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Math Circles and the Common Core State Standards: Supporting Systemic Change in K-12 Math Education

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Introduction

I was first introduced to Math Circles when a fellow graduate student encouraged me to serve as an instructor in the Math Circle in Boston founded by Bob and Ellen Kaplan. I later joined the MSRI-sponsored trip to Russia, where a group of American mathematicians learned about Math Circles and other math outreach initiatives in St. Petersburg and Moscow. I continue to lead Math Circles and other math outreach programs for children, as well as professional development workshops and classroom demonstrations for teachers in South Bend, Indiana.

During the last six years, I have been thrilled to see the explosion in the number of Math Circles for students and teachers across the United States. The resulting community of Math Circle leaders is developing rapidly, and I believe that this group is uniquely positioned to help with several aspects of the systemic change needed to improve K-12 math education in this country. This document outlines my own views of the possibilities. I welcome any comments!

The main ways that the Math Circle community has potential to systemically impact K-12 math education are:

1. By involving many mathematicians in informal K-12 education, the Math Circle community has the potential over time to influence teaching practices in college-level math courses. In particular, professors may use skills cultivated in Math Circles such as fostering critical thinking, discussion, project-based learning, and student-centered approaches to instruction that contrast with the prevailing lecture format.

2. The Math Circle community is in a position to highlight the social importance of teaching and outreach as practiced by mathematicians. In particular, this community may shift attitudes about professional contributions in these areas and how those contributions are recognized in matters of tenure and promotion.

3. By indirectly exposing mathematicians to the state of K-12 teacher preparation in this country, the Math Circle community has the potential to motivate mathematicians to take greater responsibility for the proper preparation of future teachers.

4. Math Circles can provide teachers with models for high quality math instruction. By developing and field testing lessons for Math Circles, the Math Circle community is able to develop much more sophisticated ideas for engaging children in mathematical thinking. Math Circle leaders are uniquely qualified to find interesting questions that can involve children in a particular content area via the exploration of a fascinating idea. They can also see interconnections between multiple areas that might not occur to someone less versed in advanced mathematics. These ideas should then be developed so that they are accessible to teachers and disseminated via web pages, Teachers' Circles, classroom demonstrations, and professional development workshops for teachers. (One caveat here is that it is far better when teachers follow the lead of their students rather than a lesson plan. However, teachers do need some support to build ideas for potentially fruitful questions, vistas of potential interest, and relevant math topics. I think that "lesson plans" should be written to point out some of the many possible directions the topic could take. Professional development workshops should encourage teachers to build more flexibility into their practice.) As technology emerges, it will be increasingly possible to make these lessons directly available to students who wish to explore interesting areas of mathematics.

5. The establishment of regional Math Circle Centers throughout the country would provide greater access to a full range of math outreach programs for people of all ages, professional development for current teachers, and support for schools and districts. Math Circle Centers would serve as regional hubs for efforts to improve the quality of math education in the communities where they are located. The Riverbend Community Math Center was founded with the goal of serving as such a center. We hope to encourage other regions to establish similar initiatives.

What is a Math Circle?

Of course, no single definition of a Math Circle will be perfectly satisfactory. Our community has created Math Circles with a wide range of sizes, audiences, purposes, and leadership. My belief is that the Math Circle movement should be as big a tent as possible. However, I want to use the term "Math Circle" in a more precise sense in what follows. Outreach programs which are not Math Circles as defined below should still be included in the Math Circle community. I also don't mean to imply that programs with other formats should not be called "Math Circles".

In my opinion, a Math Circle is a bit more than just a fun program where people play with math-related activities. To me, a Math Circle implies that the leaders of the Circle have a deep understanding and appreciation of mathematics, that the participants are interested in having a conversation about mathematical ideas, and that the group actually succeeds in doing meaningful mathematics together.

Let me give a few non-examples.

At the Riverbend Community Math Center, we offer Math Studio programs for students who have not yet realized how much fun it can be to discuss math. This program involves an attractive hands-on activity that entices children to choose to participate. While most children simply engage with the activity on a surface level at first, the facilitators try to get children to record observations, look for patterns, make conjectures, and prove theorems. The format for a Math Studio is open in that children are free to start and stop when they choose during any given session. Over the course of a semester, we often find that spontaneous Math Circles emerge as students spontaneously form amicable groups interested in working together more formally. However, the Math Studio itself is not a Math Circle, and most students in the program are not predisposed to be interested in mathematical conversations. In particular, attempts to coerce students to commit to an hour-long Math Circle before they are ready usually result in either frustration for the leader, or the necessity of retreating to a more exploratory style with limited mathematical depth. It is difficult to have a deeply satisfying discussion about mathematical ideas when some participants are not sure they want to do that. Most students can learn to acquire a taste for this kind of discourse, but they often require other kinds of experiences to arrive at that place.

As another non-example, we also commonly offer week-long summer professional development workshops with stipends for teachers. We typically offer our workshops in partnership with a school district to chosen representatives from each school, or to all math teachers at certain schools. Many of the teachers would not choose to participate in a math workshop if they were not requested to attend by an administrator, or were not paid a stipend. It is our job in these cases to persuade the teachers that mathematics can be fun, and that they can learn how to teach math effectively to students while enjoying the experience. It is also our job to design the workshop so that it will be of immediate practical use to them in their classrooms. If we do our job well, the group may begin to function a little like a Math Teachers' Circle over time, or a subset of teachers interested in forming a Math Teachers' Circle may emerge. A Math Teachers' Circle, however, requires the participants to choose to attend voluntarily because they are interested in deepening their teaching practice, developing a professional community, learning more mathematics, and/or enjoying the exploration of new mathematical ideas. (There is no problem if a Teachers' Circle offers a stipend or works with a school district – I am just saying that the nature of the enterprise changes completely if external factors provide the principle motivation for many participants.)

What Characterizes a Math Circle?

There are many formats and topics that can launch a good Math Circle. A Math Circle should include the following goals for participants.

- Participants learn new mathematical ideas or come to new insights about them.
- Participants draw on their prior experiences with math, thereby strengthening their understanding of concepts they knew before and arriving at a new sense of the interconnectedness of the field.
- Participants learn about the experience of doing mathematics. This includes learning how to ask good mathematical questions, attack problems, deal with being stuck, define new terms, use diagrams appropriately, use logic, and understand the value of proof. Participants should also become familiar with common kinds of mathematical reasoning.
- Participants learn to interact appropriately with peers in the context of a mathematical investigation. This includes communicating effectively in verbal and written form, balancing personal emotional needs against those of the group, building a collegial atmosphere capable of producing interesting mathematical insights, and enjoying the process of mathematical discovery.
- Participants are excited, engaged, and enjoy the experience overall (though frustration and

emotional turmoil is sometimes a valuable component). Participants come to feel that they are part of a group where they can safely make mistakes or propose wild ideas.

When choosing topics for a Math Circle, there are several characteristics worth striving for.

