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3B

Mrs. Johnson

# Reverse Fibonacci Sequence

## Intro

I was prompted to do this project when in math this year we had to write a report about a mathematician of our choice. Some of the other students in my class wrote about Fibonacci. I found the Fibonacci sequence to be interesting and wanted to do something with it. First, when I began my research, I found that in 1202 Leonardo Pisano, known today to the world as Fibonacci, published the first modern algebra book called *Liber Abaci*. For hundreds of years, it was considered the best math textbook that had been written since the end of the ancient world. In it, Fibonacci asked the following question: "How many pairs of rabbits will be produced in a year, beginning with a single pair, if in every month each pair bears a new pair which becomes productive from the second month on, and no death occurs?" The equation for this is $a_n = a_{n-1} + a_{n-2}$, where $a_n$ denotes the pairs of rabbits in the $n^{th}$-month and where $a_0 = 0$ and $a_1 = 1$. Based on this $a_2 = a_1 + a_0 = 1 + 0 = 1$, $a_3 = a_2 + a_1 = 1 + 1 = 2$, $a_4 = a_3 + a_2 = 2 + 1 = 3$, and we obtain the original Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, \ldots.,$$

where it is easy to compute the first 60 Fibonacci numbers with an Excel spreadsheet (see Table 1 below). The 60th Fibonacci number is $a_{60} = 1{,}548{,}008{,}755{,}920$ but shortly after n = 60, Excel runs out of precision and starts rounding the Fibonacci numbers. To compute the Fibonacci numbers $a_n$ for $n \geq 60$, one can use the amazing *Binet Formula*, which is

$$a_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

With it one can compute the $n^{th}$ Fibonacci number (starting with 0 and 1) directly without computing first all the previous ones. The number $\frac{1+\sqrt{5}}{2} = 1.6180339 \ldots$ is the golden ratio which can be found in many unexpected places. One is in a standard credit card where it represents the ratio of the lengths of the sides of the card. Some other places are your overall height divided by the height of your navel and the ratio of the length of your elbow to your wrist to the length of your hand. It is found throughout nature in places such as spirals of seed heads and leaf arrangements.

The values of the Fibonacci sequence will change when $a_0$ and $a_1$ have different starting values. For example if we start the Fibonacci sequence with $a_0 = 1$ and $a_1 = 3$, then we obtain the so-called *Lucas numbers*

$$1, 3, 4, 7, 11, 18, 29, 47, 76, 123, 199, \ldots .$$

When I started the project, I was wondering if there is something like Binet's Formula for any starting values $a_0$ and $a_1$. The first main result of my project is that the answer is yes and an explanation of how it works is given below.

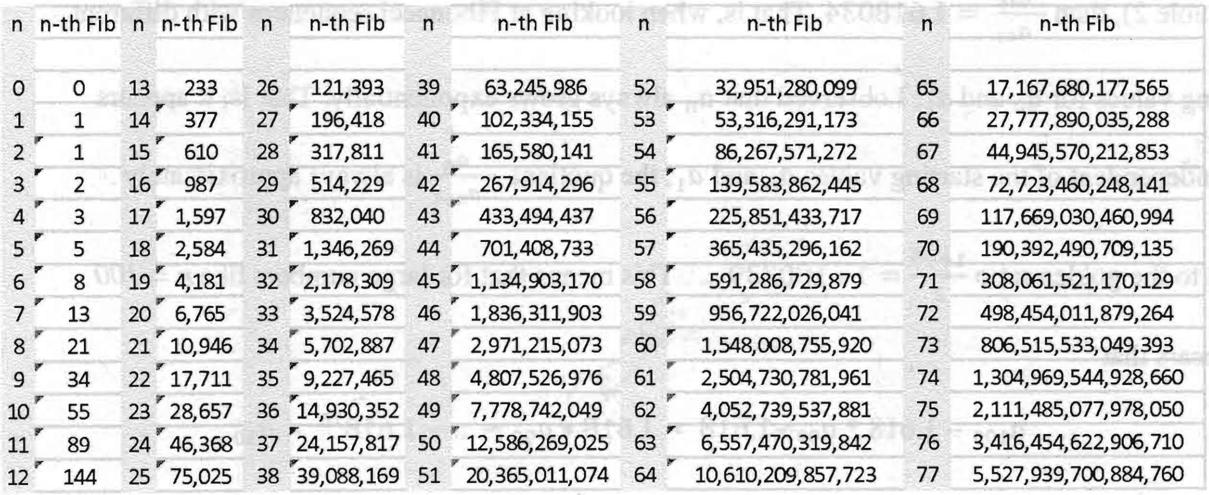
The second main result of my project concerns the reversal of the Fibonacci sequence where I start with two consecutive Fibonacci numbers $a_{N-1}$ and $a_N$ and then try to find the beginning numbers $a_0$ and $a_1$. I was testing the question, "The values of the Fibonacci sequence depend on the two initial values chosen. Can the Fibonacci sequence always be reversed or will things such as rounding affect your results?" From this, my hypothesis was that the Fibonacci sequence cannot always be reversed because after rounding multiple times, too much information is lost about your original numbers. I hoped for and finally achieved in finding a formula that allows the reversal of the Fibonacci sequence for any starting values.

