Impartial Combinatorial Games

An *impartial combinatorial game* is one in which there are two players who alternate moves. The rules for what is a legal move don't depend on which player has the next move, and the game must end after finitely many moves (no matter how it is played). Impartial combinatorial games don't involve chance or hidden information, and no ties are allowed. In *normal play*, the last player to move wins. (In *misère play* the last player to move loses.)

Game positions can be classified as either *winning*, in which the next player has a winning strategy of play, or *losing*, in which the previous player has a winning strategy. In normal play, the player who finds she can't move is facing a losing position, and in fact has just lost.

From each winning position, at least one move is available that leads to a losing position. From each losing position, every available move leads to a winning position. This seems self-referential, but it is not. (why?) Winning and losing positions are also called *unsafe* and *safe*, and N and P.

The Fundamental Theorem of Impartial Combinatorial Games must surely be the Sprague-Grundy Theorem, which states that every impartial combinatorial game is equivalent to a one-heap game of Nim. The term *equivalent* is used in a technical sense, and is the start of *Nimber Theory*.

Take-Away, Game 0

Here's a great warm-up game! There is one pile of 21 counters. On your turn, you can take 1, 2 or 3 counters. You win if you take the last counter. In a *losing* position, the next player will lose if her opponent plays strategically. Challenge: find and $\boxed{\text{box}}$ all *losing positions*!

$\boxed{0}$ 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21

Take-Away, Game 1

There is one pile of 21 counters. The player to move first can remove up to 20 counters. From there on, each player can remove any number of counters, but no more than as many as their opponent just took. Which player has a winning strategy?

Take-Away, Game 2

There is one pile of 21 counters. The player to move first can remove up to 20 counters. From there on, each player can remove any number of counters, but no more than *twice as many* their opponent just took. Which player has a winning strategy?

Some Challenges

One way to analyze the games above is to represent positive integers in specific ways.

0. Show that each integer is the sum of a multiple of 4 and either 0, 1, 2 or 3.
1. What positive integers are sums of distinct powers of 2?
   For example, $19 = 16 + 2 + 1$. Limit yourself to using just 1, 2, 4, 8, and 16.
2. What positive integers are sums of distinct Fibonacci numbers? For example, $19 = 13+5+1$. Limit yourself to just 1, 2, 3, 5, 8, and 13. Why don't you need to use consecutive terms?

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What is a Math Circle and what is SIGMAA-MCST?

A math circle is broadly defined as a semi-formal, sustained enrichment experience that brings mathematics professionals in direct contact with pre-college students and/or their teachers. Circles foster passion and excitement for deep mathematics.

The goal of this SIGMAA is to give math circles greater visibility and a permanent presence in the mathematical and educational landscape. This SIGMAA supports MAA members who share an interest in developing, supporting and running math circles. It works to facilitate vertical integration of elementary, middle and high school students, their teachers, undergraduate and graduate students, and faculty up through high-level research mathematicians.

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To join SIGMAA on Circles, you must first be a member of the MAA. For information on joining the MAA, please visit `http://www.maa.org/SIGMAA/joining.html`. The MAA is considering less expensive associate memberships for K–12 Teachers and their students.

The annual SIGMAA on Circles membership fee for MAA members is $12 (US dollars).

There is a a check-off box on MAA membership forms for SIGMAA on Circles, and we ask you to check this box when you pay your MAA membership dues.

For Further Reading

Charles L. Bouton, *Nim, A Game with a Complete Mathematical Theory*, The Annals of Mathematics Vol. 3, No. 1/4 (1901 to 1902), 35–39. This is *the* historic research paper, where a proof of the winning strategy of Nim and its mathematical theory was first published.

A. J. Schwenk, *Take-away games*, Fibonacci Quart. 8, (1970), 225–234. This technical article provides a detailed solution of general Take-away games, including those considered here.

Elwyn R. Berlekamp John H. Conway and Richard K. Guy, *Winning Ways for your mathematical plays,* Academic Press, London, 1982. Both accessible and tantalizingly difficult, *Winning Ways* lays out the full theory of two-person, full information games. From Nimbers to Surreal Numbers, this classic text can take you to the research level, if you are prepared to work hard!

The Sprague-Grundy Theorem is fundamental to the theory of impartial games, and states that every impartial game (played under normal conditions) is equivalent (in a technical sense) to a one-pile game of Nim, and thus has a *nimber* value. It was proved independently by R. P. Sprague (*Über mathematische Kampfspiele*, Tohoku Mathematical Journal 41, (1935–6), 438–444) and P. M. Grundy (*Mathematics and games*, Eureka 2, (1939), 6–8). Challenge yourself to prove it!

Zeckendorf's Theorem: Every positive integer has a unique representation as the sum of distinct, non-consecutive Fibonacci numbers. For example, $45 = 34 + 8 + 3$. This was proved by Zeckendorf (E. Zeckendorf, *Representation des nombres naturels par une somme de nombres de Fibonacci ou de nombres de Lucas*, Bull. Soc. Royale Sci. Liege 41 (1972):179–182.) and apparently earlier by Lekkerkerker (C. G. Lekkerkerker, *Voorstelling van natuurlijke getallen door een som van getallen van Fibonacci*, Simon Stevin 29 (1951–1952):190–195.) Can you find your own proof?

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