- I prefer to select topics where I am aware of many possible directions the group might take, and where there are probably directions which I have not considered. This allows the process of discovery and exploration to be genuine rather than contrived, which sometimes happens if I have a particular conclusion I would like the group to reach. That said, there is no problem with serving as sherpa to a group by letting them know that there is an interesting scenic overlook near the path they are taking.
- Another worthwhile characteristic to strive for is to select a topic which has great depth and connections to many areas of mathematics. This is difficult for me as a leader to always achieve, because even if there are lots of questions that we might investigate in a given corner of mathematics, I may not know enough about the connecting fields of mathematics to understand how we could take it farther. It can also happen that a topic is soon exhausted. A great topic invites at least some students to continue on their own to see where a given path can lead.
- Yet another consideration is the aesthetics of the topic. It is important to me as a Math Circle leader to choose an opening question that is of interest to me personally, and which I think will also interest the participants. A topic may attract me because it is quirky or playful, because it leads to awesome vistas of mathematical beauty, because knowledge of the topic will be of social utility in discussions with others, or because the topic has applications of potential use to our society.
- It is important to consider the interests of the group. However, it is usually not appropriate to simply let the participants choose topics. A Math Circle has a leader with more mathematical sophistication for a reason. The leader will usually be aware of topics that are new to the group, will have more understanding of the social and intellectual norms of the larger mathematical community, will be able to steer a group clear of some pitfalls when they would not contribute postively to the experience (or steer them into the pitfall when it is of value), and will be able to fashion the experience so that it is fun and benefits the participants at multiple levels.

Obviously, there is nothing easy about being a good Math Circle leader, but I find it tremendously satisfying to strive towards these goals even when I fall short of my hopes.

Audience, Goals, and Topics In Classroom Instruction and Math Circles

What are the differences between teaching in a Math Circle and teaching math in a classroom? Three potential differences between the two formats are the audience, participant goals, and topics chosen.

There is some difference in the audience for a math classroom and a Math Circle. My definition of a Math Circle implies that the students are voluntary participants. This is sometimes true in a classroom, and sometimes very far from the case. High school and college students sometimes have the chance to choose courses or work to enroll in more advanced courses. There are even elementary schools where students have chosen to be there. However, most students in K-12 do not have any choice about enrolling in a given course, and many pre-service teachers do not have options about which math courses they take. This means that classroom instruction requires great skill on the part of the teacher to build the motivation of participating students. It also means that topics that could work very well for the subset of students who are ready will not necessarily work with the larger group at the beginning of the course. In some cases, the only thing that can be attempted within the time alotted for the course is to try to improve student attitudes towards learning in general, and math in particular.

Despite the potential difference in audience, I think that the participant goals listed for Math Circles listed in the previous section also apply to a math classroom. Several of these goals are explicitly described in the new Common Core State Standards. Even though these goals are not currently realized on a wide-spread basis, I think that great teachers have always strived towards these goals, and the new standards make it clear that these goals represent the prevailing direction the math education community would like to take in the future.

There is a larger difference in the nature of the topics chosen for a Math Circle and for a classroom. In a classroom, the topics to be included are usually dictated to some extent by a syllabus, a district plan, or a set of grade-level standards. However, teachers usually have quite a bit of freedom in deciding how they will implement the topics. Even in cases where an administration wishes to keep all students of a certain age on the same section throughout the district, many teachers still exercise quite a lot of independent choice about the approach taken. However, even with freedom to choose the implementation, the restraint of "covering" certain topics is enough to make it difficult to also freely explore topics just because they might be interesting or because they help to reveal the nature of mathematics. In other words, while some topics commonly explored in Math Circles are perfectly suited to the goals of a math classroom, others are less likely to be chosen for that purpose.

What Would Happen If Math Classes Were Replaced By Math Circles?

Math Circles provide students with a rich and multi-faceted view of mathematics which is often much more rewarding than a traditional math classroom. Many mathematicians have asked me in all seriousness whether it would not be better to simply replace math classroom experiences with Math Circles for all ages. While I do think that Math Circle ideas have applicability in classrooms, I see a few drawbacks to this notion. For students who are not psychologically ready for Math Circles, another approach would be necessary before a Math Circle could occur. While the traditional classroom approach is also not usually effective for such students, the classroom approaches now advocated by the wider math education community (or math outreach programs with other formats) would be more appropriate. Since most students in this country are currently in this category, these intermediate outreach programs would need to be included in any initiative wishing to have a systemic impact.

There are also some drawbacks to the general notion of replacing formal math instruction with informal programs. It is important that students are exposed to a full range of math topics. A math class is designed so that students are exposed to ideas systematically. A Math Circle may draw on many of these ideas over time, and it may give a richer view of those ideas, but there is no guarantee that any of the students have a chance to achieve mastery of the full spectrum of topics they should know. Another difficulty comes in upper elementary and middle school when students need to begin acquiring some discipline with respect to their written mathematical work. Students need to learn to express their mathematical thinking so that it is neat and comprehensible to others. I have known several quite sophisticated home-schooled students who participated in our Math Circles for many years. Some of them were afraid of the word "algebra" and were not sure they were ready to start learning it even though they had been playing quite capably with ideas from algebra, trigonometry, calculus, and group theory for quite a while, and we had introduced those terms as appropriate. They had not been exposed to a formal algebra course in school, and so they had no way to evaluate or categorize what they knew. It is important as a student to acquire the meta-knowledge of how mathematical knowledge is traditionally organized, and there comes a time when a student needs to discipline themselves enough to work through a standard textbook. I can imagine other ways of achieving the larger goal, but there is something to be said for learning how other people in our society understand standard math topics.

How Many Math Circles Could the Mathematical Community Sustain?

I believe that the Math Circle movement may be able to develop to the point where most university math departments sponsor at least one Math Circle, and possibly several Circles for various groups of students and teachers. Ideally, this would place a Math Circle within the reach of any student or teacher in the country who is sufficiently determined to join one. As a practical matter, I think it will only be possible for a tiny proportion of the teachers and students in any given area to participate in a Circle. In other words, I believe that the number of people directly impacted by participation in a Math Circle will always be quite small.

How Can the Math Circle Movement Support Systemic Change?

From the discussion above, we might suppose that the potential of the Math Circle movement to effect systemic change is vanishingly small. I nevertheless believe that the Math Circle movement is uniquely positioned to help with certain aspects of the systemic change many of us would like to see in K-12 math education. I believe that the Math Circle movement can help to influence teaching practices in math departments, to change attitudes among mathematicians about professional contributions towards teaching and outreach, to improve the preparation of future teachers, and to provide current teachers with models for engaging children in mathematical thinking. In addition, the Math Circle community could work to establish regional Math Circle Centers throughout the country. These Centers would offer a full range of math outreach programs, professional development, and support for schools. They would serve as regional hubs for efforts to improve the quality of math education in the communities where they are located.

Influencing Teaching Practices in Math Departments

One of the most critical changes that I think the Math Circle movement may be able to effect, relates to its impact on the community of mathematicians rather than its impact on teachers or children. Because Math Circles are fundamentally about fostering delight in mathematics, most mathematicians can relate to the mission more easily than they can relate to movements centering around specific pedagogical techniques. Math Circles of any variety inevitably bring mathematicians into at least indirect contact with the K-12 math education system. Raising awareness among mathematicians about the importance of teacher preparation is a prerequisite for systemic changes in the attention given to future teachers.