## Body

In this project, the materials needed are simple. You will need a pack of paper, pencils, and a computer with Excel. The paper and pencils will be used to do the mathematics, and Excel is used to compute Fibonacci sequences and test related equations and formulas.

STEP 1: Binet's Formula for the Fibonacci Sequence. Because Excel cannot be trusted to compute Fibonacci numbers $a_n$ for large *n* due to a lack of precision (see Table 1; $a_{74}$ is incorrect since it is not $a_{73} + a_{72}$) the first step in my project is to develop an explicit formula for the Fibonacci sequence that holds for any starting values (extension of Binet's formula).

Table 1



| n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 0 | 13 | 233 | 26 | 121,393 | 39 | 63,245,986 | 52 | 32,951,280,099 | 65 | 17,167,680,177,565 |
| 1 | 1 | 14 | 377 | 27 | 196,418 | 40 | 102,334,155 | 53 | 53,316,291,173 | 66 | 27,777,890,035,288 |
| 2 | 1 | 15 | 610 | 28 | 317,811 | 41 | 165,580,141 | 54 | 86,267,571,272 | 67 | 44,945,570,212,853 |
| 3 | 2 | 16 | 987 | 29 | 514,229 | 42 | 267,914,296 | 55 | 139,583,862,445 | 68 | 72,723,460,248,141 |
| 4 | 3 | 17 | 1,597 | 30 | 832,040 | 43 | 433,494,437 | 56 | 225,851,433,717 | 69 | 117,669,030,460,994 |
| 5 | 5 | 18 | 2,584 | 31 | 1,346,269 | 44 | 701,408,733 | 57 | 365,435,296,162 | 70 | 190,392,490,709,135 |
| 6 | 8 | 19 | 4,181 | 32 | 2,178,309 | 45 | 1,134,903,170 | 58 | 591,286,729,879 | 71 | 308,061,521,170,129 |
| 7 | 13 | 20 | 6,765 | 33 | 3,524,578 | 46 | 1,836,311,903 | 59 | 956,722,026,041 | 72 | 498,454,011,879,264 |
| 8 | 21 | 21 | 10,946 | 34 | 5,702,887 | 47 | 2,971,215,073 | 60 | 1,548,008,755,920 | 73 | 806,515,533,049,393 |
| 9 | 34 | 22 | 17,711 | 35 | 9,227,465 | 48 | 4,807,526,976 | 61 | 2,504,730,781,961 | 74 | 1,304,969,544,928,660 |
| 10 | 55 | 23 | 28,657 | 36 | 14,930,352 | 49 | 7,778,742,049 | 62 | 4,052,739,537,881 | 75 | 2,111,485,077,978,050 |
| 11 | 89 | 24 | 46,368 | 37 | 24,157,817 | 50 | 12,586,269,025 | 63 | 6,557,470,319,842 | 76 | 3,416,454,622,906,710 |
| 12 | 144 | 25 | 75,025 | 38 | 39,088,169 | 51 | 20,365,011,074 | 64 | 10,610,209,857,723 | 77 | 5,527,939,700,884,760 |

Maybe the most important observation when dealing with Fibonacci numbers is that when one divides two Fibonacci numbers $a_{n-1}$ and $a_n$ (the larger n, the better), one obtains the golden ratio, which is $\frac{a_n}{a_{n-1}} \sim \frac{1+\sqrt{5}}{2} = 1.6180339 \ldots$. Using Excel spreadsheets, I found out that this is true for any Fibonacci sequence independent of the particular choice of when $a_0$ and $a_1$.

