Supercomposite Numbers, Exponent Worms, and Lattice Rules

by Dan Bach - *www.dansmath.com*

MAA MATHFEST 2025 | SACRAMENTO, CA

MathFest Aug 9, 2025

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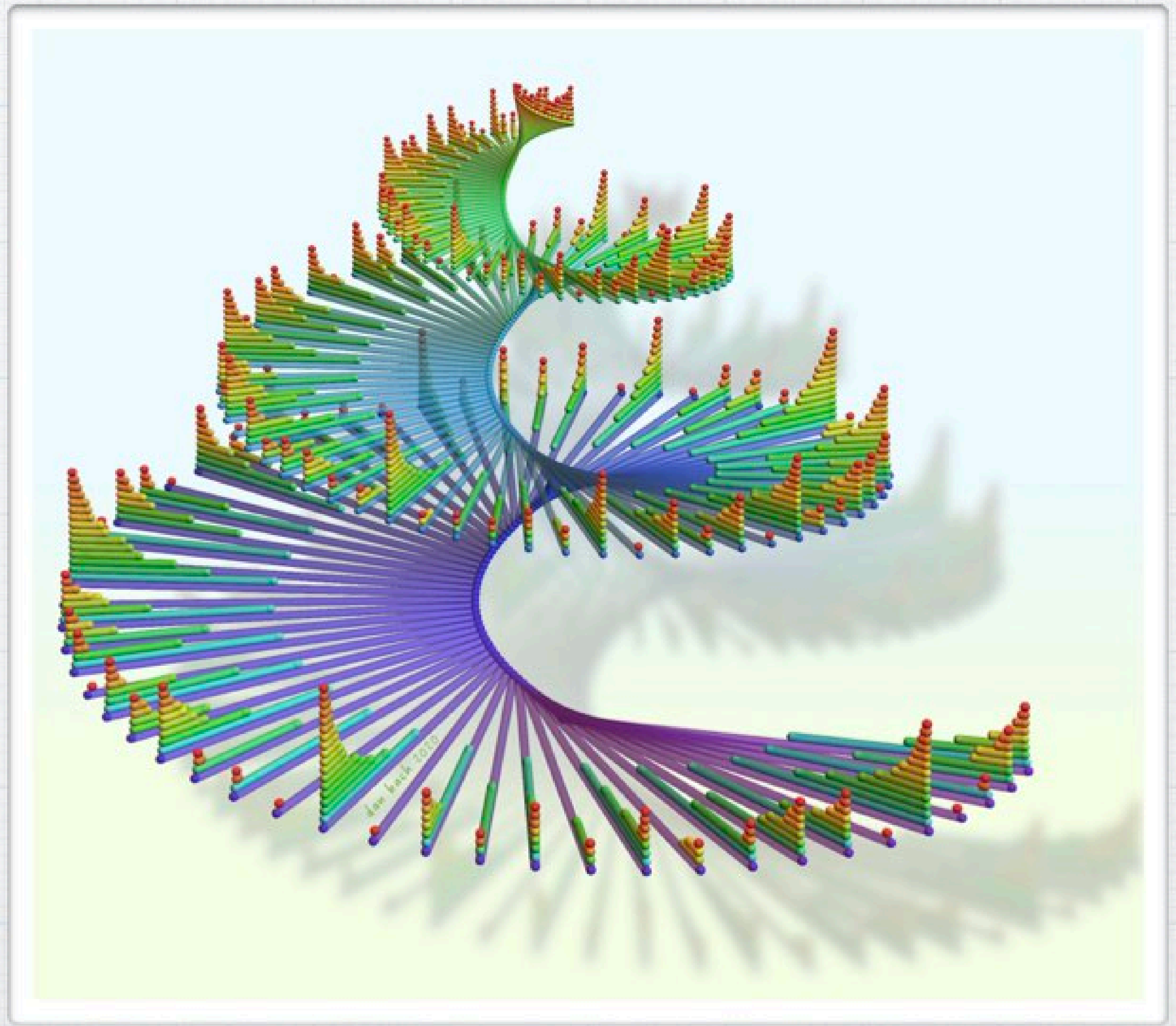

