Contrasting Pedagogical Goals in Different Settings

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Riverbend Community Math Center

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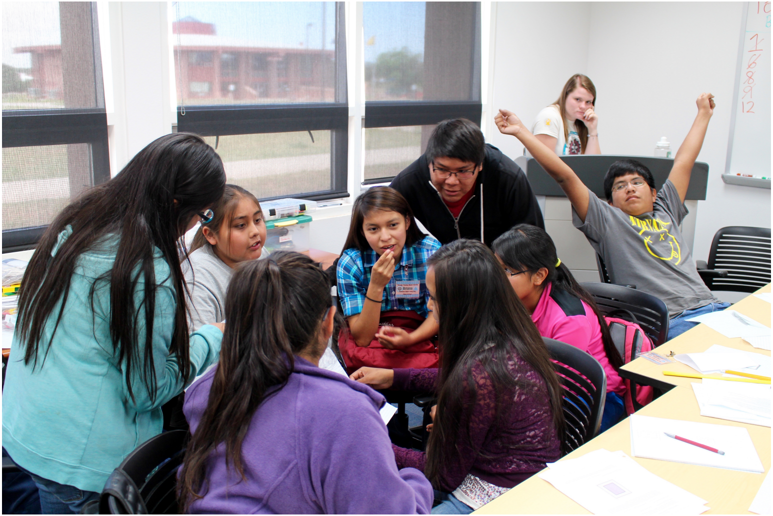
Go to any school, community center, or neighborhood and you will find . . .

Students who aspire to be scholars

Students who are curious

Students who want to change the world

Students who want to help their communities



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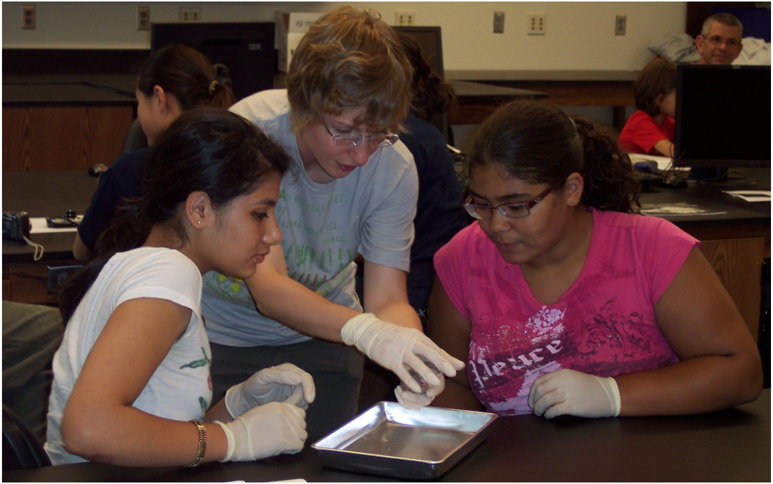
Students with big dreams for the future



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- Students need opportunities to feel the power of their own minds.
- Students need opportunities to be intellectually creative, and to have their ideas and their education taken seriously.

Four Philosophies of Instruction

- Traditional Math
- Conceptual Math
- Inquiry / Project Based Learning
- Math Circles

Traditional Math

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- Tries to broaden the pool of people able to accurately perform specific computations.
- Students learn that being good at math means not needing to think and being quick to compute correct answers.

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- Students discuss different approaches to computational tasks, debating correctness and efficiency.
- Students learn that being good at math means being willing to think carefully and explain their reasoning using graphical representations, mathematical notation, and words.

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- Students learn that being good at math means struggling with a complex problem and being able to contribute to a team effort.

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- Instructors stay on the stage to enforce the norms of the discipline of mathematics and to model enthusiasm for interesting questions.
- Instructors welcome misconceptions as a sign that the confusing idea is interesting.
- Students learn that mathematics is a vast field and that it is not practical to worry about “being good” at it.

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Mathematical Creativity

- Create a new playground.
- Ask a question about an existing playground.
- Explore a question.
- Find an example or counter-example.
- Devise a method for organizing data.
- Make a conjecture.
- Prove a theorem.
- Communicate findings.
- Find connections to other playgrounds.
- BE EAGER TO FLAIL AND FAIL!

Examplars of fraction lessons

Let's look at some examples of fraction lessons adhering to each of the pedagogical approaches.

We will consider what characterizes each of these as the given approach and not one of the others.

Exemplars of fraction lessons — Traditional

- Equivalent fraction team hunt.
- Student-designed fraction quiz games.
- Fraction scavenger hunt where the letters corresponding to the correct answers at each station spell out where to go next.
- Illustrated, student-made, vocabulary pages for math journals.
- Procedure posters.