Table 2

| n | n-th Luc | n | n-th Luc | n | n-th Luc | n | n-th Lucas | n | n-th Lucas | n | n-th Lucas |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | |
| 0 | 1 | 13 | 843 | 26 | 439,204 | 39 | 228,826,127 | 52 | 119,218,851,371 | 65 | 62,113,250,390,418 |
| 1 | 3 | 14 | 1,364 | 27 | 710,647 | 40 | 370,248,451 | 53 | 192,900,153,618 | 66 | 100,501,350,283,429 |
| 2 | 4 | 15 | 2,207 | 28 | 1,149,851 | 41 | 599,074,578 | 54 | 312,119,004,989 | 67 | 162,614,600,673,847 |
| 3 | 7 | 16 | 3,571 | 29 | 1,860,498 | 42 | 969,323,029 | 55 | 505,019,158,607 | 68 | 263,115,950,957,276 |
| 4 | 11 | 17 | 5,778 | 30 | 3,010,349 | 43 | 1,568,397,607 | 56 | 817,138,163,596 | 69 | 425,730,551,631,123 |
| 5 | 18 | 18 | 9,349 | 31 | 4,870,847 | 44 | 2,537,720,636 | 57 | 1,322,157,322,203 | 70 | 688,846,502,588,399 |
| 6 | 29 | 19 | 15,127 | 32 | 7,881,196 | 45 | 4,106,118,243 | 58 | 2,139,295,485,799 | 71 | 1,114,577,054,219,520 |
| 7 | 47 | 20 | 24,476 | 33 | 12,752,043 | 46 | 6,643,838,879 | 59 | 3,461,452,808,002 | 72 | 1,803,423,556,807,920 |
| 8 | 76 | 21 | 39,603 | 34 | 20,633,239 | 47 | 10,749,957,122 | 60 | 5,600,748,293,801 | 73 | 2,918,000,611,027,440 |
| 9 | 123 | 22 | 64,079 | 35 | 33,385,282 | 48 | 17,393,796,001 | 61 | 9,062,201,101,803 | 74 | 4,721,424,167,835,360 |
| 10 | 199 | 23 | 103,682 | 36 | 54,018,521 | 49 | 28,143,753,123 | 62 | 14,662,949,395,604 | 75 | 7,639,424,778,862,810 |
| 11 | 322 | 24 | 167,761 | 37 | 87,403,803 | 50 | 45,537,549,124 | 63 | 23,725,150,497,407 | 76 | 12,360,848,946,698,200 |
| 12 | 521 | 25 | 271,443 | 38 | 141,422,324 | 51 | 73,681,302,247 | 64 | 38,388,099,893,011 | 77 | 20,000,273,725,561,000 |

For example, if we look at the Fibonacci sequence that starts with 1 and 3 (the Lucas Numbers, see Table 2), then $\frac{a_{45}}{a_{44}} = 1.618034$. That is, when looking at Fibonacci sequences with different starting values for $a_0$ and $a_1$, I observed that $a_n$ always grows exponentially. That is, it appears that independent of the starting values $a_0$ and $a_1$, the quotient $\frac{a_n}{a_{n-1}}$ is always approximately equal to the golden ratio $\frac{1+\sqrt{5}}{2} = 1.6180339 \ldots$. This means that for large numbers like $n = 100$ it appears that

$$a_{100} \sim 1.618 * a_{99} \sim 1.618 * 1.618 * a_{98} \sim \ldots \sim 1.618^{70} * a_{30}.$$

This leads to the idea to look for a number $x$ and a constant $c$ such that

$$a_n = c * x^n.$$

Substituting this into the Fibonacci equation $a_n = a_{n-1} + a_{n-2}$ gives

$$c * x^n = c * x^{n-1} + c * x^{n-2}.$$

After dividing this equation by $c$ and by $x^{n-2}$, I obtain the characteristic equation

$$x^2 = x + 1.$$

Now bring $x+1$ to the other side, which gives you the quadratic equation

$$x^2 - x - 1 = 0$$

whose two solutions are $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$. Now we know that for all numbers $c_1$, $c_2$ the sequences $a_n = c_1 x_1^n$ and $a_n = c_2 x_2^n$ satisfy the Fibonacci equation $a_n = a_{n-1} + a_{n-2}$. Therefore, for all $c_1$, $c_2$, the sequence

$$a_n = c_1 x_1^n + c_2 x_2^n$$

will also solve the Fibonacci equation $a_n = a_{n-1} + a_{n-2}$. Next we choose $c_1$, $c_2$ so that $a_0 = c_1 + c_2$ and $a_1 = c_1 x_1 + c_2 x_2$, where $a_0$ and $a_1$ are the given starting values. Because we have two equations for the two unknowns $c_1$, $c_2$, we can solve for $c_1$ and $c_2$ and get

$$c_1 = \frac{a_1 - a_0 x_2}{x_1 - x_2} \text{ and } c_2 = \frac{a_0 x_1 - a_1}{x_1 - x_2}.$$

Plug these values into the equation $a_n = c_1 x_1^n + c_2 x_2^n$ to get the *Generalized Binet Formula*

$$a_n = \frac{a_1 - a_0 x_2}{x_1 - x_2} * x_1^n + \frac{a_0 x_1 - a_1}{x_1 - x_2} * x_2^n,$$

where $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$. With this formula one can compute the Fibonacci sequence $a_n$ for any starting values $a_0$ and $a_1$. In order to better understand this formula, I will denote from now on the standard Fibonacci sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, .... by

