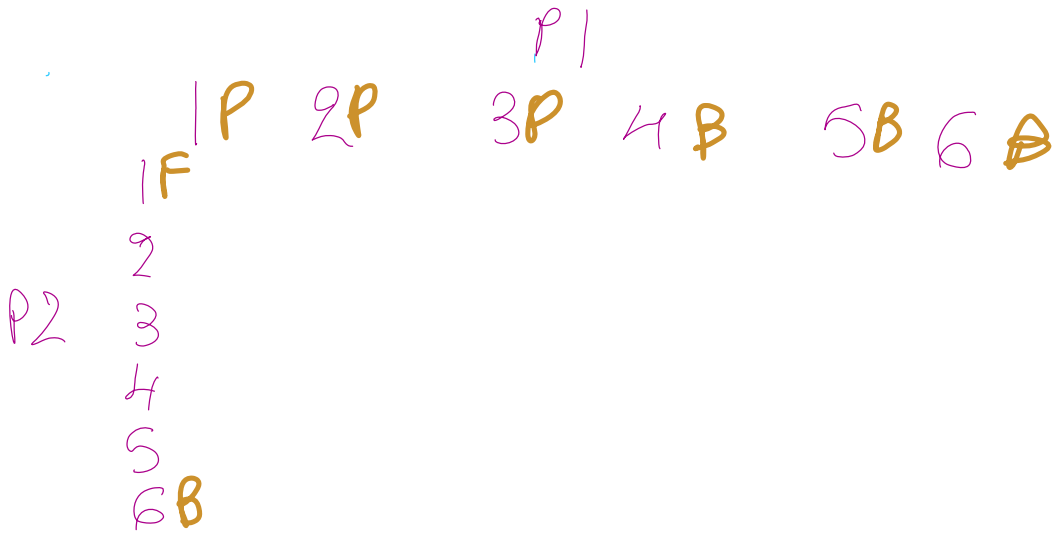
- Session is a bit more "red" than many I lik

See the the award winning article:
Kaity Parsons, Peter Tingley & Emma Zajdelda (2021)
When to Hold Em
Mathematics Magizine,94:3, 201-212
DOI:10.1080/0025570X.2021.1908785

