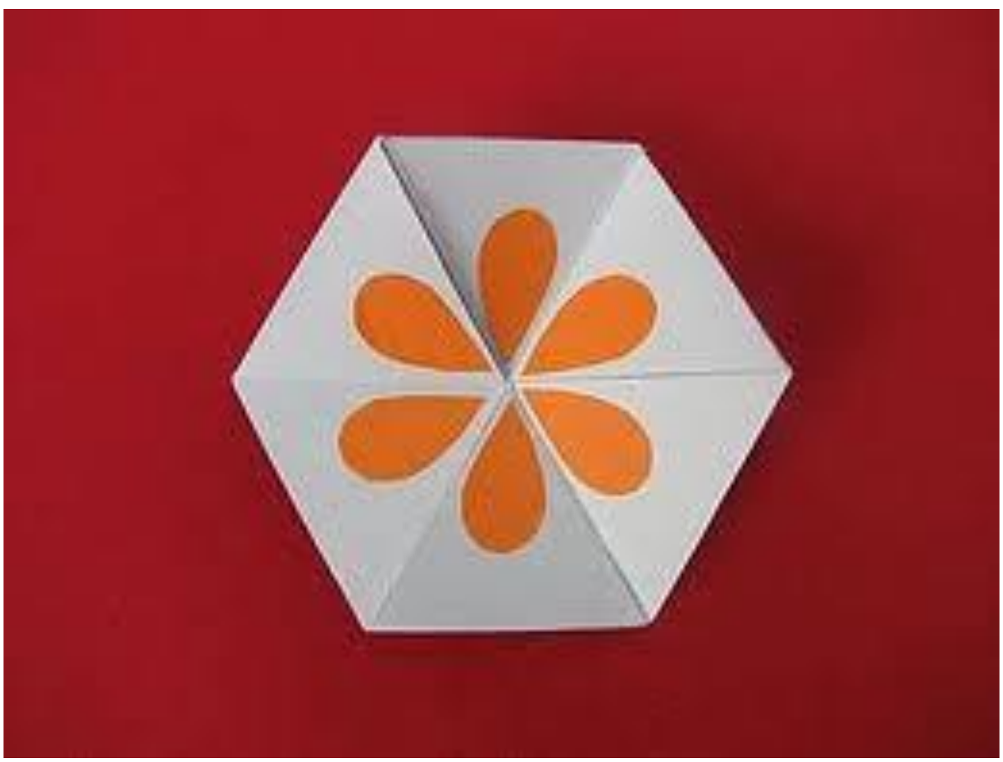
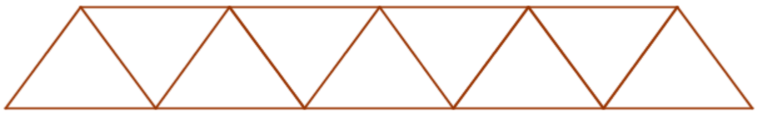
The Trihexaflexagon

and its symmetry group

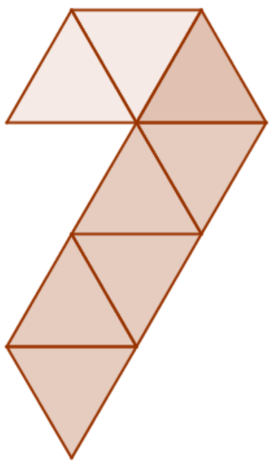


Constructing a trihexaflexagon:

To create a trihexaflexagon, start with a strip composed of nine equilateral triangles.

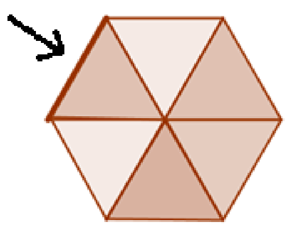


Fold carefully in both directions along each of the triangle edges. With the strip of paper in the position shown above, fold the fourth triangle *down* over the third, as shown below:



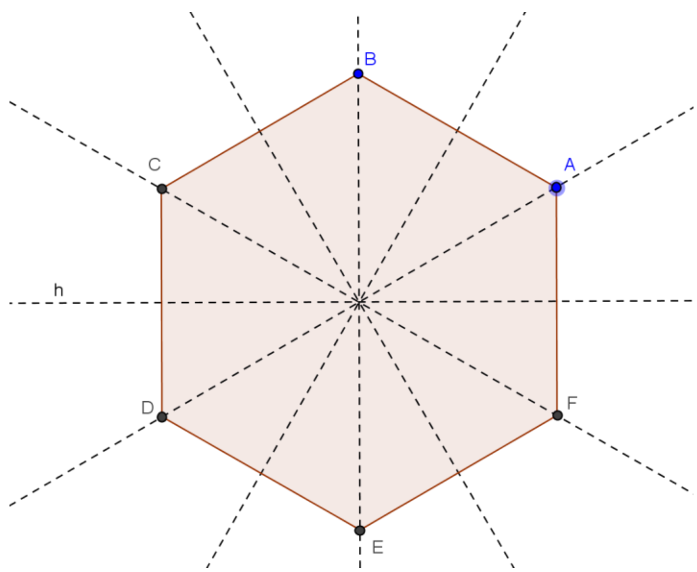
Next fold the seventh *up* over the sixth to form a hexagon. Tuck the ninth triangle *under* the first one, and tape along the common edge of triangles 1 and 9 on the perimeter of the hexagon.