Traditional Fraction Lesson Characteristics

While students may be engaged, creative, and empowered in these lessons, they are only practicing rote skills.

Exemplars of fraction lessons — Conceptual
Xavier's Pizza

Xavier's family made two square pizzas at home, one 8 inches on a side and the other 12 inches on a side. He ate 1/4 of the small pizza and 1/6 of the larger pizza. Xavier says he ate 2/10 of the pizza altogether. However, his friend Jessica thinks he ate 5/12 of the pizza.

- How do you think that Xavier and Jessica found their answers?
- Make a model that illustrates the story and use it to determine the correct answer.
- Could Xavier's approach ever work? Can you change the scenario so that his approach would give a correct answer?
- Could Jessica's approach ever work?

Xavier's Pizza (continued)

- Groups tackle the questions independently.
- Students do a gallery walk to view the work of other groups.
- A class conversation resolves differences of opinion.
- Students share sentences summarizing their conceptual take-aways.

Xavier's Pizza Characteristics

- Students use models to understand fractions as parts of a whole. The key issue is that the size of the whole is different between the two pizzas and between both combined.
- Critical thinking is required in several of the questions.
- The scenario is not realistic — no one would ever care about this question other than as an academic exercise.
- Even though Farey addition of fractions is RIGHT THERE, the group will not go off into that marvelous land to explore its wonders. This investigation is well-fenced.

Exemplars of fraction lessons — Conceptual Fractional Area/Perimeter Challenge

- Groups of students receive kits of card stock pieces and dowel rods.
- The dowel rods come in one-foot or half-foot pieces.
- There are three kinds of card stock pieces — $1' \times 1'$ squares, $1' \times \frac{1}{2}'$ rectangles, and $\frac{1}{2}' \times \frac{1}{2}'$ squares.
- The instructor introduces the components and reviews how they relate to area and perimeter of rectangles.

Fractional Area/Perimeter Challenge (continued)

- Groups arrange the pieces on the floor as they work through a scaffolded sequence of challenge problems. They shade their answers onto a grid on the handout and write equations computing the perimeter and area in terms of the side lengths.
- The first level asks students to give multiple correct solutions with whole numbers. Later levels require students to exchange unit lengths and areas for the fractional pieces, and the final level includes impossible problems and challenges to maximize or minimize the area or perimeters given constraints.

Fractional Area/Perimeter Challenge Characteristics

- This lesson includes active problem solving that allows students to develop comfort with a model for fraction multiplication.
- Students encounter and untangle misconceptions such as the idea that a square with half the side lengths should have half the area of the original.
- This activity is not realistic and students do not drive the investigation or make the questions.
- The puzzles require students to use only the three kinds of pieces in the kit, and so open-ended creative solutions are not possible.

Exemplars of fraction lessons — Conceptual

Find four distinct positive single-digit numbers a, b, c, and d so that the sum of $\frac{a}{b}$ and $\frac{c}{d}$ is less than 1 but as large as possible.

- As students find any results, the instructor encourages them to write what they found on the board.
- After several answers are up, the class discusses which meet the criteria and how their sizes compare, using a variety of fraction comparison methods which have been discussed in prior lessons.
- When students arrive at the optimal answer, the instructor asks how they know this is the best possible answer.

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Characteristics of the $\frac{a}{b} + \frac{c}{d}$ Lesson

- This lesson is a delightful romp of an activity as students work together to find better and better solutions.
- Although the final step does invite a proof, that proof is a straightforward application of basic concepts about fractions.
- Students practice rote addition of fractions with unlike denominators. However, this lesson is conceptual rather than traditional because of the discussions about fraction comparison and the final justification of the optimal solution at the end.
- This activity is closed in that there is not a clear direction to go after finding the optimal answer. Students have no real choices in the direction of the inquiry.

Exemplars of fraction lessons — Inquiry
$\frac{1}{n} + \frac{1}{n+1}$ Patterns

- The instructor posts the prompt: $\frac{1}{2} + \frac{1}{3} = \frac{5}{6}$ and $\frac{1}{3} + \frac{1}{4} = \frac{7}{12}$.

- Students who are newer to inquiry may be given a choice of “regulatory cards” to select at various points in the lesson, while more experienced students already know and claim many of these options. These include choices like “decide if the prompt is true”, “inquire with another student”, “ask the instructor to explain”, “discuss conjectures as a group”, “find more examples”, “practice a procedure”, “find connections”, or “make a generalization”.