Divisor Stack Spiral
dan bach 2020

Supercomposite Numbers, Exponent Worms, and Lattice Rules

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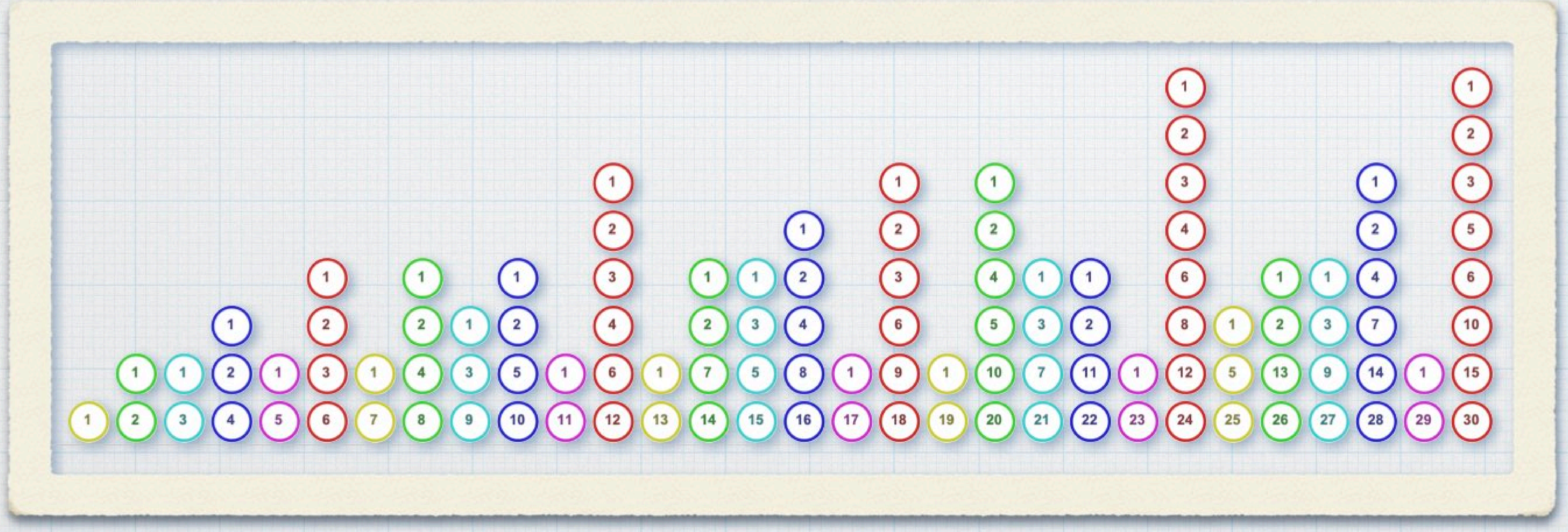
- A ***supercomposite*** number ***n*** is one with more divisors than any smaller number
- The number of divisors of ***n*** can be calculated from the ***exponents*** of its prime factorization
- Better exponent structures give more ***efficient n***; 'better' is judged by inequalities

Popularly called **highly composite numbers** (HCN)

Divisors - Stacks & Stats

Some numbers have lots of divisors but *primes* don't.

