Folding Regular Polygons

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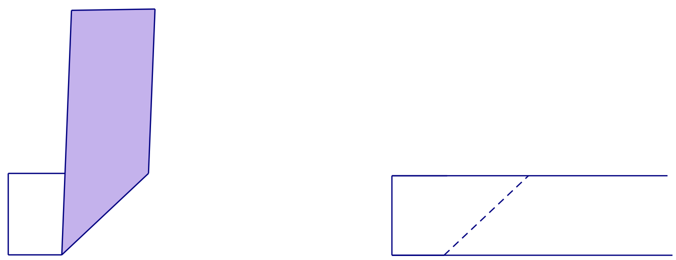
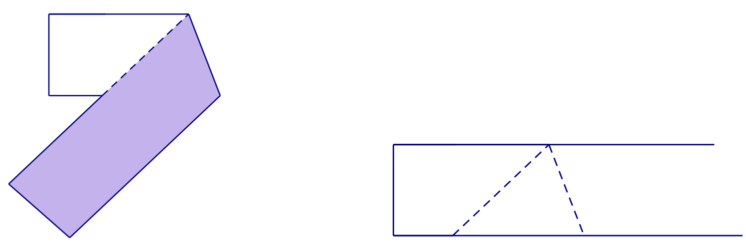
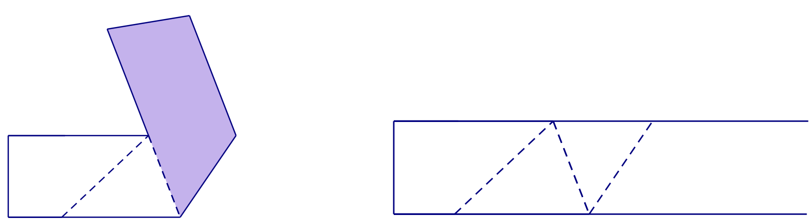
Take a long, thin strip of paper and begin by folding up to form any angle you like, then unfolding it to see the angle formed by the crease line and the bottom edge of the paper.



Now fold down so that the top edge of the paper falls along the crease line you just made, and unfold to see the new crease line.



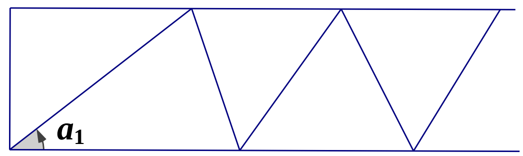
Next fold up along the newest crease line, then unfold.



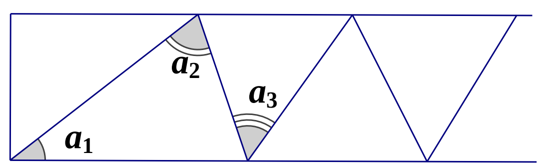
We will call this folding pattern U^1D^1. Notice that the crease line made by an "up" fold has positive slope, whereas one made by a "down" fold has a negative slope. Continue folding in this way, alternating folding up and down along the crease lines. What do you notice about the triangles that are formed by this procedure? Make a conjecture about these triangles.

Proving the conjecture: It appears that the triangles formed by the folding procedure are equilateral. Therefore, we would like to prove that the measure of the angle approaches the value $\frac{\pi}{3}$ as the number of folds increases.

Label the angles formed at each stage of the process, using as few different variables as possible. For example, if the angle created by the first fold is a_1, what can we say about angles formed by subsequent folds? What relationship will they have to a_1?

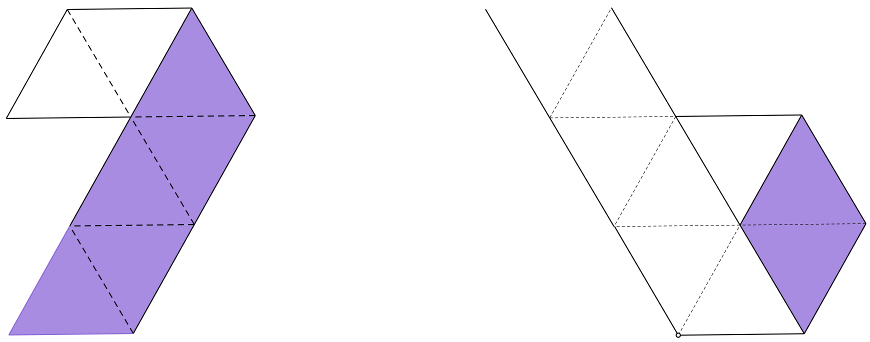


Consider the three angles labeled a_1, a_2, a_3 in the diagram below. What is the relationship between a_1 and a_2? Between a_1 and a_3? In general, how would you define a_n? What is the relationship between a_n and a_{n+1}? Between a_1 and a_n? Using either of these relationships, and some properties of sequences and/or series, you can prove that as *n* gets large, a_n approaches $\frac{\pi}{3}$.