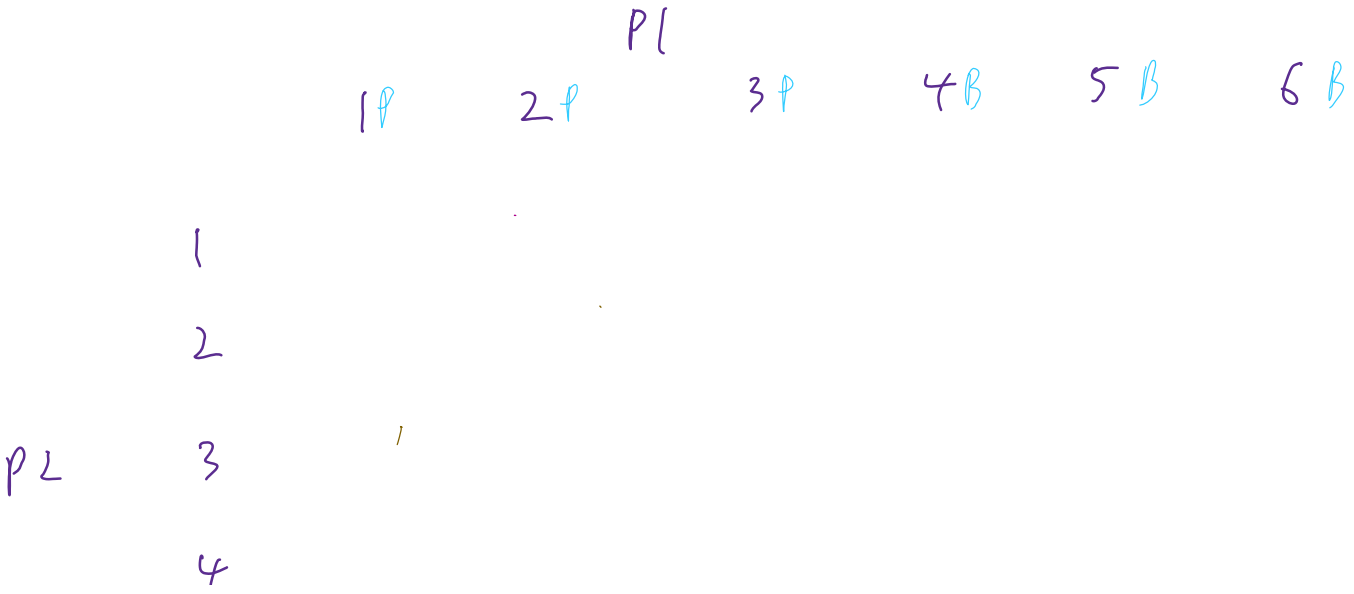
The (main) game

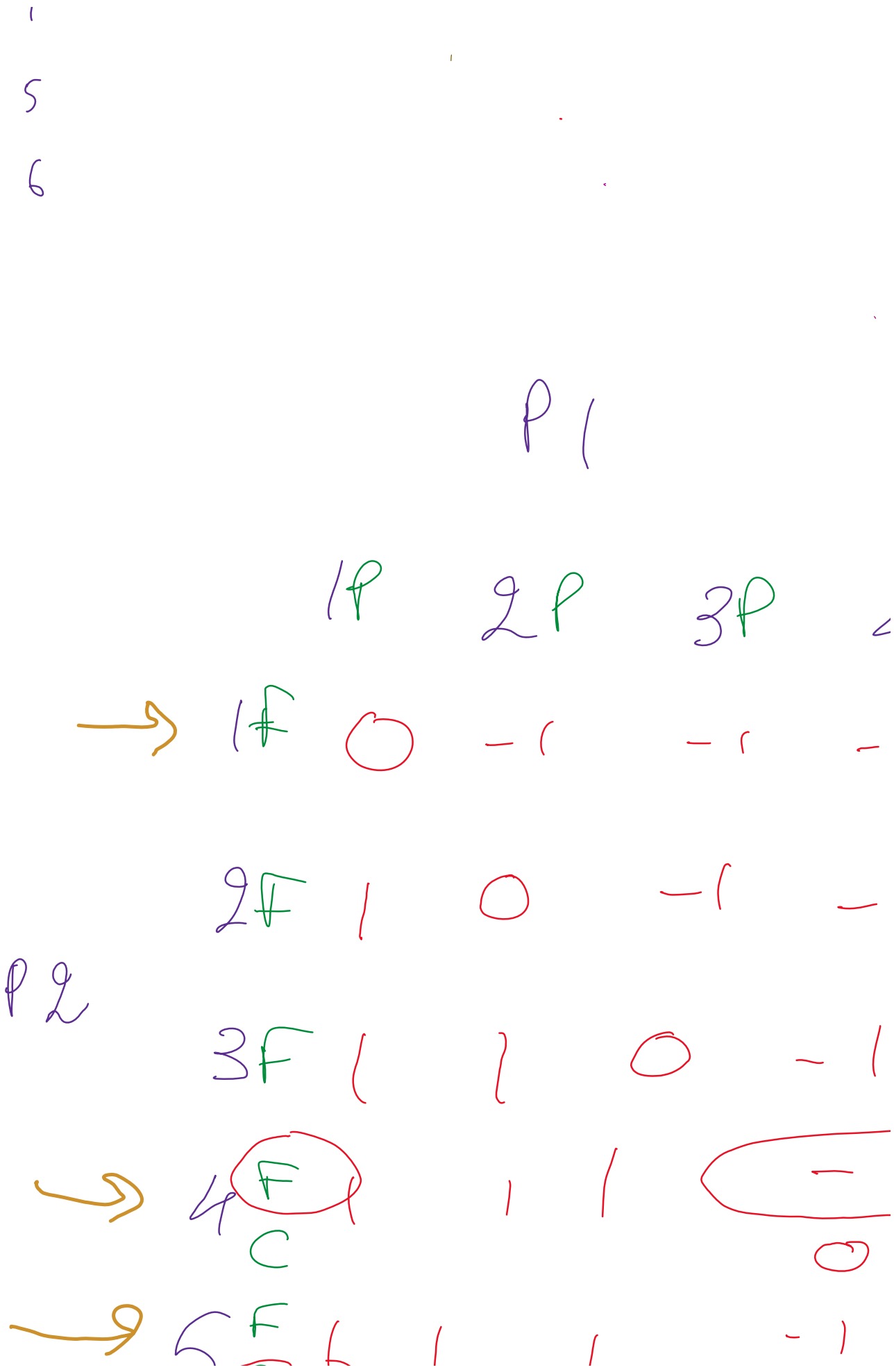
- There are two players P1 and P2.
- Each puts $1 in the pot (the “ante”).
- Each player’s hand is determined by rolling a die, so the possible hands are 1,2,3,4,5,6 and all are equally likely. Roll is hidden from the other player.
- After seeing their hand, P1 can either bet another $1 or pass.
- If P1 bet, P2 can either call by also placing another $1 in the pot, or fold, in which case P1 gets the money in the pot.
- If P1 passed or P1 bet and P2 called they compare hands and the higher number gets all the money in the pot. If there is a tie, they split the pot.