Additionally, because the Math Circle community promotes informal and conceptual approaches to mathematical topics, mathematicians who participate in Math Circles are likely to learn and practice new kinds of teaching techniques which may impact their teaching practices in the classroom. One example of such an improvement would be the inclusion of more critical thinking tasks for students at all levels (as opposed to requiring only the memorization of procedures or proofs). Another example is careful thinking about the evolving needs of students (rather than simply following course designs that may be outdated). Still another example is including more project-based learning and other forms of active classroom engagement (rather than following a lecture-based format). Math Circle leaders often learn to foster discussion and debate among the students as opposed to telling students how to work a given type of problem. Incorporating this approach is often beneficial in a college classroom as well.

Influencing Attitudes About Professional Contributions Towards Teaching and Outreach

The field of mathematics grows through new research results, and universities have always attracted students by hiring faculty members on the cutting edges of their fields. It therefore makes sense that research contributions constitute the principle criterion for tenure and promotion of mathematicians (at least at larger universities). There is an increasing tendency in some math departments to give at least some weight to the teaching and university service components of a candidate's curriculum vitae.

One difficulty is that mathematicians typically receive very little professional credit for service beyond the university. One of the goals of the Math Circle movement is to make it more possible for the math outreach efforts of faculty members to be acknowledged formally by their colleagues. I believe that a change in this direction would help to build a more supportive community of mathematicians. Because Math Circles also encourage leaders to explore more fields of mathematics than they otherwise might, I also think that Math Circles will benefit our community mathematically.

Responsbility For the Preparation of Teachers

What institution should be responsible for training pre-service math teachers? I argue that the answer to this question from a systemic perspective can only be tenured mathematicians at universities.

All pre-service teachers obtain their degrees and certifications from universities. Which departments within the university structure have the responsibility to do this? The answer to this must either be schools of education or math departments, and the balance of the responsibility varies from institution to institution. It is a tragic reality that very few universities have strong working relationships between members of these two groups, and that there are very few instances of collaborative efforts between national organizations representing mathematicians and math educators.

If the responsibility for training future math teachers lies with schools of education, whose job is it to actually see to it that the preparation aligns with the needs of K-12 schools? Who is responsible for making sure that pre-service coursework aligns with changing requirements such as the new Common Core State Standards? There simply are not very many people with doctorates in math education, and most universities do not have any tenured math education faculty members. In those universities, adjunct faculty or lecturers teach all of the math methods courses, and they often lack the authority (not to mention the pay) to consider systemic matters. And what about those universities fortunate enough to have tenured math education faculty? Unfortunately, it is often not the job of such faculty members to consider systemic math education issues. This is because tenure and promotion in math education is based on research just as it is in math departments. A well-controlled statistical study of the pros and cons of a given pedagogical approach for a given age-level tends to cause the researcher to focus narrowly on a specific educational question. If the researcher interacts with local teachers and students at all, it will typically be restricted to a small segment of the K-12 math education system (for example, to conceptions of equality among 5-year-olds or the emergence of algebraic thinking among 12-year-olds.

Future teachers usually receive most of their math-related training from math departments. Elementary teachers typically take 3 or 4 courses in the math department before taking the one required math methods course in the school of education. Middle school and high school teachers take many more courses in the math department before taking their math methods course. Who sees to it that these future teachers receive high-quality instruction and coursework that aligns with their future needs? I argue that mathematicians are ultimately responsible.

It is common to delegate math courses for elementary teachers to non-tenured faculty, because it is not usually considered prestigious for tenure-track faculty members to teach non-major courses. Future high school math teachers usually take the same courses as math majors who plan to pursue graduate school. But it is too common for faculty members to feel more affinity for students bound for graduate school than for students who plan to teach. As a result, courses are often more suited to the needs of future graduate students than to the needs of future teachers. To be fair, it is also true that mathematicians are more likely to understand the needs of future graduate students. Future teachers need to experience learning environments like those they will be expected to create for their own students. The new Common Core State Standards explicitly require more active engagement on the part of students than is provided by traditional lecture-based courses. The Common Core State Standards also require specialized content knowledge that is not necessarily included automatically in a math major. It is no longer appropriate to suppose that the math department will impart content knowledge only to future teachers, and that the one methods course they take will fill in all of the pedagogy they need to know.

It is my hope that by bringing more mathematicians into direct contact with teachers and K-12 students, Math Circles will increase awareness of teacher preparation issues among the mathematicians who are in positions to improve pre-service teacher courses.

Providing Teachers With Models For High Quality Math Instruction

The Math Circle community is rapidly developing a body of math lesson ideas which engage children in mathematical thinking. These ideas are developing at a time when teachers desperately need new approaches to math instruction that satisfy the requirements of the Common Core State Standards. Some of the ideas already developed by the Math Circle community are a perfect fit for the needs of math classrooms. Math Circle leaders are uniquely qualified to develop new lesson ideas that explore given topics while allowing young students to experience what it means to do mathematics. Math Circle leaders can try these ideas out with a small group of enthusiastic students first, and can then work on adapting them for classroom use.

In order to make this possibility into a reality, members of the Math Circle community need to educate themselves about the Common Core State Standards, become familiar with the needs of teachers and students in classrooms, work purposefully to develop the needed lesson and course ideas, and package them into forms that are accessible and useful to teachers (possibly including in-class demonstrations, lesson plans, and videos). This last step is antithetical to the approaches taken in Math Circles – we usually follow the lead of the students rather than following a predetermined script. However, it is still possible to describe the launching sequence and explain some of the terrain likely to be explored. Professional development workshops for teachers should emphasize the importance of implementing any lesson plan with great flexibility and should foster teacher confidence in developing their own ideas and following the lead of their students.

The Common Core State Standards

The Common Core State Standards have emerged out of recommendations by the National Council of Teachers of Mathematics (NCTM) made in the early 1990s. Many states implemented their own sets of math standards based on these recommendations. Several years ago, governors from many states felt that the time had come to unify these standards into a single approach, so that textbooks and tests would work equally well in any state and to bring unity to the math education of citizens of the United States. The Common Core State Standards are slated for implementation in most states by 2014. They contain an explicit emphasis on critical thinking. For each computational skill, the standards ask that students first work with the skill concretely, then develop flexible approaches to that kind of computation, and only later memorize facts and algorithms. The mastery of algorithms is accordingly delayed in the Common Core State Standards; algorithms for multi-digit addition and subtraction do not appear until 4th grade (rather than 2nd grade). However, students in 2nd and 3rd grade do work with multi-digit addition and subtraction using first concrete models, and later a variety of strategies. Similarly, the standard algorithm for long division does not appear until 6th grade, but students in 4th and 5th grade still perform multi-digit division. At the same time, the seeds of algebraic thinking are to be cultivated in students even from Kindergarten, with unknowns, the meaning of equality, and properties of operations listed explicitly in the standards.