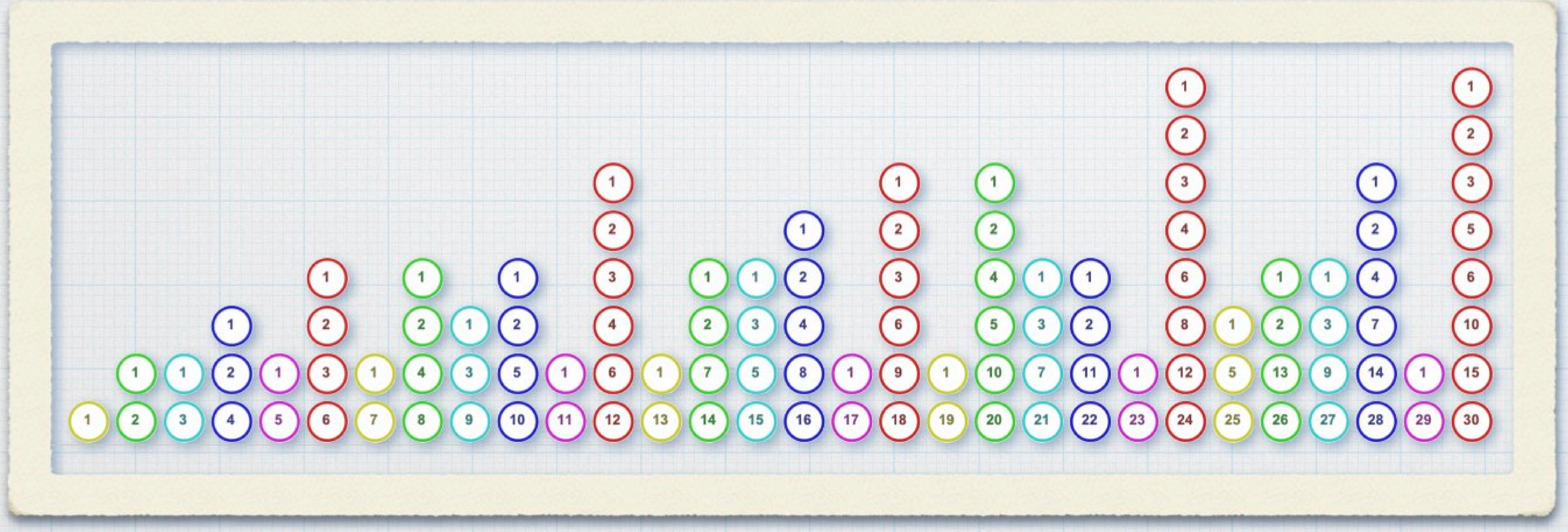


Divisors - Stacks & Stats

Some numbers have lots of divisors but *primes* don't.

Which numbers set a record for *most divisors so far?*

d(n) = number of positive divisors of n.



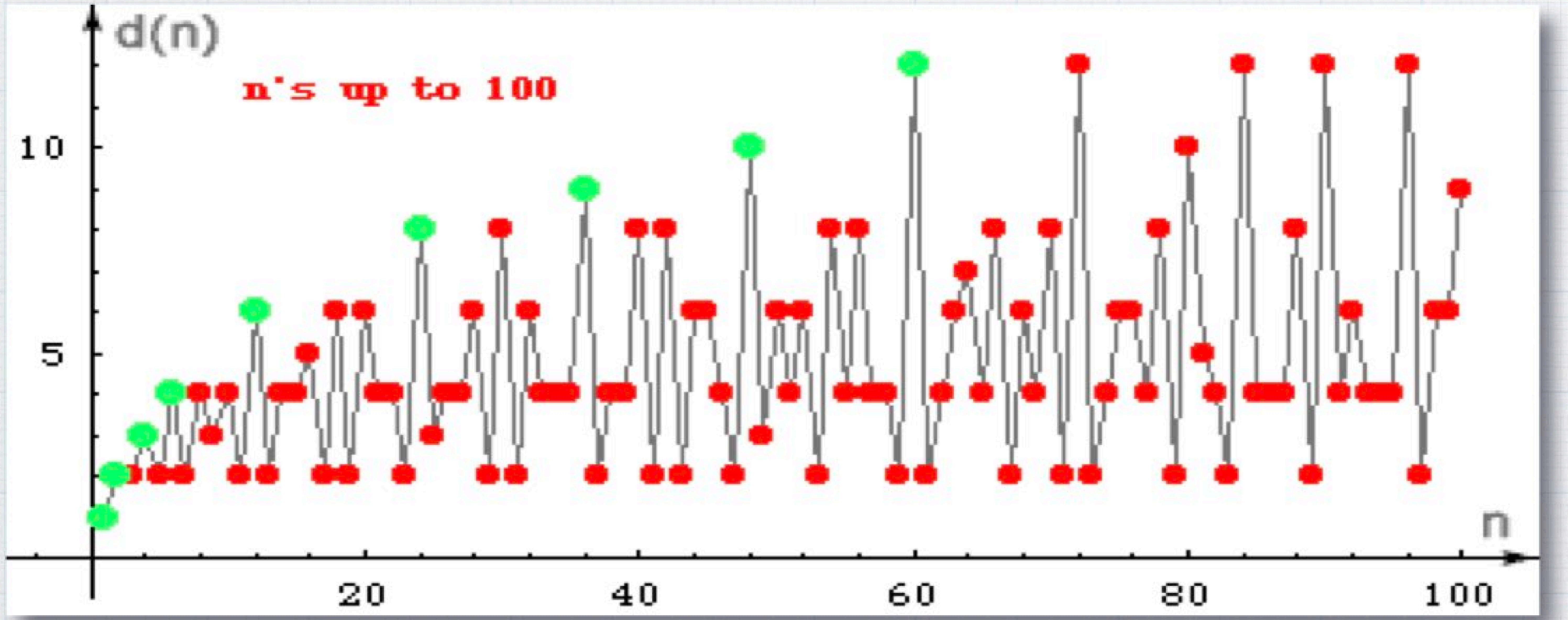
n	divisors	d(n)
1	1	**1**
2	1, 2	**2**
3	1, 3	2
4	1, 2, 4	**3**
5	1, 5	2
6	1, 2, 3, 6	**4**
7	1, 7	2
8	1, 2, 4, 8	4
9	1, 3, 9	3
10	1, 2, 5, 10	4
11	1, 11	2
12	1, 2, 3, 4, 6, 12	**6**
13	1, 13	2
14	1, 2, 7, 14	4
15	1, 3, 5, 15	4
16	1, 2, 4, 8, 16	5
17	1, 17	2
18	1, 2, 3, 6, 9, 18	6