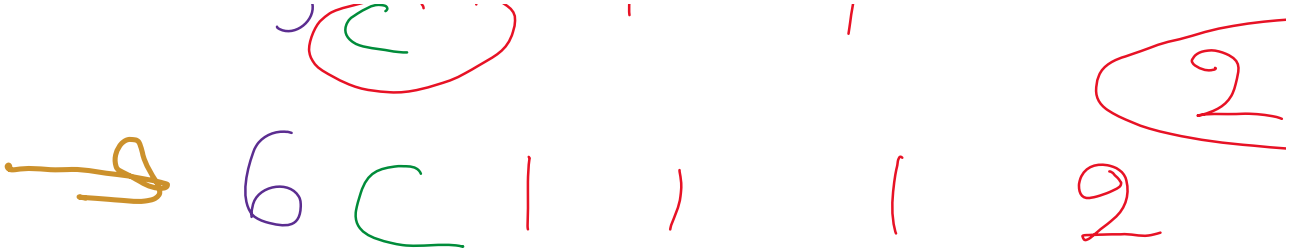
Q1: what is a strategy (for P1)



Q2: How do we tell if a P1 strategy is good? We need to know how P2 is playing! We can assume they are playing well….but what does that mean? Gets circular? To break the circle: P2 gets to know P1's strategy (but not P1's hand) and responds in the best possible way. Now we can evaluate a P1 strategy.




1
5
6
P1
1P 2P 3P
1F 0 -1 -1 -
2F 1 0 -1 -
P2
3F 1 1 0 -1
4F 1 1 -
C 0
5F 1 1 -1



We forgot what we were doing! Try a bluffing strategy!