$f_n$ instead of $a_n$. In the standard Fibonacci sequence one has $a_0 = 0$ and $a_1 = 1$ and then the formula above becomes

$$f_n = \frac{1}{x_1 - x_2}\,[x_1^n - x_2^n] = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right),$$

which is the old formula of Binet that I mentioned already above. Going back to the equation

$$a_n = \frac{a_1 - a_0 x_2}{x_1 - x_2} * x_1^n + \frac{a_0 x_1 - a_1}{x_1 - x_2} * x_2^n,$$

and breaking it into pieces, I get

$$a_n = \frac{a_1}{x_1 - x_2}\,[x_1^n - x_2^n] - \frac{a_0}{x_1 - x_2}\,[x_2 x_1^n - x_1 x_2^n].$$

Since $x_1 * x_2 = -1$, it follows that

$$a_n = a_1 * \frac{1}{x_1 - x_2}\,[x_1^n - x_2^n] + a_0 * \frac{1}{x_1 - x_2}\,[x_1^{n-1} - x_2^{n-2}]$$

which I can reduce to the following simplified form of the *Generalized Binet Formula*

$$a_n = a_1 f_n + a_0 f_{n-1}$$

for the solution of the Fibonacci equation $a_n = a_{n-1} + a_{n-2}$ with given starting values $a_0$ and $a_1$. To see how my formula works, I need the original Fibonacci numbers $f_n$given in Table 1 (which are all correct up to at least n = 60; for a correct list of Fibonacci numbers for n up to 300, see http://www.maths.surrey.ac.uk/hosted-sites/R.Knott/Fibonacci/). If one wants to find the Fibonacci sequence that starts with $a_0 = 1$ and $a_1 = 3$, then according to the formula above

$$a_n = 3f_n + f_{n-1}$$

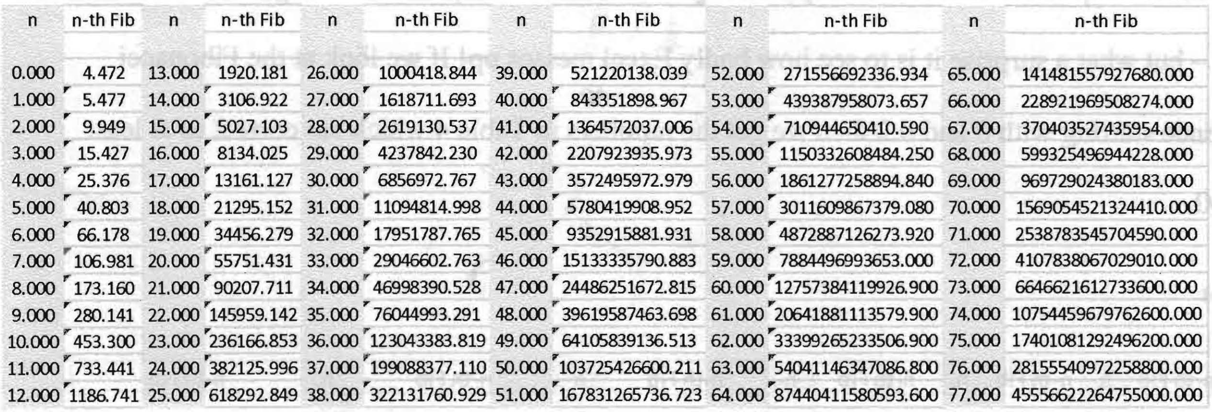
and one can get the sequence $a_n$ by multiplying one Fibonacci number by 3 and adding to it the previous one. This yields, for example that the 60th Lucas number (see Table 2) is three times the 60th plus the 59th Fibonacci number (see Table 1):

$$5{,}600{,}748{,}293{,}801 = 3*1{,}548{,}008{,}755{,}920 + 956{,}722{,}026{,}041.$$

As the next table shows (Table 3), Excel cannot compute $a_{60}$ if we start with $\sqrt{20} = 4.472\ldots$ and $\sqrt{30} = 5.477\ldots.$ The result is not $a_{60} = 12{,}757{,}384{,}119{,}926.900\ldots.$ as Excel says it is, but

$$a_{60} = \sqrt{30} *1{,}548{,}008{,}755{,}920 + \sqrt{20} *956{,}722{,}026{,}041 = 12{,}757{,}384{,}119{,}926.92854265\ldots..$$

