Math Circle Poster and Activity Session – JMM2013– San Diego

Pool Table Geometry

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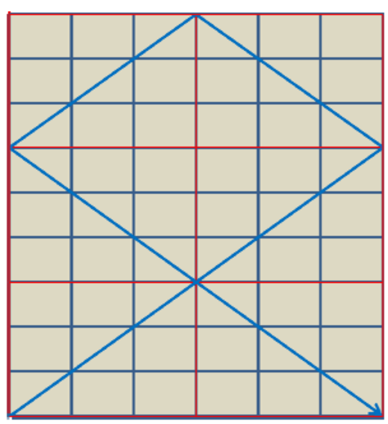
Problem: Imagine a pool table with pockets only in the corners. Suppose a ball starts at the bottom left corner and moves up at an angle of 45°. It will pass through diagonals of squares and bounce off the sides of the table and eventually fall into a pocket.

For example, in a 6×9 (width 6 and height 9) table, the ball bounces off the sides twice and once off the top before falling into the pocket on the bottom right corner as shown in the figure below. We count this as a total of 3 bounces. Let's not include starting from a pocket and falling into a pocket as bounces/hits.



One way to lead the discussion is to begin by asking the class to come up with questions for this problem. After listing the questions, the teacher can choose a few questions for the class to explore. The following list is based on some key questions that arise.

Questions

1) Draw tables of various sizes and make a chart listing the dimensions of the table, how many times the ball bounces in total, how many of those bounces are on the left/right sides, how many of those bounces are on the top/bottom sides, and into which pocket the ball eventually lands. Let's call the pockets in the top left, top right and bottom right corners as A, B, and C, respectively. Below is a sample chart with blank rows in the end so that one can include other

sizes as needed (see [1]).

(Width, Length)	Top/bottom # of bounces	Sides # of bounces	Total # of bounces	Pocket
(1,3)				
(1,4)				
(1,6)				
(3,5)				
(6,10)				
(9,15)				
(1,12)				
(2,12)				
(3,12)				
(4,12)				
(5,12)				
(6,12)				

2) Why do some of the rows in your table give the same answers? (Hint: Think of the greatest common divisor)

3) What formulas do the patterns in the table suggest for calculating number of bounces on the top/bottom, on the sides and the total number of bounces for an $m \times n$ table? Can you explain why this formula holds?

4) Can the ball ever return to the starting pocket?

5) Can the ball enter into an "infinite loop" and never fall into a pocket?

6) Given an $m \times n$ table can you predict into which pocket the ball will eventually fall? Why? Hint: Color the grid points of the $m \times n$ table alternately as black or blue. Then a ball that starts at a black pocket, say, will always follow the black grid points and fall into the other black pocket. Think about what mathematical property is shared by all black points and likewise all blue points.

7) Predict the correct numbers and pocket for each table given in the following chart:

(Width, Length)	Top/bottom # of bounces	Sides # of bounces	Total # of bounces	Pocket
(67,105)				
(67,102)				
(68, 103)				
(68,102)				

8) Can you construct a pool table for which the ball bounces exactly 4 times on the top/bottom and exactly 8 times on the sides? How about bouncing an arbitrary number of times on the sides and an arbitrary number of times on the top/bottom?

9) How many squares does the ball cross in its path in an $m \times n$ table? Can you find a formula for the number of squares in terms of m and n?

10) What kinds of symmetries do the pool tables exhibit?

Extension question (see [2]): Suppose the ball starts at <u>any</u> grid point of an $m \times n$ pool table and moves up at an angle of 45°. Assume that all four corners of the table have pockets. What can we say about the behavior of the ball?

Acknowledgements/References

Many thanks to Blake Thornton, James Tanton and Joshua Zucker for posting lesson plans on this topic! Much of the material in this document is taken directly or adapted from their websites that are listed below as 1, 2, and 3, respectively.

1. http://www.dehn.wustl.edu/~blake/circles/teacher_circle/talks/2008-Apr24-Blake_Thornton-gcd.pdf

2. http://www.jamestanton.com/wp-content/uploads/2010/02/reflection-math.pdf

3. http://www.mathteacherscircle.org/resources/materials/numbertheory.pdf

4. http://illuminations.nctm.org/LessonDetail.aspx?ID=U125