$\frac{1}{n} + \frac{1}{n+1}$ Patterns (continued)

- The instructor watches for misconceptions that arise. The instructor confronts these misconceptions by pushing students to reconcile conclusions with prior knowledge, asking how they know they are correct, or by giving examples that clearly do not fit their patterns.

- This lesson may lead to student-generated lines of inquiry including "Would there still be a pattern if you switched the numerators and denominators?", "Does the numerator increase by 2 each time?", "Should the answer to the first computation actually be $\frac{2}{5}$?", "What if the numbers in the numerator were both 2 instead?"

Characteristics of $\frac{1}{n} + \frac{1}{n+1}$ Patterns

- Students develop metacognitive skills because they control the flow of the inquiry.
- Students will make and attempt to prove many conjectures.
- The prompt nudges students to guess the general rule for adding fractions (or at least unit fractions) based on equations that parachute in from the sky rather than from models or reasoning. This makes it something of a guessing game. However, they would be encouraged to use prior knoweldge to investigate whether the equations seem correct as one of their options.
- The investigation is closed in scope. While students may pursue variations, they are unlikely to wander beyond any conclusions that could not be proven with a bit of high school algebra.

Exemplars of fraction lessons — Inquiry Lemonade Stand

The class plans and implements a lemonade stand business which they will operate every day after school for a week. The components of this project include:

- Designing and selecting an ideal lemonade recipe
- Building the Stand
- Quantity planning
- Cost Analysis

Lemonade Stand (continued)

The instructor might:

- Let the students complete the steps however they want and hope some math sneaks in.
- Give the students warm-up worksheets each day they work on the unit to make sure the math gets in there somewhere even if they don't use it for the actual project.
- Carefully structure the prompts to require enough precision that the students will definitely need to use fraction concepts.
- Challenge the students to generate a list of fraction concepts they have learned this year, find ways to incorporate those skills, and make products depicting how they have done so.

Lemonade Stand Characteristics

- The students will have fun and will remember this experience.
- Depending on how the instructor leads this, students may benefit from thinking about how what they have learned about fractions can be used to accomplish a very real task. However, this will mostly be an exercise in remembering common applications of each fraction operation encountered earlier in prompts and adapting them to this situation.
- The actual mathematics in this task consists of straightforward applications of rote computational skills completed a very small number of times.

Exemplars of fraction lessons — Math Circles
Numbers Between Numbers

- This Math Circle is for 5 to 10 young students (4 or 5 year olds) who have not been formally taught anything about fractions.
- A conversation begins with the instructor asking about counting. What number do we start with when we count? Is it 17 or 0 or 1, I forget. How does counting go? 1, 2, 3, 17, No? Oh, 4! Right. Then what again? OK.
- So are there any numbers between those numbers? Maybe 1 and then something else before we get to 2? No? So there is nothing between 3 and 4 or between 4 and 5? Definitely not? How old are you? 4 and a half? Wait, does that number come after 17? Between 4 and 5 you say? (The instructor draws 5 small lily pads onto the entire board with plenty of space in between.)

Numbers Between Numbers (continued)

- So 1 goes here, then what, 2? Then 17, right? Oh, no 3? Then what? OK. Where does 4 1/2 go? Somewhere between 4 and 5. (The instructor draws the lily pad comically close to the 5 and far from the 4.) Oh, that is not right? Why not? How can we draw pictures of these numbers? And how about for 4 1/2? Do you think there are any other numbers in here between the 4 and the 5?
- This conversation continues for five to ten sessions, involving many discoveries about fractions, including various kinds of playful fraction equivalence (including compound fractions and possibly decimal-fraction hybrids), the density of the rationals, fraction addition and subtraction. Negative whole numbers and fractions may also join the party. The group might explore really large numbers and where they live relative to these little numbers.

Numbers Between Numbers (continued)

- Sessions might be interspersed with a game of Nim or the Function machine game to allow for brain breaks when the stamina of the group wanes. At some point, these might morph to include Fraction Nim (where students start at 0 and can add 1/2, 1/4, or 1/8 and the goal is be the one who reaches 3), or Function machines may emerge that involve fractions.

How does this compare/contrast with the various philosophies?

Exemplars of fraction lessons — Math Circles
Rational Tangles

- This Math Circle is for 3rd and 4th graders.
- The instructor begins by finding volunteers to hold the four ends of the two ropes and leads them through what the Do-Si-Do and Turn 'Em Round commands mean in terms of the Rational Tangles dance and as functions acting on rational numbers. The group also establishes notation that can be used to record the dances.
- The students work through several untangling challenges as a group, with the instructor avoiding interference.