Table 3



| n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.000 | 4.472 | 13.000 | 1920.181 | 26.000 | 1000418.844 | 39.000 | 521220138.039 | 52.000 | 271556692336.934 | 65.000 | 141481557927680.000 |
| 1.000 | 5.477 | 14.000 | 3106.922 | 27.000 | 1618711.693 | 40.000 | 843351898.967 | 53.000 | 439387958073.657 | 66.000 | 228921969508274.000 |
| 2.000 | 9.949 | 15.000 | 5027.103 | 28.000 | 2619130.537 | 41.000 | 1364572037.006 | 54.000 | 710944650410.590 | 67.000 | 370403527435954.000 |
| 3.000 | 15.427 | 16.000 | 8134.025 | 29.000 | 4237842.230 | 42.000 | 2207923935.973 | 55.000 | 1150332608484.250 | 68.000 | 599325496944228.000 |
| 4.000 | 25.376 | 17.000 | 13161.127 | 30.000 | 6856972.767 | 43.000 | 3572495972.979 | 56.000 | 1861277258894.840 | 69.000 | 969729024380183.000 |
| 5.000 | 40.803 | 18.000 | 21295.152 | 31.000 | 11094814.998 | 44.000 | 5780419908.952 | 57.000 | 3011609867379.080 | 70.000 | 1569054521324410.000 |
| 6.000 | 66.178 | 19.000 | 34456.279 | 32.000 | 17951787.765 | 45.000 | 9352915881.931 | 58.000 | 4872887126273.920 | 71.000 | 2538783545704590.000 |
| 7.000 | 106.981 | 20.000 | 55751.431 | 33.000 | 29046602.763 | 46.000 | 15133335790.883 | 59.000 | 7884496993653.000 | 72.000 | 4107838067029010.000 |
| 8.000 | 173.160 | 21.000 | 90207.711 | 34.000 | 46998390.528 | 47.000 | 24486251672.815 | 60.000 | 12757384119926.900 | 73.000 | 6646621612733600.000 |
| 9.000 | 280.141 | 22.000 | 145959.142 | 35.000 | 76044993.291 | 48.000 | 39619587463.698 | 61.000 | 20641881113579.900 | 74.000 | 10754459679762600.000 |
| 10.000 | 453.300 | 23.000 | 236166.853 | 36.000 | 123043383.819 | 49.000 | 64105839136.513 | 62.000 | 33399265233506.900 | 75.000 | 17401081292496200.000 |
| 11.000 | 733.441 | 24.000 | 382125.996 | 37.000 | 199088377.110 | 50.000 | 103725426600.211 | 63.000 | 54041146347086.800 | 76.000 | 28155540972258800.000 |
| 12.000 | 1186.741 | 25.000 | 618292.849 | 38.000 | 322131760.929 | 51.000 | 167831265736.723 | 64.000 | 87440411580593.600 | 77.000 | 45556622264755000.000 |

The result is more striking if we compare Excel's $a_{77} = 45{,}556{,}622{,}264{,}755{,}000$ with the true result which is, according to my formula, $a_{77} = a_1 f_{77} + a_0 f_{76} = \sqrt{30} f_{77} + \sqrt{20} f_{76}$. As we will see, Excel's number is over 2 trillion larger than the true number! Since $f_{77}$ is known to be 5,527,939,700,884,757 and $f_{76} = 3{,}416{,}454{,}622{,}906{,}707$, the true value is

$$a_{77} = \sqrt{30} f_{77} + \sqrt{20} f_{76} = 43{,}434{,}390{,}529{,}984{,}436.34\ldots\ldots$$

<u>STEP 2: Binet's Formula for the Reversed Fibonacci Sequence.</u> The second step in my project is to develop an explicit formula for the reversed Fibonacci sequence. To do so, we must first have an equation for the reversed Fibonacci sequence. This can be done easily if one looks at an example. If we know that 309 and 500 are two consecutive Fibonacci numbers, then we can go backwards by subtracting 309 from 500 to get 191. Continuing in this manner we get the

reverse Fibonacci sequence 500, 309, 191, 118, 73, 45, 28, 17, 11, 6, 5, 1. Now we know that the Fibonacci numbers 309 and 500 are generated by a Fibonacci sequence with starting values of 1 and 5 and we know that the reverse Fibonacci sequence is given by the equation

