Pool Table Geometry

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1. Fairfield County Math Teachers' Circle (FCMTC)

The FCMTC launched its first summer immersion workshop in July, 2012 at Sacred Heart University (SHU) in Fairfield, CT. The three day workshop was enthusiastically attended by 17 middle school math teachers. The theme for the workshop was "Open-Ended-Problem-Solving." Topics covered at the workshop ranged from prime numbers and divisibility to logic, geometry, fractions and approximations.

The Pool Table Geometry Problem, in particular, was very well received by our teachers. It was rated excellent and fun! Many of the teachers have already implemented lesson plans that included this problem in their middle school math classrooms with great success! In the context of the pool table geometry problem, teachers and students have the opportunity to practice a rich variety of elementary and middle school math skills and concepts from algebra, number theory, and geometry.

2. The Problem

Imagine a pool table with pockets only in the corners. Start with a ball at the bottom left corner moving up at an angle of 45 degrees.

What questions can we ask about the path of the ball?

3. Illustrative Examples

Figures A, B and C below show the path of a ball in a 3 x 6, 6 x 9 and a 6 x 10 table respectively.

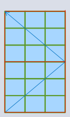

Figure A
3 x 6
or 1 x 2

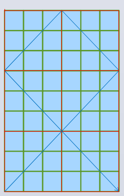

Figure B
6 x 9
or 2 x 3

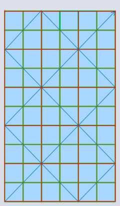

Figure C
6 x 10
or 3 x 5

The number of hits in Figures A, B and C are given below:

Figure	Top/Bottom	Sides	Total hits	Pocket
A	0	1	1	top left
B	1	2	3	bottom right
C	2	4	6	top right

4. Some Questions

1. Draw tables of various sizes and the path of the ball in each table. Record the number of hits on the top/bottom, on the sides as well as the total number of hits. Also, record the pocket in which the ball lands. Can you explain why some sizes give the same answers?
2. For an m x n table (width m, length n), what formula in terms of m and n do the patterns suggest for the number of hits on the top/bottom, on the sides and the total number of hits? Can you explain why this formula holds for any m x n table?
3. Does the ball always fall into a pocket? Can the ball fall back into the starting pocket?
4. Based on the observed patterns, can you predict which pocket the ball will fall into given an m x n table and why?
5. How many cells, in terms of m and n, does the ball traverse before going into a pocket?

5. Examples to illustrate the formula for the number of cells traversed by the ball in an *m* x *n* table

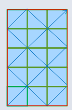

Figure D
3 x 5

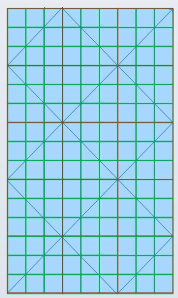

Figure E
9 x 15

In Figure D, the ball traverses each of the 15 cells before falling into a pocket. For, the ball has to travel 5 units up before it can come down and vice versa. Likewise the ball has to move three units to the right before moving left and vice versa. So the total number of cells traversed must be a common multiple of 3 and 5. As the lcm(3, 5) = 15, and the ball cannot traverse any cell twice (why?), the ball must traverse each of the 15 cells.

If gcd(m,n) = 1, then the ball traverses m x n cells.

In Figure E, the ball traverses 3 x 5 = 15 square blocks of size 3. So the ball traverses 3 x 5 x 3 = 45 cells.
Note that 45 = (9/3) x (15/3) x 3 = (9 x 15)/3 = lcm(9,15).

In Figure C, the number of cells traversed by the ball in a 6 x 10 table is 30 = 3 x 5 x 2 = (6/2) x (10/2) x 2 = (6 x 10)/2 = lcm(6, 10).

If d = gcd(m,n), then the ball traverses
(m/d) x (n/d) x d = (m x n)/d = lcm(m,n) number of cells.

6. Example to illustrate the formula for the total number of hits in an *m* x *n* table

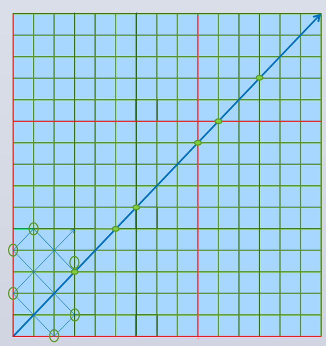

Figure F
15 x 15

It is enough to consider m x n tables, with gcd(m,n) = 1 (Why?).

A 15 x 15 table can be packed with three rows and five columns of 3 x 5 tables. A ball that starts at the bottom left corner goes straight into the diagonally opposite pocket traversing 15 cells. In its path it will hit the sides of four 3 x 5 tables and the tops/bottoms of two 3 x 5 tables.

Therefore, by reflection, the number of hits on the sides of a 3 x 5 table is 4 and on the top/bottom is 2 which gives a total of 6 hits.

For an m x n table with gcd(m,n) = 1,
the total number of hits on the top/bottom is m -1.
the total number of hits on the sides is n -1
the total number of hits is m + n -2.

7. Which pocket does the ball fall into?

A key fact to note is that the ball does not pass through any cell more than once! So the ball must fall into a pocket other than the starting pocket.

Let's consider a 3 x 4 table as shown below:

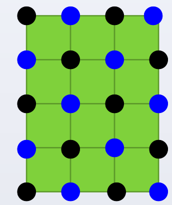

Figure G
3 x 4

A ball starting at a black pocket passes only through the black grid points and so will fall into the other black pocket.

We can explain this mathematically using the following ideas:
(i) When the ball passes through a cell along its diagonal, it travels 1 unit horizontally and 1 unit vertically.

(ii) Assign coordinates to the grid points. Say, the bottom left black corner represents (0,0). Then both coordinates of each black point have the same odd/even parity and the coordinates of each blue point have opposite parity.

8. Conclusion and Extension

The Pool table Geometry problem is very simple to state and yet generates many interesting questions! Through this problem, middle school students will learn important problem solving skills such as analyzing patterns and creating visual representations in a fun way!

Here is a question for further exploration from James Tanton:

Suppose the ball starts at any grid point of an m x n pool table and moves up at an angle of 45 degrees. Assume all four corners of the table now represent pockets. What can one say about the behavior of the ball?

Acknowledgements

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