Overall, I find the Common Core State Standards to be a realistic portrayal of the kinds of math content that students of a given age should have the chance to learn. It seems helpful to me to have a list of some topics that students of a given age should be exploring. It is less clear to me whether every 10 year-old should be expected to master all of the 4th grade standards before they are 11, for example. I would prefer to see some allowance for students who need to learn at different paces. Some students need more time to achieve mastery of certain concepts, and students sometimes contend with personal and family issues that make learning difficult at a given time. Other students master the concepts associated with the standards quite quickly and need additional challenges to remain intellectually engaged in mathematics. It would be nice if there were sufficient flexibility in the system to allow students to learn at the pace most appropriate for them. I do think it is helpful for both students and teachers to have some indication of a student's current level of mastery of math concepts. There is a tendency to simply promote students along the grade levels because of their chronological age. This approach often robs students of opportunities to learn critical thinking skills, because teachers who are desperate to make up for lost time resort to simply teaching rules and algorithms students who are not prepared for the content to be taught. It takes much more time (in the short run) to allow a student to understand a concept concretely, and then to devise flexible alternate approaches. However, in the long run, students do not remember or trust rules and algorithms without the necessary conceptual foundations. They also become increasingly anxious as the subject makes less and less sense to them. In the long run, algorithmic approaches taught in the name of efficiency actually impede the learning of mathematics. There is no royal road to mathematical understanding.

One huge challenge of the new Common Core State Standards is that most adults in our society are not equipped to understand, and would not know how to teach, the math standards beyond a 2nd grade level. This is equally true of elementary (and sometimes middle school) teachers, because they were never trained that way as children, and because traditional classes for pre-service teachers do not sufficiently address the deficiencies. While it is true that students come to school with many personal challenges that can make learning difficult, my experiences visiting many schools suggest that the true root cause of our nation's poor math education lies in the fact that most 3rd through 6th grade teachers are not comfortable with the mathematical content they are teaching. This is particularly true of concepts such as models of rational numbers, multi-step problem solving of any kind, concepts involved in the transition to algebraic thinking, and geometric reasoning.

Even in cases where 3rd through 6th grade teachers do know how to perform the appropriate algorithms, the new Common Core State Standards require teachers to have a much deeper understanding of both the content and the pedagogy relating to these topics than they have traditionally been prepared to have. In other words, the successful implementation of the Common Core State Standards will require a significant nationwide effort to retrain and support 3rd through 6th grade math teachers. Additionally, while there are some K-5 curricula which could be used to implement the new Common Core State Standards, there are very few 6th to 12th grade curricula that are even somewhat adequate (and those few are either expensive or incomplete).

It is rare for 3rd through 6th grade teachers to specialize so that they teach only one subject. I think that having the opportunity to focus on a single topic is one straight-forward way to improve math instruction in those grade levels, because it would allow teachers to seek out the professional development they need to be successful. In recent decades, it has been common in some school districts to assign teachers with general elementary licenses to teach 7th and 8th grade math. This practice is now being legislated away in most states, with teachers scrambling to acquire emergency licenses or quick add-ons to meet the requirements. The result of this is that at this time, too few qualified 7th and 8th grade math teachers are available, and so the systemic math education difficulties that begin in 3rd through 6th grade are compounded in those two grade levels.

High school is usually the first grade level when students come into contact with teachers who have ever taken Calculus (actually high school math teachers generally have a complete math major). High school math teachers are usually quite capable in terms of their content knowledge. However, because new legislation in most states prohibits students from earning high school credit for courses below Algebra 1, students who arrive at high school with gaps in their understanding of pre-algebraic fundamentals are not given much opportunity to remedy the situation. Many schools offer a remedial pre-algebra course that is taken concurrently with Algebra 1, but this course is sometimes taught by teachers who do not have a secondary math license. Even in cases where it is taught by a fully qualified teacher, it is difficult to make up 3rd through 8th grade content in one year when the students are already demoralized. Among students who are able to "pass" pre-algebra and go on to the usual high school sequence, it is quite common for gaps in pre-algebraic fundamentals to persist through high school. It is these fundamental skills with rational numbers, multi-step problem solving, and geometric reasoning which are missing in most graduating high school students. It is these skills which are most critical for our workforce, and it is these skills which are preventing people in this country from finding work when there are so many skilled and semi-skilled positions unfilled.

The Process-Oriented Common Core State Standards

In addition to content-specific standards, the Common Core State Standards include 8 Standards For Mathematical Practice that educators at all levels should seek to develop in their students. These process standards are well-supported by the approach to mathematics commonly used in Math Circles, and are not as well integrated into most existing school curricula.

1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Critical Math Content Areas

Critical Concepts in Grades K-2

In the past few years, I have had the chance to work with many teachers as they prepare to implement the Common Core State Standards in math. The following concepts are the most critical in grades K-2 in terms of the amount of time students need to spend to learn them. My belief is that we need to encourage a shift away from thinking that math instruction consists of following a textbook from one end to the other. A textbook or other book can be a resource to help teach some topics, but even when it is followed for part of math instruction, there should be other components that do not come from the textbook. There are some topics that take a full year of instruction for students to become sufficiently sophisticated with them. Time should be allotted every week for work with these topics outside of the progressing sequence of topics in the rest of the curriculum. I am appending a draft of charts for the pacing these topics in grades K-7.

These topics are also areas where teachers frequently ask for additional training and ideas for lessons.

Flexible operations with, and decompositions of, numbers: Most K-2 teachers have many ideas for representing numbers concretely, but they tend to skip quickly from that stage to memorizing basic facts. The Common Core State Standards ask that students have time for a middle stage where they learn to operate with numbers more flexibly and creatively. Students should be experiencing challenges which ask them to analyze number patterns, devise their approaches to operations, and evaluate which of several possible approaches might be best in a given situation.

Authentic use of geometric vocabulary, with emphasis on attributes: Many K-2 teachers have asked me whether it is really necessary for students to learn that the same shape (a square) can be called both a square and a rectangle. The Common Core State Standards expect students to understand that geometric vocabulary describes the attributes of a shape that are of interest at the moment. This idea is new to some teachers. I am often asked about issues such as whether irregular polygons are still shapes, distinctions between convex and concave polygons, simple polygons and those with intersecting edges, whether the point at the tip of a cone counts as a vertex, and other similar issues relating to geometric vocabulary.

Autonomous problem solving: Students need exposure to a variety of problem solving opportunities. Teachers who are uncomfortable with math themselves have a tendency to control the problem solving process for their students, presenting a simple one-step problem and then guiding the students through every mark to be placed on the page. Students should be exposed to multi-step and open-ended problem solving situations even starting in Kindergarten, and they need chances to struggle with the problems individually and in small groups without being told "the answers". Problem solving is much more than solving word problems, and ideas for bringing some of these other types of problem solving to classrooms are much needed.

Logic puzzles, strategy games, and brain teasers: These kinds of activities are NOT mentioned in the Common Core State Standards. It is important for mathematicians working with teachers to realize that these kinds of activities will not be immediately applicable to their classrooms (other than as enrichment activities). However, my opinion is that experience with these kinds of ideas at an early age is quite important to the attitudes that students develop about mathematics. They also develop critical thinking skills and habits of mind that build a foundation for later mathematical experiences. Therefore, I mention these topics here as a critical kind of enrichment that K-2 students may only get to see during Math Circles or on special occasions in the classroom.