$$b_n = -b_{n-1} + b_{n-2}$$

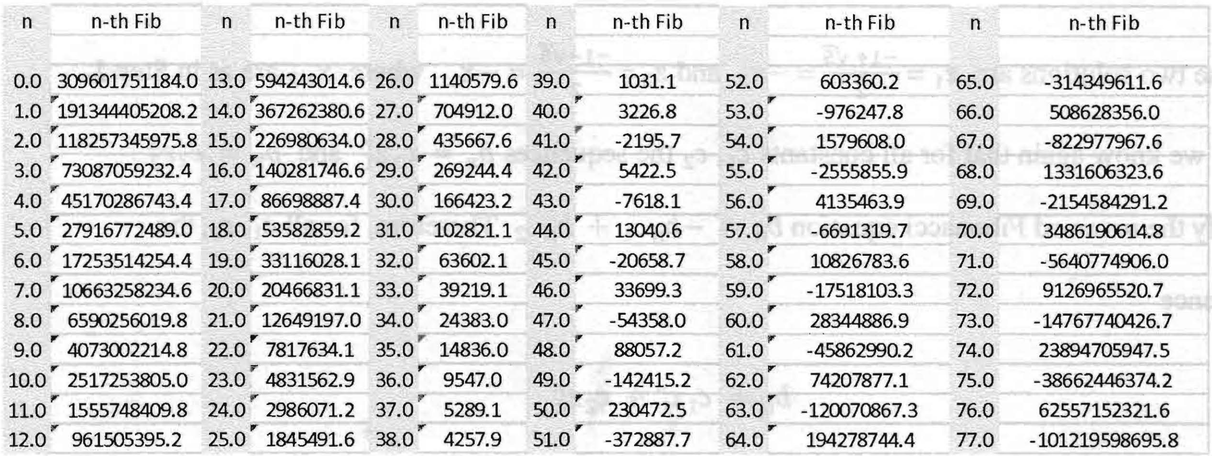
with starting values $b_0 = a_N$ and $b_1 = a_{N-1}$, where $a_N$ and $a_{N-1}$ are two consecutive values of a Fibonacci sequence. This looks simple enough to use an Excel spreadsheet to get the work done – but what a surprise it is to see how badly Excel messes up! If we look at the Fibonacci sequence starting with 0 and 0.2 then we get the numbers in Table 4 which are correct to at least n = 60.

Table 4

| n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | | | | | | | | | | |
| 0.0 | 0.0 | 13.0 | 46.6 | 26.0 | 24278.6 | 39.0 | 12649197.2 | 52.0 | 6590256019.8 | 65.0 | 3433536035513.0 |
| 1.0 | 0.2 | 14.0 | 75.4 | 27.0 | 39283.6 | 40.0 | 20466831.0 | 53.0 | 10663258234.6 | 66.0 | 5555578007057.6 |
| 2.0 | 0.2 | 15.0 | 122.0 | 28.0 | 63562.2 | 41.0 | 33116028.2 | 54.0 | 17253514254.4 | 67.0 | 8989114042570.6 |
| 3.0 | 0.4 | 16.0 | 197.4 | 29.0 | 102845.8 | 42.0 | 53582859.2 | 55.0 | 27916772489.0 | 68.0 | 14544692049628.2 |
| 4.0 | 0.6 | 17.0 | 319.4 | 30.0 | 166408.0 | 43.0 | 86698887.4 | 56.0 | 45170286743.4 | 69.0 | 23533806092198.8 |
| 5.0 | 1.0 | 18.0 | 516.8 | 31.0 | 269253.8 | 44.0 | 140281746.6 | 57.0 | 73087059232.4 | 70.0 | 38078498141827.0 |
| 6.0 | 1.6 | 19.0 | 836.2 | 32.0 | 435661.8 | 45.0 | 226980634.0 | 58.0 | 118257345975.8 | 71.0 | 61612304234025.8 |
| 7.0 | 2.6 | 20.0 | 1353.0 | 33.0 | 704915.6 | 46.0 | 367262380.6 | 59.0 | 191344405208.2 | 72.0 | 99690802375852.8 |
| 8.0 | 4.2 | 21.0 | 2189.2 | 34.0 | 1140577.4 | 47.0 | 594243014.6 | 60.0 | 309601751184.0 | 73.0 | 161303106609879.0 |
| 9.0 | 6.8 | 22.0 | 3542.2 | 35.0 | 1845493.0 | 48.0 | 961505395.2 | 61.0 | 500946156392.2 | 74.0 | 260993908985731.0 |
| 10.0 | 11.0 | 23.0 | 5731.4 | 36.0 | 2986070.4 | 49.0 | 1555748409.8 | 62.0 | 810547907576.2 | 75.0 | 422297015595610.0 |
| 11.0 | 17.8 | 24.0 | 9273.6 | 37.0 | 4831563.4 | 50.0 | 2517253805.0 | 63.0 | 1311494063968.4 | 76.0 | 683290924581341.0 |
| 12.0 | 28.8 | 25.0 | 15005.0 | 38.0 | 7817633.8 | 51.0 | 4073002214.8 | 64.0 | 2122041971544.6 | 77.0 | 1105587940176950.0 |

As we see, $a_{60}$ = 309,601,751,184 and $a_{59}$ = 191,344,405,208.2. Subtracting as above, we get $a_{58}$ = 118,257,345,975.8. Handing the next 57 steps over to Excel, we see that Excel can compute $a_{57}$ correctly and starts messing up from $a_{56}$ on (with the mistakes starting out small and then getting bigger and bigger).

