Fun with Coins and Dice

JMM 2013 Math Circle Demo

Facilitator: Paul Zeitz
(zeitzp@usfca.edu)

These are “math circle style" questions for a demo circle targeted at random JMM participants, i.e., a fairly sophisticated audience. If the audience were mathematically “younger," for example, high school students, then there would be fewer questions, and the questions would be unpacked into many more “stepping-stone" questions. In fact, each numbered problem would most likely be a single math circle session lasting a couple of hours.

1 *Coins to Dice*.

(a) Can you use a fair coin to randomly and uniformly pick from one of three choices? In other words, can you simulate a three-sided fair die with a two-sided fair coin? If so, are there any issues involving number of tosses or expected number of tosses?

(b) Likewise, can you use a fair six-sided die to simulate a fair five-sided die? Generalize!

(c) Can you use a *biased* coin (bias unknown) to simulate a *fair* coin?

(d) Can you use *two* fair coins to simulate a fair 3-sided die with a guaranteed *finite* number of flips?

(e) Can you use a single *biased* coin (you get to chose the bias) to simulate a fair 3-sided die with a guaranteed *finite* number of flips?

(f) Can you generalize (d) and (e) to simulate fair dice with more than three sides, with a guaranteed *finite* number of flips?

2 *Strings of Heads*. When you flip a fair coin many times, you'd expect to see strings of heads. In particular, you'd expect to see two heads in a row. Let u_n be the number of flips of length n for which it takes n flips to see two heads in a row for the first time. For example, if $n = 5$, one such flip would be $HTTHH$.

(a) Find a formula for u_n.

(b) Since you expect to see two heads with probability 1, you get a nice infinite series. Can you evaluate it directly (without knowing that it equals 1)?

(c) Now generalize to strings of three heads, four heads, etc.

(d) What does this have to do with dice? Remember that “dice" don't have to have six sides.

3 *A Problem from Bay Area Math Meet*. Consider the following experiment:

- First a random number p between 0 and 1 is chosen by spinning an arrow around a dial which is marked from 0 to 1. (This way, the random number is “uniformly distributed"—the chance that p lies in the interval, say, from 0.45 to 0.46 is exactly $1/100$; and the chance that p lies in the interval from 0.324 to 0.335 is exactly $11/1000$, etc.)
- Then an unfair coin is built so that it lands “heads up" with probability p.
- This coin is then flipped 2000 times, and the number of heads seen is recorded.

What is the probability that exactly 1000 heads were recorded?