Critical Concepts in Grades 3-5

Meanings, applications, and flexible use of basic operations: Teachers usually are aware of concrete ways of modeling the four basic operations. However, they often are not aware of different classes of applications where those operations emerge. A variety of applications for addition, and to some extent multiplication, are commonly presented in classrooms. However, subtraction and division applications are commonly not explored as fully. Subtraction is frequently modeled only as "take away". It is much less common to see classroom lessons using subtraction to model situations involving comparison or distance between two numbers. Students commonly have limited command of language relating to additive versus multiplicative comparisons, though this topic is now required in the Common Core State Standards for 3rd grade. Students and teachers both need to be aware that $12 \div 3$ could be interpreted as either 12 items divided into three equal groups or as 12 items divided into groups of three. Students need a large repertoire of situations where the various operations apply.

Conceptual transitions to computational algorithms: The Common Core State Standards encourages students to use a variety of ad hoc methods to add, subtract, multiply, and divide multi-digit numbers before exposing them to a standard algorithm. Teachers need ideas for helping students to develop these methods and ultimately analyzing why various algorithms work.

Problem solving requiring measurements with rational numbers: Students need lots of meaningful hands-on measurement experiences with fractions and decimals. They need concrete experiences with length, area, capacity, mass/weight, temperature, time, and angles.

Conceptual approaches to rational numbers: One of the most critical needs is to find interesting math questions that familiarize students with concrete representations of fractions and decimals. Students should work with area models, set models, and number line models. Although there are many lesson ideas for teachers that address this need, most of them do not simultaneously also meet the goal of being intrinsically compelling, related to other areas of math, or showing a fascinating glimpse into what it means to do math. I believe that the Math Circle community is able to generate ideas that would allow 3rd to 5th grade students to acquire a deeper understanding of both rational numbers and mathematics.

Critical Concepts in Grades 6-8

Understanding number systems: Middle school math classrooms desperately need a more sophisticated and interesting approach to understanding number systems. The traditional approach simply lists some properties of real numbers and shows a Venn diagram depicting natural numbers, whole numbers, integers, rational numbers, irrational numbers, real numbers, and complex numbers. Middle school students should be invited to think like anthropologists or lawyers, discovering properties of both familiar and strange number systems. They should know how to prove that the square root of two is irrational and should hear the tales about how that proof shattered the preconceptions of the Pythagoreans. They should play with groups and other number structures that lack commutativity or other properties. They should learn to discover properties of unfamiliar systems and use those properties to make conclusions about what is possible or impossible. They should be exposed to the audacity of Cantorian Set Theory, thereby coming to a deeper understanding of the nature of the infinite.

Algebraic modeling: Middle school students need to learn how to devise formulas and equations to fit novel multi-step contexts. Students should learn to use variables in spreadsheets and programming languages to model situations with independent and dependent variables or evolving dynamical systems. Students especially need exposure to concrete examples that involve rates, ratios, and slope.

Proportional reasoning: Middle school students should work with problems requiring use of scale, percents, unit conversions, and other forms of proportional reasoning.

Critical Concepts in High School

The Common Core State Standards require quite a few changes in the way high school math is traditionally taught. Students should see more concrete applications of topics, should use more higher level critical thinking skills rather than simply memorizing algorithms, and should come to understand the nature of mathematics as a discipline, especially including proofs.

Modeling: The Common Core State Standards include an emphasis on mathematical modeling in every high school course. Very few existing high school textbooks support this need. High school teachers are usually quite well-trained, but the new approach will involve a huge amount of effort to find ideas for reinventing traditional classes to incorporate modeling with sufficient depth.

New topics: The new Common Core State Standards list many specific topics that are not traditionally part of the high school curriculum. These include comparing and contrasting straightedge and compass construction capabilities with other kinds of construction tools (such as origami or graphic design programs), applications of matrix operations, complex analysis, combinatorics, probability, and statistics. It would be helpful for mathematicians and teachers to look at the list of new standards together to brainstorm possible approaches to the new requirements. Many topics commonly explored in Math Circles would fit very nicely with the new approach.

Kindergarten Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Oral Counting	*** By 1s to 30.** *** By 10s to 30**	*** By 1s starting at any # up to 60.** *** By 10s up to 60.**	*** By 1s starting at any # up to 100.** *** By 10s up to 100.** Count backwards from 20.	*** By 1s starting at any # up to 100.** *** By 10s up to 100.** Count backwards from any point up to 100.
Object Counting	Count up to 20 ordered items. *** Show up to 5 fingers with counting.** Count up to 10 scattered objects.	*** Count out up to 20.** *** Show up to 5 fingers without counting.** Introduce number talk for dot cards.	Count out up to 60, forming 10s. *** Show up to 10 fingers with counting.** Develop number talk for dot cards.	Count out up to 100, forming 10s. *** Show up to 10 fingers without counting.** Fluent number talk for dot cards.
Number Recognition	*** Up to 10.**	*** Up to 20.**	Up to 100.	Compare two written numbers up to 10.
Addition and Subtraction		Guided +/− within 10 using fingers, objects, drawings, acting, sounds. Introduce diverse scenarios and equation forms for +/−. Guided problem solving Acting, Pictures, #s, Words.	Independent +/− within 10 using fingers, objects, drawings, acting, sounds. Guided problem writing for equations. Develop independent small group solving, writing, presenting.	Independent +/− within 20 using choice of tools Fluent +/− within 5. Independent small group problem writing for equations. Fluent independent small group solving, writing, presenting.
Shape and Position Vocabulary	Understand: triangle, square, circle, rectangle, hexagon above, behind, in front of, below, beside, next to.	State attributes of: triangle, square, circle rectangle, hexagon.	Understand: cone, cube, cylinder, sphere Classify 2-D and 3-D objects.	Describe scenes in terms of 2-D/3-D shapes and their relative positions.