Tape
here



Now that we have constructed the trihexaflexagon, let's investigate its *symmetries*. A symmetry can be thought of as an undetectable motion. Some symmetries of a hexagon are:

- r - counterclockwise rotation in the plane containing the hexagon, through an angle of 60°, where the center of rotation is the center of the hexagon.
- h - 180° rotation about the horizontal line through the center of the hexagon.



It is easy to see that the composition of two symmetries is a symmetry. We will write the composition of two functions f and g as fg instead of $f \circ g$, but as is standard with composition, when performing the composition fg we first apply g and then f.

Label the vertices of your hexagon and perform the symmetry hr. Now try rh. What do you notice? Now try r^5h. Can you explain what you observe?

In all there are twelve symmetries of the hexagon and each can be written in the form r^kh^l, where $0 \le k \le 6,\ 0 \le l \le 1$.

The relationships $r^6 = h^2 = e$, and $hr = r^5h$ allow us to work easily with this group.

Now the trihexaflexagon is more than just a hexagon. There are two more motions that we can perform on the trihexaflexagon that are not possible on a simple hexagon; these are the flexing motions. To understand how to flex, it helps to distinguish two types of creases on the face of the trihexaflexagon, slits and hinges. Those creases that are at the edge of a pocket formed by two triangles are called slits and the others are called hinges.

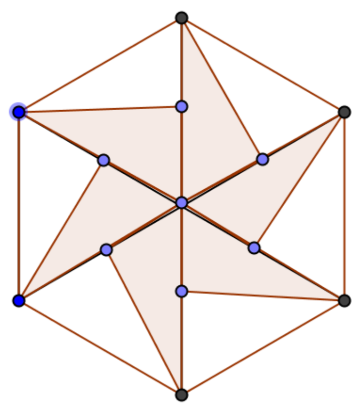
f - flex down (valley fold hinges down to meet *below* the center, open at the top of the flexagon and flatten out)

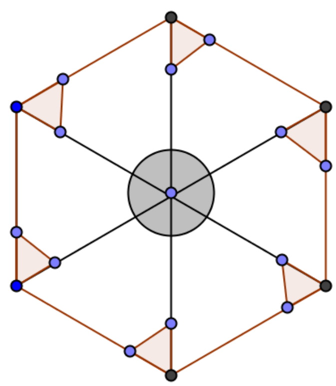
$\bar{f}$ - flex up (mountain fold slits up to meet *above* the center, open at the bottom of the flexagon and flatten out)

After you have mastered the flexing motions, you may start to investigate the relationships between flexing and the other motions we considered for the hexagon. Because symmetries are undetectable motions, in order to understand what symmetries are involved in the flexing, we must have some system, as we did when we labeled the vertices of the hexagon, to see what affect the motion has.

First, you may lightly color the top face one color and the bottom face another color. Now perform one flex. What happens? Why do you think this object is called a trihexaflexagon?

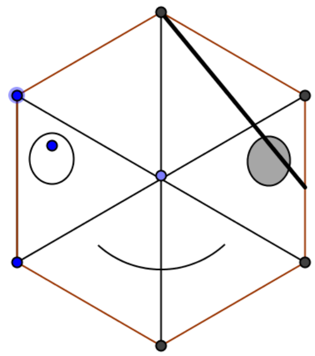
Now color the triangles that make up the face of the hexagon so as to distinguish the vertices of the three triangles. Here are a couple of possibilities, but feel free to get creative.





What happens when you perform the motion hf?

This happy pirate face was designed by Peter Hilton and Jean Pederson, authors of A Mathematical Tapestry. The smile should be drawn across a hinge.



Can you guess what will happen when you perform the motion hf?

Starting with the happy pirate face up, carefully flex down three times, always keeping the same vertex pointing up (*i.e.* being careful not to rotate the hexagon as you flex.) What happens?

Try various combinations of motions to see what relationships you can find. For example, can we write all symmetries in the form $f^k h^l$? If so, what is the range of values for k? How would you write hf?

For more information on flexagons, see:

Martin Gardner – Hexaflexagons and Other Mathematical Diversions

Les Pook – Flexagons Inside Out

Peter Hilton and Jean Pederson – A Mathematical Tapestry: Demonstrating the Beautiful Unity of Mathematics