Table 5



| n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib | n | n-th Fib |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | 309601751184.0 | 13.0 | 594243014.6 | 26.0 | 1140579.6 | 39.0 | 1031.1 | 52.0 | 603360.2 | 65.0 | -314349611.6 |
| 1.0 | 191344405208.2 | 14.0 | 367262380.6 | 27.0 | 704912.0 | 40.0 | 3226.8 | 53.0 | -976247.8 | 66.0 | 508628356.0 |
| 2.0 | 118257345975.8 | 15.0 | 226980634.0 | 28.0 | 435667.6 | 41.0 | -2195.7 | 54.0 | 1579608.0 | 67.0 | -822977967.6 |
| 3.0 | 73087059232.4 | 16.0 | 140281746.6 | 29.0 | 269244.4 | 42.0 | 5422.5 | 55.0 | -2555855.9 | 68.0 | 1331606323.6 |
| 4.0 | 45170286743.4 | 17.0 | 86698887.4 | 30.0 | 166423.2 | 43.0 | -7618.1 | 56.0 | 4135463.9 | 69.0 | -2154584291.2 |
| 5.0 | 27916772489.0 | 18.0 | 53582859.2 | 31.0 | 102821.1 | 44.0 | 13040.6 | 57.0 | -6691319.7 | 70.0 | 3486190614.8 |
| 6.0 | 17253514254.4 | 19.0 | 33116028.1 | 32.0 | 63602.1 | 45.0 | -20658.7 | 58.0 | 10826783.6 | 71.0 | -5640774906.0 |
| 7.0 | 10663258234.6 | 20.0 | 20466831.1 | 33.0 | 39219.1 | 46.0 | 33699.3 | 59.0 | -17518103.3 | 72.0 | 9126965520.7 |
| 8.0 | 6590256019.8 | 21.0 | 12649197.0 | 34.0 | 24383.0 | 47.0 | -54358.0 | 60.0 | 28344886.9 | 73.0 | -14767740426.7 |
| 9.0 | 4073002214.8 | 22.0 | 7817634.1 | 35.0 | 14836.0 | 48.0 | 88057.2 | 61.0 | -45862990.2 | 74.0 | 23894705947.5 |
| 10.0 | 2517253805.0 | 23.0 | 4831562.9 | 36.0 | 9547.0 | 49.0 | -142415.2 | 62.0 | 74207877.1 | 75.0 | -38662446374.2 |
| 11.0 | 1555748409.8 | 24.0 | 2986071.2 | 37.0 | 5289.1 | 50.0 | 230472.5 | 63.0 | -120070867.3 | 76.0 | 62557152321.6 |
| 12.0 | 961505395.2 | 25.0 | 1845491.6 | 38.0 | 4257.9 | 51.0 | -372887.7 | 64.0 | 194278744.4 | 77.0 | -101219598695.8 |

At the end Excel finds that the original Fibonacci sequence started out with $a_0$ = 28,344,886.89 and $a_1$ = -17,518,103.30. This is obviously nonsense since we know that we started out with $a_0$ = 0 and $a_1$ = 0.2.

Because Excel spreadsheets obviously do not work at all, I explored if there is also a formula similar to Binet's Formula for a reversed Fibonacci sequence $b_n = -b_{n-1} + b_{n-2}$. As in Step 1, the main idea is to look for a number *z* and a constant *c* such that

$$b_n = c * z^n.$$

Substituting this into the reversed Fibonacci equation $b_n = -b_{n-1} + b_{n-2}$ gives

$$c * z^n = -c * z^{n-1} + c * z^{n-2}.$$

After dividing this equation by *c* and by $z^{n-2}$, I obtain the characteristic reverse equation

$$z^2 = -z + 1.$$

Now bring -*z*+1 to the other side, which gives you the quadratic equation

$$z^2 + z - 1 = 0$$

whose two solutions are $z_1 = \frac{-1+\sqrt{5}}{2} = -x_2$ and $z_2 = \frac{-1-\sqrt{5}}{2} = -x_1$, where $x_{1,2}$ are as in Step 1.

Now we know again that for all constants $c_1, c_2$ the sequences $b_n = c_1 z_1^n$ and $b_n = c_2 z_2^n$ satisfy the reversed Fibonacci equation $b_n = -b_{n-1} + b_{n-2}$. Therefore, for all $c_1, c_2$, the sequence

$$b_n = c_1 z_1^n + c_2 z_2^n$$

will also solve the reversed Fibonacci equation $b_n = -b_{n-1} + b_{n-2}$. It remains to be shown that we can choose $c_1, c_2$ so that $b_0 = c_1 + c_2$ and $b_1 = c_1 z_1 + c_2 z_2$, where $b_0 = a_N$ and $b_1 = a_{N-1}$ and where $a_N$ and $a_{N-1}$ are two consecutive values of a Fibonacci sequence.