Plotting the Divisors of n

Here's a look at the divisor function!

(We dig back to 1995 for my helpful graph)

This and most other graphics made in ***Mathematica***



Primes *p* are along the bottom; d(*p*) = 2.

Can you 'spot' the supercomposite n's?

Examining the Exponents...

What's so 'super' about supercomposite numbers?
Let's take a look at their prime factorizations (pf).

List of supercomposites *n* under 100:

n	d(n)	pf of n
2	2	2^1
4	3	2^2
6	4	$2^1 \cdot 3^1$
12	6	$2^2 \cdot 3^1$
24	8	$2^3 \cdot 3^1$
36	9	$2^2 \cdot 3^2$
48	10	$2^4 \cdot 3^1$
72	12	$2^3 \cdot 3^2$
60	12	$2^2 \cdot 3^1 \cdot 5^1$

Examining the Exponents...

What's so 'super' about supercomposite numbers?
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Some necessary conditions to be super:

Saturated: No prime is skipped

not $20 = 2^{2} \cdot 3^{0} \cdot 5^{1}$ but $12 = 2^{2} \cdot 3^{1}$

Monotone: Non-increasing exponents

not $18 = 2^{1} \cdot 3^{2}$ but $12 = 2^{2} \cdot 3^{1}$

Example: $35000 = 2^{3} \cdot 5^{4} \cdot 7^{1}$; saturate to $2^{3} \cdot 3^{4} \cdot 5^{1} = 3240$, then reorder to $2^{4} \cdot 3^{3} \cdot 5^{1} = 2160$. Is it super yet?

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List of supers under a billion, with exponent lists.

n	d(n)	a	b	c	d	e
		2	3	5	7	11
2	2	1				
4	3	2				
6	4	1	1			
12	6	2	1			
24	8	3	1			
36	9	2	2			
48	10	4	1			
60	12	2	1	1		
120	16	3	1	1		
180	18	2	2	1		
240	20	4	1	1		
360	24	3	2	1		
720	30	4	2	1		
840	32	3	1	1	1	
1,260	36	2	2	1	1	
1,680	40	4	1	1	1	
2,520	48	3	2	1	1	
5,040	60	4	2	1	1	
7,560	64	3	3	1	1	
10,080	72	5	2	1	1	
15,120	80	4	3	1	1	
20,160	84	6	2	1	1	
25,200	90	4	2	2	1	
27,720	96	3	2	1	1	1
45,360	100	4	4	1	1	
50,400	108	5	2	2	1	
55,440	120	4	2	1	1	1
83,160	128	3	3	1	1	1
110,880	144	5	2	1	1	1
166,320	160	4	3	1	1	1
221,760	168	6	2	1	1	1
277,200	180	4	2	2	1	1