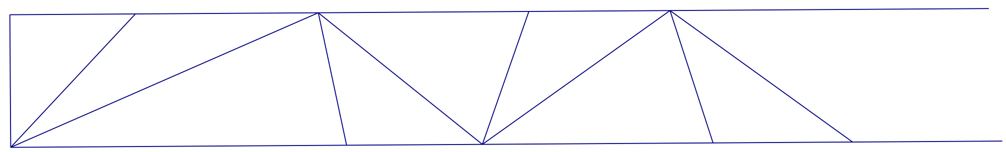


An alternate approach (using our conjecture to its fullest): Since after a few iterations, the triangles look equilateral, we may exploit the fact that the angles appear to approach $\frac{\pi}{3}$. Let $a_1 = \frac{\pi}{3} + \epsilon$. What is the relationship between a_2 and $\frac{\pi}{3}$? Between a_n and $\frac{\pi}{3}$?

Folding a hexagon: Now that we know that the measure of the angles we fold is approaching $\frac{\pi}{3}$, we can use the strip to fold a hexagon. Throw away the first few triangles on your strip of paper, then fold the paper along crease lines to form a hexagon, as illustrated below.

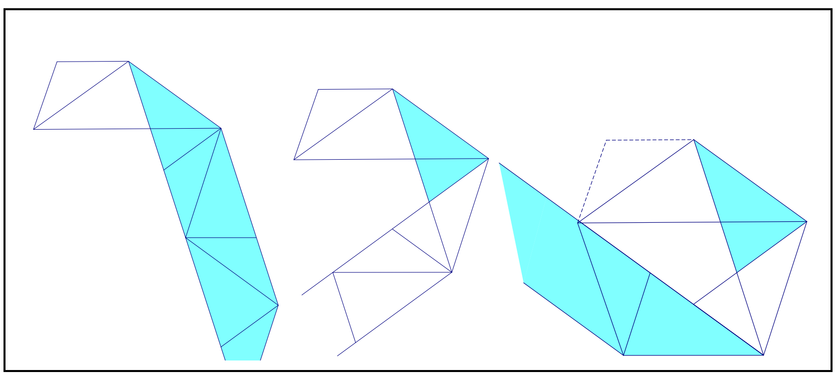


Generalizing the idea: We may consider other folding patterns similar to the U^1D^1. One possibility is to fold in the same direction twice before changing the direction of the fold. We will call this folding pattern U^2D^2. To perform the U^2D^2 folding pattern, you fold up twice, then down twice, then repeat. Take another long thin piece of paper and repeat the U^2D^2pattern until you see the triangles beginning to stabilize.

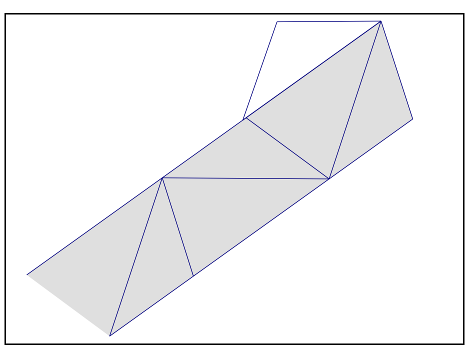


First two iterations of an U^2D^2 folding pattern

It is a bit harder to see what angles are formed now, so we can use the idea of folding polygons to identify the angles that appear after a few iterations. As before, throw away the first few triangles on the strip. Try folding a regular polygon by folding along crease congruent crease lines (there will be two choices: short lines and long lines). How many sides do you get?



Folding an U^2D^2 along the longer crease lines



Folding an U^2D^2 along the shorter crease lines

What does this tell you about the angles created by the U^2D^2 folding pattern? Can you prove your conjecture?

Generalize the idea further: Can you find the angles formed by the U^3D^3 folding pattern? Can you prove your conjecture? Is it easier to fold polygons from the strip, or analyze the sequence of angles? Now consider the U^nD^n folding pattern for an arbitrary positive integer *n.* What pattern do you notice relating the number of times you fold up (or down) and the measure of the angle you produce? What polygons can be formed from a U^nD^n strip, where *n* is any positive integer?

Generalize even more: Try folding U^2D^1. Can you fold a regular polygon from this strip? Can you determine what polygon(s) can be folded from this strip by analyzing the sequence of angles produced?

Note that we can fold in any sequence of "ups" and "downs". For example $U^2D^1U^3D^2$ means that we fold up twice, then down once, then up three times, then down twice. The sequence then repeats. Will this sequence "converge", (i.e., will the angles formed by the fold lines and the edges of the strip approach a set of fixed angles)?

Questions to consider:

1. Is there a folding sequence that will result in a strip that can be folded into a regular 7-gon?

2. Does every periodic folding sequence converge?

3. Can every regular polygon be folded using an iterative procedure to produce fold lines of the appropriate angle?

Acknowledgement: The material for this Math Teachers' Circle session is taken from the excellent book by Peter Hilton and Jean Pedersen, <u>A Mathematical Tapestry: Demonstrating the Beautiful Unity of Mathematics,</u> Cambridge University Press, 2010