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 1st grade.**

1st Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Addition and Subtraction	*** Model and solve +/− up to 20 using fingers, objects, pictures, and a variety of other manipulatives.** *** Generate equivalent forms of a # up to 20 using objects, pictures, and number sentences.** *** Students interpret and solve +/− problems within 20.** *** Fluent mastery of +/− facts within 5.**	*** Model and solve +/− up to 50 using fingers, objects, pictures, and a variety of other manipulatives.** *** Generate equivalent forms of a # up to 50 using objects, pictures, and number sentences.** *** Students flexibly interpret and solve +/− problems within 20.** *** Fluent mastery of +/− facts within 10.**	*** Model and solve +/− up to 100 using fingers, objects, pictures, and a variety of other manipulatives.** *** Generate equivalent forms of a # up to 100 using objects, pictures, and number sentences.** *** Students fluently use and articulate strategies for +/− within 20.** *** Fluent mastery of +/− facts within 10.**	*** Model and solve +/− up to 120 using fingers, objects, pictures, and a variety of other manipulatives.** *** Generate equivalent forms of a # up to 120 using objects, pictures, and number sentences.** Students use and articulate strategies for +/− within 120. *** Fluent mastery of +/− facts within 10.**
Place Value	*** Count, read, write #s to 120, grouping in 10s.** *** Arrange given numbers up to 10 in order.**	*** Count out up to 120 objects, grouping in 10s.** *** Arrange given numbers up to 50 in order.**	*** Find the # that is 1 more / 1 less.** *** Arrange given numbers up to 100 in order.**	*** Find the # that is 10 more / 10 less.** *** Arrange given numbers up to 120 in order.**
Problem Solving	Guided problem solving Acting, Pictures, #s, Words.	Develop independent small group solving, writing, presenting.	Fluent independent small group solving, writing, presenting.	Fluent independent small group problem writing to match given equation.
Linear Measurement	Order objects by length by direct comparison.	Use identical objects for non-standard measuring.	Use one short object iteratively to measure.	Compare lengths of two objects indirectly.
Geometry	Identify/draw/build cones rectangles, squares, cubes, trapezoids, half-circles, quarter-circles, prisms, triangles, cylinders.	Sort shapes by attribute Distinguish defining and non-defining attributes.	Compose and decompose shapes.	Partition circles and rectangles into halves and fourths/quarters.

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 2nd grade**

2nd Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Place Value	*** Count by 1s, 5s, 10s, and 100s forwards or backwards up to 1000.** *** Arrange given numbers up to 100 in order.**	*** Represent #s to 1000: models, drawings, #s equations.** *** Arrange given numbers up to 1000 in order.**	*** Use models, drawings, numbers, to +/− #s up to 1000.** Use $<$, $=$, and $>$ to compare numbers.	Relate a concrete approach for +/− within 1000 to mental math strategies and written methods. Use $<$, $=$, and $>$ to compare numbers.
Addition and Subtraction	*** Use / articulate mental math strategies for +/− within 20.** *** Fluent mastery of +/− facts within 10.**	*** Use / articulate mental math strategies for +/− within 100.** *** Fluent mastery of +/− facts within 20.**	*** Mentally +/− 10 or 100 to/from a given # within 1000.** *** Fluent mastery of +/− facts within 20.**	Use / articulate mental math strategies for +/− within 1000. *** Fluent mastery of +/− facts within 20.**
Problem Solving	Guided problem solving Acting, Pictures, #s, Words. Include: Count coins within \$.20 Count money within \$100 Lengths up to 20 units Line plots, tables, graphs.	Develop independent small group solving, writing, presenting, creating problems for equations. Include: Count coins within \$1 Count money within \$1000 Lengths up to 100 units Line plots, tables, graphs.	***Fluent small group solving, writing, presenting, creating problems for equations.** Include: +/− coins with exchanging +/− \$ with exchanging Lengths to 100 units Line plots, tables, graphs.	***Fluent individual solving, writing, creating problems.** Include: +/− coins with exchanging +/− \$ with exchanging Lengths with mixed units Line plots, tables, graphs.
Measurement	*** Measure/construct lengths using copies of a unit.** Tell time to the half hour.	*** Measure/construct lengths using rulers, yard/meter sticks, tape measures.** Tell time to the quarter hour.	*** +/− several lengths and construct sum or difference.** Tell time to nearest 10 min.	Measure same object with different units. Predict measurements from others. Tell time to nearest 5 min.
Geometry	Identify/draw/build cubes triangles, quadrilaterals, pentagons, hexagons.	Recognize and draw shapes with specified attributes.	Partition circles and rectangles in halves, thirds, fourths.	Decompose rectangles into unit squares.

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 3rd grade**

3rd Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Addition & Subtraction	*** Use manipulatives to +/− within 1000.** *** Fluent mastery of +/− facts within 20.**	*** Students fluently use / articulate strategies for +/− within 1000.** *** Fluent mastery of +/− facts within 20.**	*** Students fluently use / articulate strategies for +/− within 1000.** Fluent mastery of +/− facts within 20.	*** Students fluently use / articulate strategies for +/− within 1000.** Fluent mastery of +/− facts within 20.
Multiplication & Division	Interpret / model ×/÷ up to 100 using objects, pictures, 100 charts, number lines, money, rekenreks, arrays, grid paper. *** Generate equivalent forms of a # up to 100 using objects, pictures, and number sentences.**	Students use models and articulate strategies for ×/÷ within 100. Generate forms of a # up to 1000 using objects, pictures, and # sentences.	*** Students fluently use and articulate strategies for ×/÷ within 100.** *** Flexibly generate forms of a # up to 1000 using objects, pictures, and number sentences.** *** Mastery of ×/÷ facts with factors up to 5.**	Students use and articulate strategies for ×/÷ within 1000. *** Flexibly generate forms of a # up to 1000 using objects, pictures, and number sentences.** *** Mastery of ×/÷ facts with factors up to 10.**
Problem Solving	Transition from guided to independent small group Acting, Pictures, #s, Words. Create problems for equations. Include: +/ − / × /÷ interpretation Fraction problems Time Lengths with mixed units Line plots, tables, graphs Scaled picture graphs Area.	Fluent small group solving, writing, presenting, creating problems for equations. Include: +/ − / × /÷ interpretation Fraction problems Time Lengths with mixed units Line plots, tables, graphs Scaled picture graphs Area.	***Fluent individual solving, writing, creating problems.** Include: +/ − / × /÷ interpretation Fraction problems Time Lengths with mixed units Line plots, tables, graphs Scaled picture graphs Area.	***Fluent small group solving, writing, creating problems.** Include: +/ − / × /÷ interpretation Fraction problems Time Lengths with mixed units Line plots, tables, graphs Scaled picture graphs Area.

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 4th grade**

3rd Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Fractions	Introduce area, set, # line models with halves, thirds, fourths. Include money, time, rulers, hundred grids, cup measures, tokens.	* **Use both ratio and equal groups approaches fluently to model fractions with area, set, # line models. Include money, time, rulers, hundred grids, cup measures, tokens.**	* **Use area, set, # line models, and assorted manipulatives to show equivalent fractions.**	Use concrete models and reasoning to compare fraction sizes.
Measurement	* **Use up to 100 physical square inches, square cm grids to find areas and perimeters of rectangles. Construct square yards, meters, and feet and use them to measure areas.** * **Measure/construct lengths using rulers, yard/meter sticks, tape measures.** Tell time to nearest minute.	* **Use strategies (including distributive property) to find areas and perimeters of rectangles without counting every square.** * **Measure/construct lengths accurate to 1/4 inch or 1/2 cm.** Include 1/2, 1/4, 1/3 foot, yard, meter. Strategies for elapsed time.	Use strategies to find areas and perimeters involving large areas or shapes composed of rectangular regions. Measure same object with different units. Predict measurements from others. Include simple fractions. Strategies for elapsed time.	Use strategies to find areas and perimeters involving large areas or shapes composed of rectangular regions. Include figures with unknown side lengths. Measure same object with different units. Predict measurements from others. Include simple fractions. Strategies for elapsed time.
Geometry	* **Flexibly and accurately partition circles, rectangles, and other shapes into halves, thirds, and fourths, attending to accuracy.**	* **Flexibly and accurately partition circles, rectangles, and other shapes into halves, thirds, fourths, . . . twelfths, attending to accuracy.**	Decompose shapes with right angles into rectangles and unit squares.	Identify/draw triangles, quadrilaterals, pentagons, hexagons, octagons. Classify quadrilaterals.