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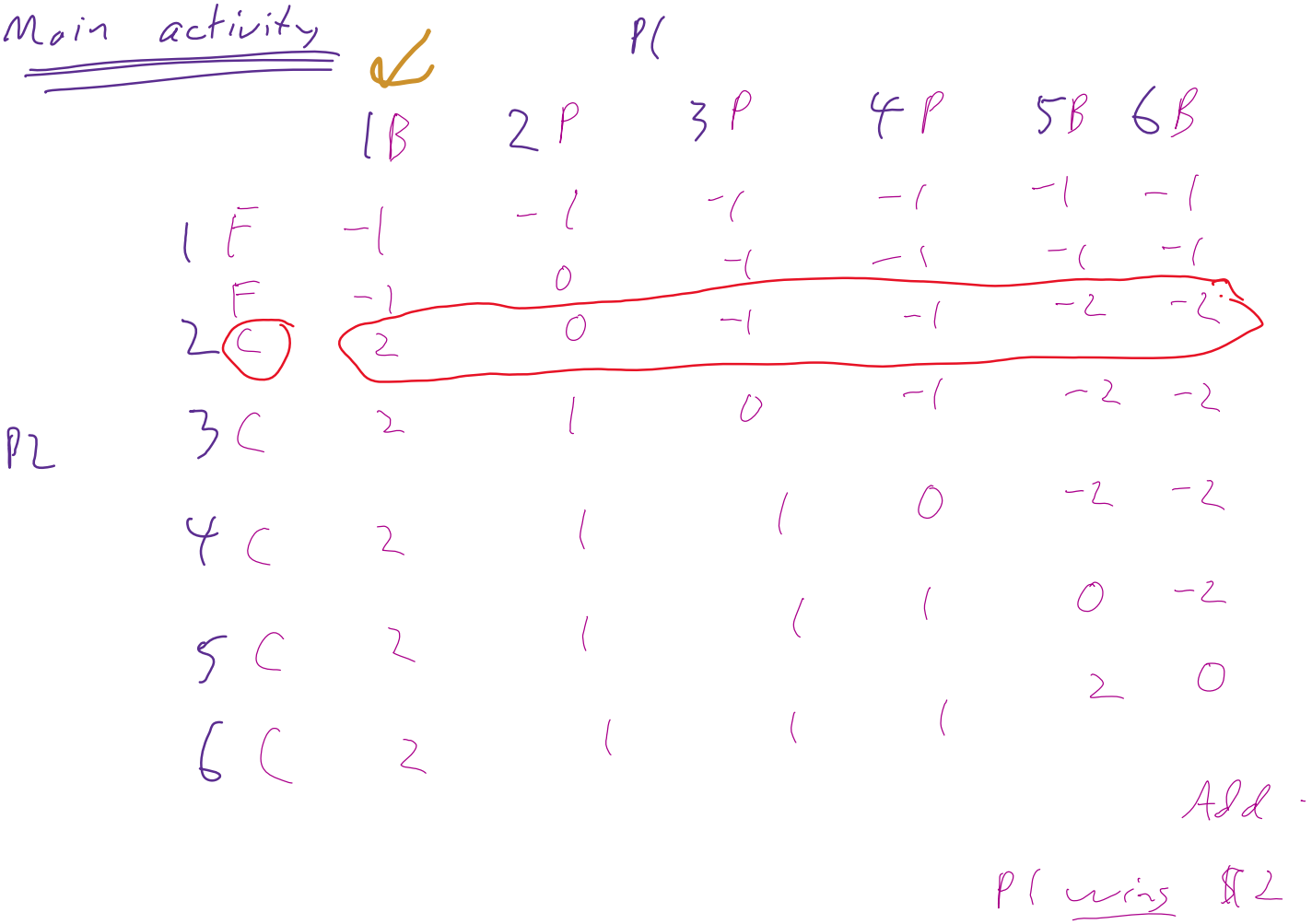
Main activity



Continuous

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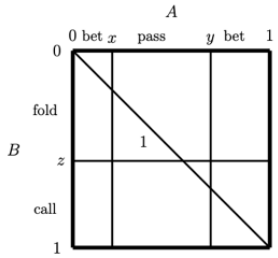
Have sometimes used a version of this worksheet after the dice poker stuff - but caution, people find the dice poker bluffing calculation the most satisfying, I would never advise rushing that to do this. Although this is pretty beautiful!

When to hold 'em

Consider the following simplified poker game: There are two players, A and B. Each "antes" \$1. Each is "dealt" a random number between 0 and 1. A can then either bet \$1 or pass. If A bets \$1, B can either call by also placing another \$1 in the pot, or fold, in which case A gets the pot. In all other cases the player with the higher number takes the pot.

This is like our "dice" game, except there are more possible hands. So, it seems reasonable that A should bet on his best and worst hands and pass on some middle hands, and B should call with hands above some "cutoff" value. The "table" of payoffs, like we used in dice poker, will be a square:



1. For each region, fill in the payoff to B. For instance, in the region marked 1, A passes and B has a better hand, so they show cards and B wins the 1 that A anteed.
2. Assume A bets on the best 20% of hands and the worst 20% (so the vertical lines are in positions $x = 0.2$ and $y = 0.8$), and B knows it.
   (a) If B has 0.5 and A bet, should she call? Hint: calculate her expected payoff (i) if she calls and (ii) if she folds.

   (b) If B gets the hand 0.15 and A bets, should she call?

   Does the answer to (b) seem reasonable? What do you think it means about A's strategy?
3. If A bets, B has hand exactly z (the position of the horizontal line), and she folds, then she loses her ante, so gets a payout of -1. Discuss why, if B is playing optimally, her expected payout if she calls should also be -1 (hint: she calls with $z + 0.001$ and folds with $z - 0.001$). You should no longer be assuming $x = 0.2, y = 0.8$.
4. Use Q3 to find an equation that must hold if B is playing optimally. This will involve some or all of the variables, x, y and z.
5. Assume $y = 0.8$ (so P1 bets on the top 20% of hands). What does the equation from Question 4 tell you about x and z? Discuss.