Rational Tangles (continued)

- As the sessions proceed, the group brainstorms conjectures about methods for untangling and try to prove whether the methods they found will always work.
- They invent their own moves (such as the “Baseball Slide”). To figure out what this move does, they put different starting fractions on the ropes, do the move, then untangle it visually using known moves, reverse engineering the numbers they got at each step after getting back to zero. This raises the question of whether they can get to any fraction and how to go about doing that.
- They brainstorm questions about the Rational Tangles dance and invent other variations.

How does this compare/contrast with the various philosophies?

Exemplars of fraction lessons — Math Circles
Repeating Decimals

- This Circle is for upper elementary or middle school students.
- The group explores remainder webs generated using division to convert fractions into decimals.
- They discuss whether rationals always result in decimals that terminate or repeat. They use this insight to find the full repeating part for the decimal expansion of $1/47$ using feeble calculators, and discuss how/why this works.
- The group explores remainder webs further, making conjectures about the patterns they find and working to prove them.

Repeating Decimals (continued)

- The group might wander off into one or more of the following: converting repeating decimals back into fractions, numbers with digits repeating to the left, explorations of repeating decimals in various bases, infinite series in general, philosophical musings about the legitimacy of allowing infinitely repeating decimals or the nature of a continuum a la Xeno's paradox.

How does this compare/contrast with the various philosophies?

Exemplars of fraction lessons — Math Circles Boomerang Fractions

- This Circle is for middle school or high school students.
- In Boomerang Fractions, we choose $\frac{a}{b}$ to generate a sequence. The first member of the sequence is 1, and the next member is $1 + \frac{a}{b}$. After that, we have a choice — we may either take the reciprocal of our last number or we may add the chosen fraction to our last number. The goal is to return to 1 as quickly as possible.
- Students brainstorm some questions they have about this and try some examples. They gather and organize some data (possibly writing code to help with this), and then make and try to prove some conjectures.

Boomerang Fractions (continued)

- Students may compare their results with what has already been posted in the Online Encyclopedia of Integer Sequences and elsewhere and consider trying to prove something that is not yet known, possibly publishing it.
- Students may consider making their own variant questions inspired by Boomerang fractions and see how the results compare.

How does this compare/contrast with the various philosophies?

Cantorian Set Theory

- Students play with the idea of making up various sets and seeing whether they can assign members of the pigeon set to the holes in a one-to-one fashion. This progresses through various sets, including the rationals.
- After being hinted into Cantor's audacious idea for showing that the set of irrationals won't fit in the holes, the students create lots of examples of infinite sets and try to determine which of them have the same cardinality. This may include a visit to Cantor's middle thirds set and its ternary representation, and students are invited to use this to inspire the creation of infinite sets that are still wilder.

How does this compare/contrast with the various philosophies?

Challenges of Math Circle Instruction

- As Math Circle Instructors, we must necessarily confront the reality of our own ignorance. (Rational Tangles, Repeating Decimals, Boomerang Fractions, Cantorian Set Theory.)

- This is much less of a factor in the other pedagogical approaches because the level and the extent of the content is carefully fenced off by the nature of the prompts given.

Challenges of Math Circle Instruction

- A Math Circle starts as a simulation of actual mathematical research and ideally becomes less and less of a simulation as students progress. This includes simulating collegial conversations among the group, simulating student choice of direction, and simulating the thrill of discovery of concepts that took years to actually emerge in human history.

- A Math Circle requires us to become sherpas with exceptional skills because otherwise we don't provide the nudge to keep going when students are tantalizing close to an amazingly cool vista. They may wander in the wilderness and arrive at no interesting conclusions. They may become stuck in the mud. They may head off into a decidedly non-mathematical vein of inquiry.

Challenges of Math Circle Instruction

- It is necessary to try to expand our own understanding of the topics we lead, but it is impossible to ever know enough about all of them.
- Our abilities to be as creative and clever as we hope to train our students to become are necessarily not as great as we might wish.
- If we dedicate our lives to exploring one Math Circle topic very thoroughly, and teach only that, we can get closer to understanding the nuances of the terrain around that idea. However, then we may fall into the trap of leading our students up the well-cleared trail to the nicest vistas, thereby ruining their chance of feeling the thrill of discovery or developing their own powers of thought.

Challenges of Math Circle Instruction

This is why learning to teach Math Circles is endlessly fascinating, humbling, and enriching. Learning to contend with our own egos as we confront the reality of our own ignorance and explore ideas of stunning beauty is at the heart of what it means to be a mathematician.