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 4th grade**

4th Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Numbers and Operations with Algebraic Thinking	*** Generate equivalent forms of a # up to a million using objects, pictures, words, and # sentences with mixed operations.** *** Demonstrate mastery of place-value up to a million and for decimals to 1/100's.** Use concrete models for multi-digit $\times/\div$. *** Demonstrate mastery of basic $+/-/\times/\div$ facts.**	*** Flexibly generate true numerical equations using objects, pictures, words, # sentences.** *** Use $+/-$ algorithms with whole #s and explain the connection with a concrete model.** Use concrete models for multi-digit $\times/\div$. *** Demonstrate mastery of basic $+/-/\times/\div$ facts.**	*** Use numerical expressions to represent multi-step, multi-operation scenarios.** Use concrete models to add and subtract tenths and hundredths. *** Fluently use strategies for multi-digit $\times/\div$.** *** Demonstrate mastery of basic $+/-/\times/\div$ facts.**	Generate multi-step, multi-operation equations based on multiple ways of looking at geometric and scenario-based patterns. Use concrete models to add and subtract tenths and hundredths. *** Fluently use strategies for multi-digit $\times/\div$.** *** Demonstrate mastery of basic $+/-/\times/\div$ facts.**
Problem Solving	***Fluent small group solving, writing, presenting, creating problems for multi-step expressions.** Include: Multiplicative and additive comparisons Mixed operations Fraction problems Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter and area.	***Fluent small group solving, writing, presenting, creating problems for multi-step expressions.** Include: Multiplicative and additive comparisons Mixed operations Fraction problems Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter and area.	***Fluent individual solving, writing, presenting, creating problems for multi-step equations.** Include: Multiplicative and additive comparisons Mixed operations Fraction problems Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter and area.	Fluent small group solving, writing, presenting, creating problems for multi-step equations including an unknown. Include: Multiplicative and additive comparisons Mixed operations Fraction problems Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter and area.

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 5th grade**

4th Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Fractions	Use ratio and equal groups approaches to demonstrate fraction equivalence with area, set, # line models. Include money, time, rulers, hundred grids, cup measures, tokens.	* **Use ratio and equal groups approaches to demonstrate fraction equivalence with area, set, # line models. Include money, time, rulers, hundred grids, cup measures, tokens.**	* **Compare fractions/ decimals and determine which of two fractions/decimals is closer to a given fraction/decimal using concrete models and strategies.**	* **Use concrete models to flexibly solve simple fraction** $+/-/\times$ **problems and generate equivalent expressions involving equivalent fractions, decimals,** $+/-/\times$**.**
Measurement	Measure to find equivalent quantities using different units for length, weight/mass, capacity, and time. Include fractions and decimals. * **Measure/construct lengths using rulers, yard/meter sticks, tape measures, to 1/8 inch or .5 cm.** * **Use physical square inches / square cm / square feet to find perimeters and areas of shapes composed of rectangles**	* **Create diagrams and use strategies to solve measurement conversion problems. Include fractions and decimals.** Measure/construct lengths using rulers, yard/meter sticks, tape measures, to 1/8 inch or .5 cm. Use strategies to find perimeters and areas of large shapes composed of rectangles.	Solve and write explanations for real-world "how much/ how many" questions involving large #s, fractions, estimates, measurements, surveys and experiments. * **Measure/construct lengths using rulers, yard/meter sticks, tape measures, to 1/8 inch or .5 cm.** Use strategies to find perimeters and areas of large shapes composed of rectangles. Include problems with unknown side lengths.	* **Solve and write explanations for real-world "how much/ how many" questions involving large #s, fractions, estimates, measurements, surveys and experiments.** * **Measure/construct lengths using rulers, yard/meter sticks, tape measures, to 1/8 inch or .5 cm.** * **Use strategies to find perimeters and areas of large shapes composed of rectangles. Include problems with unknown side lengths.**

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 5th grade**

5th Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Fractions	*** Use ratio and equal groups approaches to demonstrate fraction and decimal equivalence with area, set, # line models.** *** Use area, set, and # line models to explore interpretations of fractions as division of the numerator by the denominator.**	*** Use area, set, and # line models to add and subtract decimals and fractions with unlike denominators.** *** Use area, set, and # line models to illustrate fraction $\times$ (including interpreting $\times$ as area, repeated addition, and scaling.**	*** Use benchmark fractions to mentally estimate sums and differences.** *** Use area, set, and # line models to divide fractions by whole numbers and unit fractions.**	*** Use models to explain procedures for adding and subtracting decimals and fractions with unlike denominators.** *** Write problems and scenarios to fit given $+/-/\times/\div$ operations with fractions**
Numbers and Operations with Algebraic Thinking	*** Generate equivalent multi-step expressions correctly using parentheses and evaluate such expressions.** *** Understand how $\times/\div$ by powers of 10 changes the number of zeros in the answer and the placement of the decimal point.** *** Demonstrate mastery of algorithms for multi-digit $+/-$, and basic $\times/\div$ facts.**	*** Translate verbal calculation instructions into numerical expressions and recognize equivalent ways of writing those expressions.** *** Fluently multiply multi-digit whole numbers using the standard algorithm.** Demonstrate mastery of algorithms for multi-digit $+/-$, and basic $\times/\div$ facts.	*** Use numerical expressions to represent multi-step, multi-operation scenarios.** *** Find quotients of whole numbers using strategies, and relate approaches to models.** Demonstrate mastery of algorithms for multi-digit $+/-$, and basic $\times/\div$ facts.	*** Generate multi-operation equations based on different ways of looking at geometric and scenario-based patterns.** *** Find quotients of decimals to hundredths using strategies, and relate approaches to models.** *** Demonstrate mastery of algorithms for multi-digit $+/-$, and basic $\times/\div$ facts.**

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 6th grade**

5th Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Problem Solving	***Fluent small group solving, writing, presenting, creating problems for multi-step expressions.** Include: Fermi-style problems × and + comparisons Fraction/decimal problems Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, volume.	***Fluent small group solving, writing, presenting, creating problems for multi-step expressions.** Include: Fermi-style problems × and + comparisons Fraction/decimal problems Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, volume.	***Fluent individual solving, writing, presenting, creating problems for multi-step equations.** Include: Fermi-style problems × and + comparisons Fraction/decimal problems Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, volume.	***Fluent small group solving, writing, presenting, creating problems for multi-step equations.** Include: Fermi-style problems × and + comparisons Fraction/decimal problems Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, volume.
Volume	Measure/construct lengths using rulers, yard/meter sticks, tape measures, to 1/16 inch or mm. Use physical cubic inches, cubic cm, and cubic feet to find volumes of structures composed of rectangular solids.	Measure/construct lengths using rulers, yard/meter sticks, tape measures, to 1/16 inch or mm. *** Measure containers and rooms composed of rectangular solids and use whole-number estimates to find volumes.**	*** Measure/construct lengths using rulers, yard/meter sticks, tape measures, to 1/16 inch or mm.** Measure containers and rooms composed of rectangular solids using fractional measurements to find more precise volumes.	Measure/construct lengths using rulers, yard/meter sticks, tape measures, to 1/16 inch or mm. Measure containers and rooms composed of rectangular solids using fractional measurements to find more precise volumes.