6. Use the ideas from Q3 to find two more equations that x, y, z should satisfy if both strategies are optimal. Hint: consider payoffs if A bets/passes with hands exactly x or exactly y.

7. Solve to find x, y, z. Do you think these are really good strategies? Do you find them surprising?

More betting

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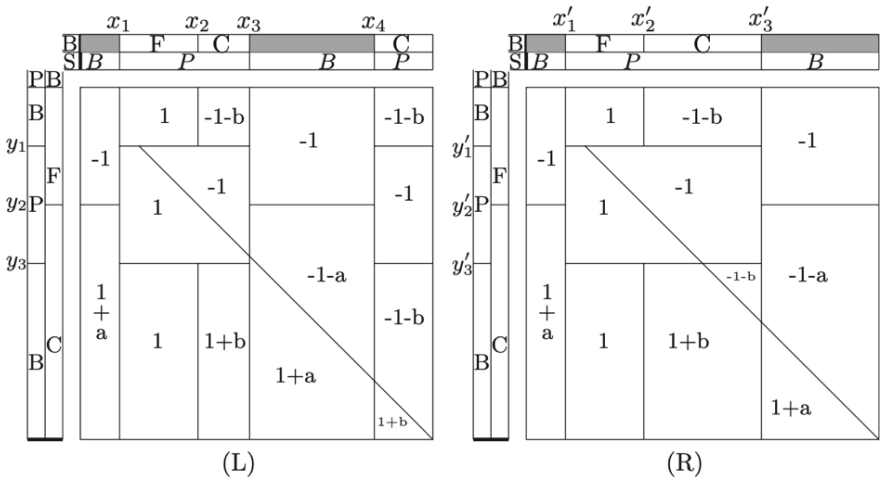
Make the game more complicated!

- P1 bets an amount a, which may not be 1 (but is fixed).
- If P1 passes then P2 can choose to bet a fixed amount b.
- If P2 bets then P1 can call or fold.



L is the answer a<b, R is the answer a>b….a=b both work!

$$-y_2+(1+a)(1-y_2)=1$$

$$y_1+(1-y_3)=-(1+b)y_1+(1+b)(1-y_3)$$

$$-(1+b)y_1-(x_3-y_1)+(y_3-x_3)+(1+b)(1-y_3)=$$
$$-y_2-(1+a)(x_3-y_2)+(1+a)(1-x_3)$$

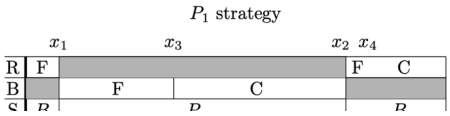
$$-y_2-(1+a)(x_4-y_2)+(1+a)(1-x_4)=$$
$$-(1+b)y_1-(y_3-y_1)-(1+b)(x_4-y_3)+(1+b)(1-x_4)$$

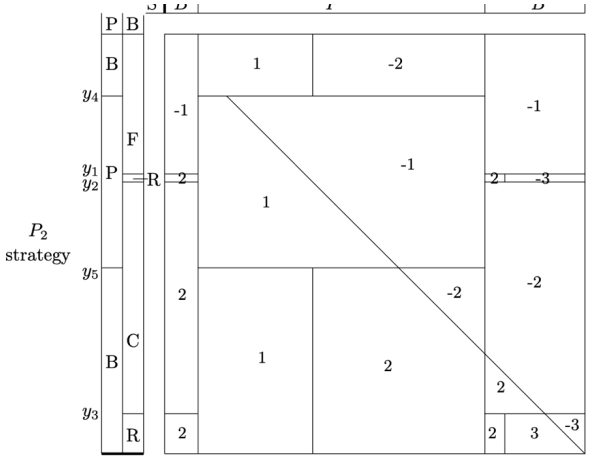
$$(x_2-y_1)-(1+b)(x_3-x_2)-(1+b)(1-x_4)=-(x_3-y_1)-(1-x_4)$$

$$-x_1-(x_4-x_3)=(1+a)x_1-(1+a)(x_4-x_3)$$

$$(x_3-x_2)-(1-x_4)=(1+b)(x_3-x_2)-(1+b)(1-x_4).$$

We did bigger games with raising and things!