Solving the two equations for the two unknowns $c_1, c_2$ one gets

$$c_1 = \frac{b_1 - b_0 z_2}{z_1 - z_2} \text{ and } c_2 = \frac{b_0 z_1 - b_1}{z_1 - z_2}.$$

Plug these values into the equation $b_n = c_1 z_1^n + c_2 z_2^n$ and use again that $x_1 * x_2 = -1$ to get the *Reversed Binet Formula*

$$b_n = \frac{b_1 - b_0 z_2}{z_1 - z_2} * z_1^n + \frac{b_0 z_1 - b_1}{z_1 - z_2} * z_2^n,$$

$$= (-1)^n\left(\frac{b_1 + b_0 x_1}{x_1 - x_2} * x_2^n + \frac{-b_0 x_2 - b_1}{x_1 - x_2} * x_1^n\right)$$

$$= (-1)^n\left(\frac{-b_1}{x_1 - x_2}\,[x_1^n - x_2^n] + \frac{b_0}{x_1 - x_2}\,[x_1 x_2^n - x_2 x_1^n]\right)$$

$$= (-1)^n\left(\frac{-b_1}{x_1 - x_2}\,[x_1^n - x_2^n] + \frac{b_0}{x_1 - x_2}\,[x_1^{n-1} - x_2^{n-1}]\right)$$

$$= (-1)^n(-b_1 f_n + b_0 f_{n-1})$$

$$= (-1)^n(-a_{N-1} f_n + a_N f_{n-1})$$

where $x_1 = \frac{1+\sqrt{5}}{2}$ and $x_2 = \frac{1-\sqrt{5}}{2}$ and where $f_n$ denotes the standard Fibonacci sequence starting with 0 and 1 as given in Table 1. To see how my formula works, let us go back to the Fibonacci numbers starting with 0 and 0.2 as in Table 4. Then

$$a_{60} = 309{,}601{,}751{,}184 \text{ and } a_{59} = 191{,}344{,}405{,}208.2,$$

$$f_{59} = 956{,}722{,}026{,}041 \text{ and } f_{60} = 1{,}548{,}008{,}755{,}920.$$

Therefore my formula yields that

$$a_0 = b_{60} = -a_{59} f_{60} + a_{60} f_{59}$$

= -191,344,405,208.2*1,548,008,755,920 + 309,601,751,184*956,722,026,041= 0

## Conclusion

My hypothesis was proved in the sense that rounding did cause Excel to mess up the reversion of the Fibonacci sequence; that is, Excel is unable to compute the reverse Fibonacci sequence $b_n$ accurately. However, I disproved my hypothesis in the sense that the equation

$$b_n = (-1)^n(-a_{N-1} f_n + a_N f_{n-1})$$

will always work to reverse the Fibonacci sequence if the following three conditions are true:

I. The numbers $a_N$ and $a_{N-1}$ are known without any error.

II. The original Fibonacci numbers $f_n$ are known without any error for $0 \le n \le N$.

III. You work with a machine that knows how to multiply the large numbers in the formula

$$b_n = (-1)^n(-a_{N-1}f_n + a_N f_{n-1}).$$

There are many ways to build upon this project. The first thing to do is to find out how one can use Excel to multiply large numbers accurately. The next step is to investigate the generalized Fibonacci sequence

$$a_n = Aa_{n-1} + Ba_{n-2},$$

where A, B are given values and $a_0$ and $a_1$ are given starting values (this equation is also called a second order linear difference equation).

A practical application of this project is the following banking problem: You want to make sure (for whatever reason) that no one can find out how much money you put in a savings account where the money gets *r%* interest every month. In order to hide from everyone how much money you put in the bank initially, you strike the following deal with the bank:

a) The bank never reveals the past of the savings account but only what is in the bank in month N and month N-1 (where N is the number of months the money sits in the savings account).

b) The bank pays interest on what is in the account with a delay of one month. That is, if $a_n$ is the amount of money in the bank at month *n*, then $a_n = a_{n-1} + ra_{n-2}$.

c) The bank always rounds $a_n$ to the nearest penny.

In this situation, because of the rounding of $a_n$ to the nearest penny, it should be impossible to reverse the sequence $a_n$. That is, given $a_N$ and $a_{N-1}$ for sufficiently large N, it should be impossible for anyone to ever find out for sure what $a_0$ and $a_1$ were.

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