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 6th grade**

6th Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Ratios and Rates	* **Use area, set, and # line models to $+/-/\times$ fractions. Use ratio and equal groups approaches to fraction models. Illustrate a quotient interpretation of fractions using drawings.**	* **Use area, set, and # line models to $+/-/\times/\div$ fractions.**	* **Use ratio language to describe relationships between quantities, and use ratio and rate reasoning with models to work with tables/graphs, unit rates, percents, and unit conversions.**	* **Fluently $+/-/\times/\div$ fractions using algorithms, and explain why procedures work using models.**
Algebraic Thinking	* **Generate equivalent multi-step expressions correctly using parentheses and exponents. Evaluate numerical expressions using order of operations conventions. Use understanding of order of operations to use 4-function or scientific calculators to evaluate multi-step numerical expressions.**	* **After creating a multi-step expression for a given scenario, replace specific numbers with variables and use the expression for a similar situation involving different numbers. Use variable expressions to create spreadsheets.**	* **Use variables to express general properties of numbers. Analyze geometric patterns from multiple perspectives to create equations relating different-looking algebraic expressions. Use dissection methods to discover formulas for areas of non-rectangular regions.**	* **Write equations to express relationships between two real-world quantities. Use guess and check to solve equations and inequalities.**

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 7th grade**

6th Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Numbers and Operations	*** Demonstrate mastery of algorithms for multi-digit and decimal** $+/-/\times/\div$. *** Use positive and negative numbers to represent quantities in real-world contexts.**	*** Demonstrate mastery of algorithms for multi-digit and decimal** $+/-/\times/\div$. *** Understand positive and negative rational numbers as points on the number line, and plot rational numbers on number lines and coordinate planes.**	*** Demonstrate mastery of algorithms for multi-digit and decimal** $+/-/\times/\div$. *** Understand positive and negative rational numbers as points on the number line, and plot rational numbers on number lines and coordinate planes.**	*** Demonstrate mastery of algorithms for multi-digit and decimal** $+/-/\times/\div$. *** Understand ordering and absolute value of rational numbers.**
Problem Solving	***Fluent small group solving, writing, presenting, creating problems for multi-step expressions.** Include: Fermi-style problems Ratio and rate problems Fraction/decimal/%s Positive/negative numbers Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, Time, length, mass/weight, surface area, volume Statistics and probability.	***Fluent small group solving, writing, presenting, creating problems for multi-step expressions.** Include: Fermi-style problems Ratio and rate problems Fraction/decimal/%s Positive/negative numbers Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, Time, length, mass/weight, surface area, volume Statistics and probability.	***Fluent individual solving, writing, presenting, creating problems for multi-step equations.** Include: Fermi-style problems Ratio and rate problems Fraction/decimal/%s Positive/negative numbers Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, Time, length, mass/weight, surface area, volume Statistics and probability.	***Fluent small group solving, writing, presenting, creating problems for multi-step equations.** Include: Fermi-style problems Ratio and rate problems Fraction/decimal/%s Positive/negative numbers Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, Time, length, mass/weight, surface area, volume Statistics and probability.

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 7th grade**

7th Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Proportions	* **Fluently $+/-/\times/\div$ fractions using algorithms, and explain the procedures using area, set, and # line models.**	* **Use ratio language to describe relationships between quantities, and use ratio and rate reasoning with models to work with tables/graphs, unit rates, percents, and unit conversions.**	* **Recognize and represent proportional relationships between quantities. Connect ideas of proportion to slope of lines through the origin, to tables, and to the form of equations. Understand slope as a unit rate.**	* **Use proportions to solve multi-step ratio and percent problems.**
Algebraic Thinking	* **Generate equivalent multi-step expressions (including exponents) correctly using parentheses. Evaluate numerical expressions using order of operations conventions. Use understanding of order of operations to use 4-function or scientific calculators to evaluate multi-step numerical expressions.** * **Visualize equations as quantities that are in balance. Solve equations by honing guess-and-check strategies.**	* **After creating a multi-step expression for a given scenario, replace specific numbers with variables and use the expression for a similar situation involving different numbers. Use variable expressions to create spreadsheets.** * **Solve equations using a combination of guess-and-check and an understanding of properties of operations including inverse relationships.**	* **Use variables to express general properties of numbers. Analyze geometric patterns from multiple perspectives to create equations relating different-looking algebraic expressions. Use dissection methods to discover formulas for areas/volumes.** * **Solve equations using guess-and-check and operation properties. Learn to record steps using a formal algebraic approach.**	* **Write equations to express relationships between two real-world quantities.** * **Fluently solve linear equations using a formal algebraic approach, retaining the practice of checking answers.**

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 8th grade**

7th Grade Benchmarks and Pacing For Year-Long Standards

	Quarter 1	Quarter 2	Quarter 3	Quarter 4
Numbers and Operations	* **Use positive and negative rational #s numbers to represent quantities in real-world contexts.** * **Fluently +/ − / × /÷ rational #s.**	* **Connect fractions with division using a concrete model and convert rational #s to decimals using long division.** * **Fluently +/ − / × /÷ rational #s.**	* **Represent +/− of positive and negative rational #s using # line models. Use models to derive rules for +/− of positives and negatives.** * **Fluently +/ − / × /÷ rational #s.**	* **Use the distributive property to derive the rules for products and quotients of positives and negatives.** * **Fluently +/ − / × /÷ rational #s.**
Problem Solving	***Fluent small group solving, writing, presenting, creating problems for multi-step expressions.** Include: Fermi-style problems Generalized quantities Finding patterns Unknown quantities Ratio and rate problems Fraction/decimal/%s Positive/negative numbers Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, Time, length, mass/weight, surface area, volume Statistics and probability.	***Fluent small group solving, writing, presenting, creating problems for multi-step expressions.** Include: Fermi-style problems Generalized quantities Finding patterns Unknown quantities Ratio and rate problems Fraction/decimal/%s Positive/negative numbers Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, Time, length, mass/weight, surface area, volume Statistics and probability.	***Fluent individual solving, writing, presenting, creating problems for multi-step equations.** Include: Fermi-style problems Generalized quantities Finding patterns Unknown quantities Ratio and rate problems Fraction/decimal/%s Positive/negative numbers Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, Time, length, mass/weight, surface area, volume Statistics and probability.	***Fluent small group solving, writing, presenting, creating problems for multi-step equations.** Include: Fermi-style problems Generalized quantities Finding patterns Unknown quantities Ratio and rate problems Fraction/decimal/%s Positive/negative numbers Time, length, mass/weight, capacity with mixed units Line plots, tables, graphs Scaled picture graphs Perimeter, area, Time, length, mass/weight, surface area, volume Statistics and probability.

*** Indicates recommended formative assessment content, and skills that MUST be mastered before 8th grade**