This gives the optimal cutoffs and the payout for bet sizes a=1,b=1,c=1. We read the equations off the picture: each says that the payout to one of the players is the same for two possible strategies at a cutoff. We also read the payout p to P2 off the picture: it is the sum over all regions of the payout in that region times the size of the region. In fact, the indifference equations are also obtained by setting the partial derivatives of p with respect to x1,x2,x3,x4,y1,y2,y3,y4,y5 equal to 0, which we used to check that the values were entered correctly. This should only be accurate in some region where b+c>=a>=b.

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Clear[p]
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E1 = 2 y1 == a (1 - y1)
E2 = b y4 + (a + 2) (y2 - y1) + b (x2 - y5) + a (1 - x2) == a (x2 - y2) + b (1 - x2)
E3 = y4 (b + 2) == b (1 - y5)
E4 = (2 a + c + 2) (y2 - y1) == c (1 - y3)
E5 = (a + 2) (x1 + x4 - x2) == (a + c) (1 - x4)
E6 = (2 a + 2) (x4 - x2) == c (1 - x4)
E7 = (y3 - x4) == (1 - y3)
E8 = 2 (x3 - y4) == b (x2 - x3)
E9 = (y5 - x3) == (x2 - y5)

EF = p == (1 + a) x1 - (2 + a) x1 y1 + (x3 - x1) y4 - (1 + b) (x2 - x3) y4 + (y4 - x1 - x2 + y5) (y5 - y4) + (x3 - x1) (1 - y5) +
    (1 + b) (x2 - x3) (1 - y5) - (1 + b) (x2 - y5) ^ 2 - y1 (1 - x2) + (1 + a) (x4 - x2) (y2 - y1) - (1 + a + c) (1 - x4) (y2 - y1) -
    (1 + a) (y3 - y2) (1 - x2) + (1 + a) (y3 - x2) ^ 2 + (1 + a) (x4 - x2) (1 - y3) + (1 + a + c) (1 - y3) (y3 - x4)