n	d(n)	a	b	c	d	e	f	g	h
		2	3	5	7	11	13	17	19
332,640	192	5	3	1	1	1			
498,960	200	4	4	1	1	1			
554,400	216	5	2	2	1	1			
665,280	224	6	3	1	1	1			
720,720	240	4	2	1	1	1	1		
1,081,080	256	3	3	1	1	1	1		
1,441,440	288	5	2	1	1	1	1		
2,162,160	320	4	3	1	1	1	1		
2,882,880	336	6	2	1	1	1	1		
3,603,600	360	4	2	2	1	1	1		
4,324,320	384	5	3	1	1	1	1		
6,486,480	400	4	4	1	1	1	1		
7,207,200	432	5	2	2	1	1	1		
8,648,640	448	6	3	1	1	1	1		
10,810,800	480	4	3	2	1	1	1		
14,414,400	504	6	2	2	1	1	1		
17,297,280	512	7	3	1	1	1	1		
21,621,600	576	5	3	2	1	1	1		
32,432,400	600	4	4	2	1	1	1		
36,756,720	640	4	3	1	1	1	1	1	
43,243,200	672	6	3	2	1	1	1		
61,261,200	720	4	2	2	1	1	1	1	
73,513,440	768	5	3	1	1	1	1	1	
110,270,160	800	4	4	1	1	1	1	1	
122,522,400	864	5	2	2	1	1	1	1	
147,026,880	896	6	3	1	1	1	1	1	
183,783,600	960	4	3	2	1	1	1	1	
245,044,800	1008	6	2	2	1	1	1	1	
294,053,760	1024	7	3	1	1	1	1	1	
367,567,200	1152	5	3	2	1	1	1	1	
551,350,800	1200	4	4	2	1	1	1	1	
698,377,680	1280	4	3	1	1	1	1	1	1
735,134,400	1344	6	3	2	1	1	1	1	

Calculating d(n) using Divisor Charts

1	3

1
2
4
8
16
32

- If p is prime, $d(p) = 2$

  $d(3) = \mathbf{2}$, since $\{1, 3\}$ only.

- If p is prime, $d(p^k) = k + 1$

  $d(32) = d(2^5) = 5 + 1 = \mathbf{6}$.

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- If $\gcd(m, n) = 1$, then

$d(m \cdot n) = d(m) \cdot d(n)$

$$d(96) = d(32 \cdot 3)$$
$$= d(32) \cdot d(3)$$
$$= 6 \cdot 2 = 12$$

1	3
2	6
4	12
8	24
16	48
32	96

Calculating d(n) using Divisor Charts

$n = 2^a \cdot 3^b$

1	3

1
2
4
8
16
32

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$d(96) = d(32 \cdot 3) \quad = d(2^5 \cdot 3^1)$

$= d(32) \cdot d(3) = (5+1)(1+1)$

$= 6 \cdot 2 = 12 \quad = 6 \cdot 2 = 12$

In general if $n = p^a q^b r^c \ldots$

then $d(n) = (a+1)(b+1)(c+1) \ldots$

* 3 →

* 2 ↓

1	3
2	6
4	12
8	24
16	48
32	96

Calculating d(n) using Divisor Charts

1	3

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2
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In general if $n = p^a q^b r^c \ldots$

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$n = 2^a \cdot 3^b$

* 3 →

* 2 ↓

a \ b	0	1	2	3	4	5	6
0	1	3	**9**	27	81	**243**	**729**
1	2	6	18	54	**162**	**486**	1458
2	4	12	36	**108**	**324**	**972**	2916
3	8	24	72	**216**	**648**	1944	5832
4	16	48	**144**	**432**	1296	3888	**11664**
5	32	96	**288**	**864**	2592	7776	**23328**
6	64	**192**	**576**	1728	5184	**15552**	**46656**
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8	**256**	**768**	2304	6912	**20736**	**62208**	186624
9	**512**	1536	4608	**13824**	**41472**	124416	373248

For more primes, we need more dimensions!

Calculating d(n) using Divisor Charts

The number $n = 16 = 2^4$ is not s.c. because it has 'too many' 2s.
See, $m = (3/4) \cdot n = 12 = 2^2 \cdot 3^1$ has at least as many divisors:
$d(16) = (4+1) = 5$, while
$d(12) = (2+1)(1+1) = 6$.

$n = 2^a \cdot 3^b$

* 3 →

* 2 ↓

× 3/4

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Let $n = 2^a \cdot 3^b$
and $m = 3n / 4$
$= 2^{a-2} \cdot 3^{b+1}$

Then $d(n) = (a+1)(b+1)$
and $d(m) = (a-1)(b+2)$

If n is supercomp then $d(m) < d(n)$, meaning
$(a-1)(b+2) < (a+1)(b+1)$
$ab+2a-b-2 < ab+a+b+1$
$a < 2b + 3$; $a \le 2b + 2$
is nec. if n is super!