NSolve[E1 && E2 && E3 && E4 && E5 && E6 && E7 && E8 && E9 && EF && a == 1 && b == 1 && c == 1,
  {a, b, c, x1, y4, y1, y2, x3, y5, x2, x4, y3, p}]
```

$$2\,y1 == a\,(1 - y1)$$

$$a\,(1 - x2) + (2 + a)\,(-y1 + y2) + b\,y4 + b\,(x2 - y5) == b\,(1 - x2) + a\,(x2 - y2)$$

$$(2 + b)\,y4 == b\,(1 - y5)$$

$$(2 + 2\,a + c)\,(-y1 + y2) == c\,(1 - y3)$$

$$(2 + a)\,(x1 - x2 + x4) == (a + c)\,(1 - x4)$$

$$(2 + 2\,a)\,(-x2 + x4) == c\,(1 - x4)$$

$$-x4 + y3 == 1 - y3$$

$$2\,(x3 - y4) == b\,(x2 - x3)$$

$$-x3 + y5 == x2 - y5$$

$$p == (1 + a)\,x1 - (2 + a)\,x1\,y1 - (1 - x2)\,y1 - (1 + a + c)\,(1 - x4)\,(-y1 + y2) + (1 + a)\,(-x2 + x4)\,(-y1 + y2) + (1 + a)\,(-x2 + x4)\,(1 - y3) +$$
$$(1 + a)\,(-x2 + y3)^2 + (1 + a + c)\,(1 - y3)\,(-x4 + y3) - (1 + a)\,(1 - x2)\,(-y2 + y3) - (1 + b)\,(x2 - x3)\,y4 +$$
$$(-x1 + x3)\,y4 + (1 + b)\,(x2 - x3)\,(1 - y5) + (-x1 + x3)\,(1 - y5) - (1 + b)\,(x2 - y5)^2 + (-y4 + y5)\,(-x1 - x2 + y4 + y5)$$

$$\{\{a \to 1.,\ b \to 1.,\ c \to 1.,\ x1 \to 0.0793651,\ y4 \to 0.147619,\ y1 \to 0.333333,\ y2 \to 0.352381,$$
$$x3 \to 0.352381,\ y5 \to 0.557143,\ x2 \to 0.761905,\ x4 \to 0.809524,\ y3 \to 0.984762,\ p \to 0.068254\}\}$$

Arbitrary bets?

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You might ask, what if you are allowed to bet varying amount?

Answer is you should use all bet sizes! And it helps quite a bit: payout goes from 1/9 to 1/7.

Can solve using an amazingly simple system differential equations:

- b(m) is the function telling P1 which hand to bluff the amount m with.
- v(m) tells P1 which hand to value bet m with.
- c(m) is P2's cutoff telling them whether to fold or call if P1 bet m.

Equations

v'(m)/b'(m)= -(m+2)/m (bet to bet+pot ratio)
(m + 2)c′(m) + c(m) = 1 (all bluffs should give same expected payout)
2v(m) = mc′(m) + c(m) + 1 (for any hand m, betting b(m) maximizes expected payout)
Also figure out some boundary conditions. Solving:

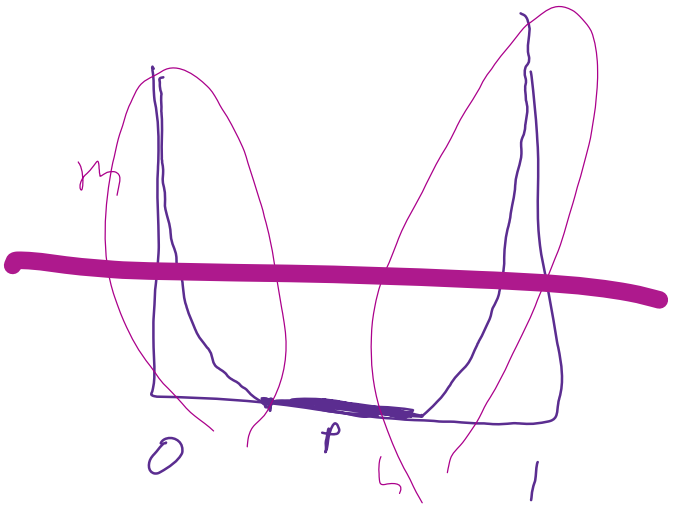
$$c(m) = 1 - \frac{12}{7(m+2)}, \quad v(m) = 1 - \frac{12}{7(m+2)^2}, \quad \text{and} \quad b(m) = \frac{4}{7}\frac{3m+2}{(m+2)^3}$$

Everything actually integrates using rational function!

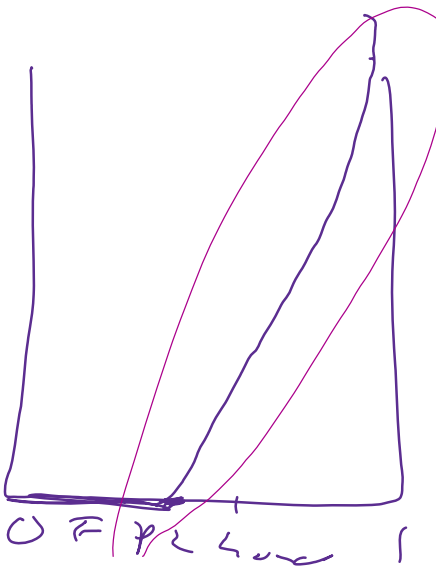
Note: This answer is also in

Mazalov, Vladimir. Mathematical game theory and applications. John Wiley & Sons, Ltd., Chichester, 2014.

And other places. But usually taking a limit of the answer for n fixed bet sizes, not directly with differential equations.



Bet in terms of p



Bet size in p2 calls

History of probability and games

Friday, August 5, 2022 10:40 AM

- Gambler's dispute in 1654 led to the creation of a mathematical theory of probability.
- This problem and others posed by de Méré led to an exchange of letters between Pascal and Fermat in which the fundamental principles of probability theory were formulated for the first time.

Antoine Gombaud, Chevalier de Méré, French nobleman with an interest in gaming and gambling questions.

Blaise Pascal (1623 – 19 August 1662) was a French mathematician, physicist, inventor, writer and Catholic theologian

Pierre de Fermat (1601–1665) was a French lawyer and a mathematician

Game theory

Antoine Augustin Cournot
Cournot duopoly model introduced the concept of a (pure strategy) Nash equilibrium

Morgenstern and Neumann
Authors of 1944 book on Theory of Games and Economic Behavior

John Nash
American mathematician who made fundamental contributions to game theory