$n = 2^a \cdot 3^b$

* 3 → * 2 ↓ x 3/4

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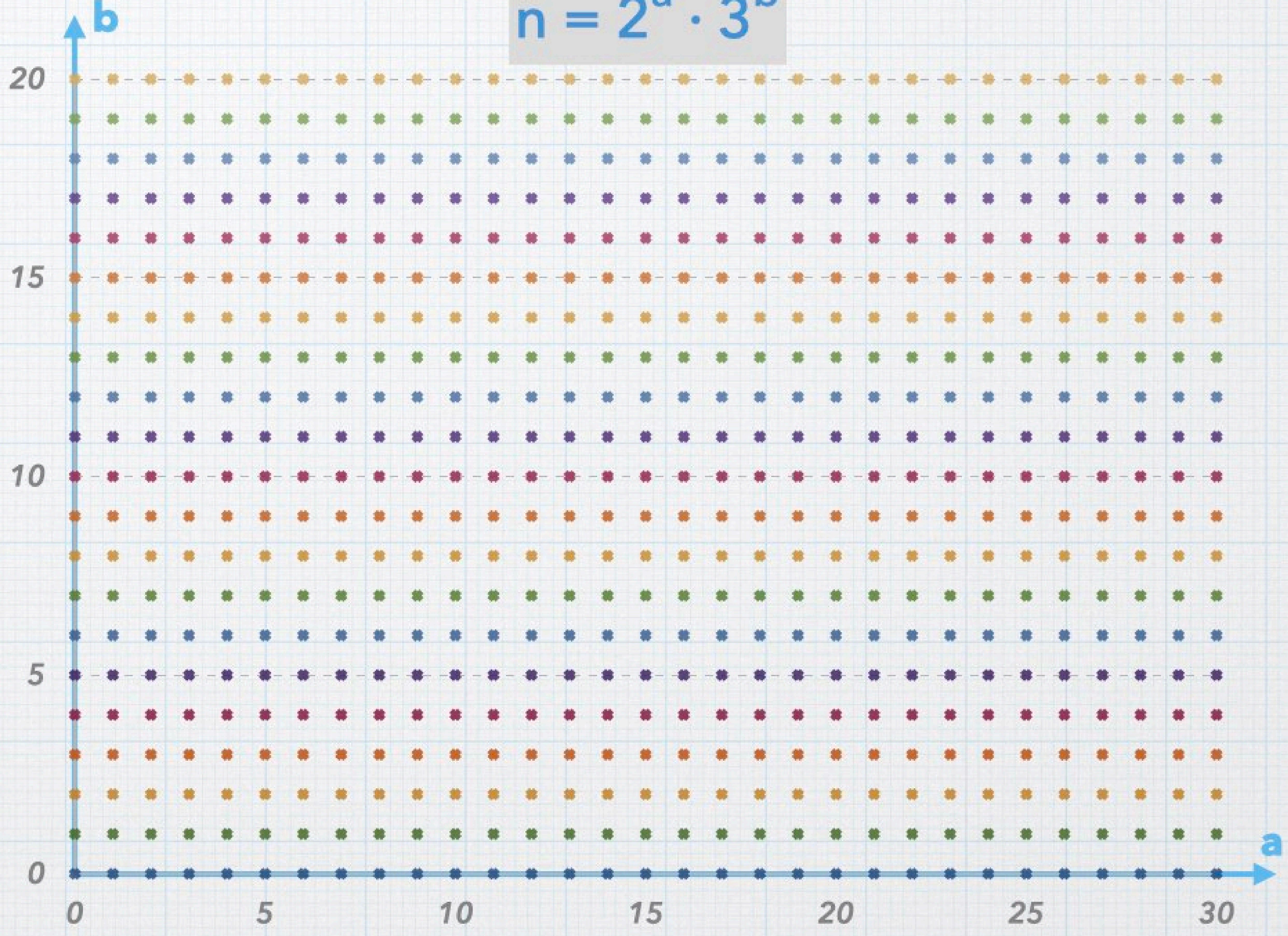
light blue cells are eliminated

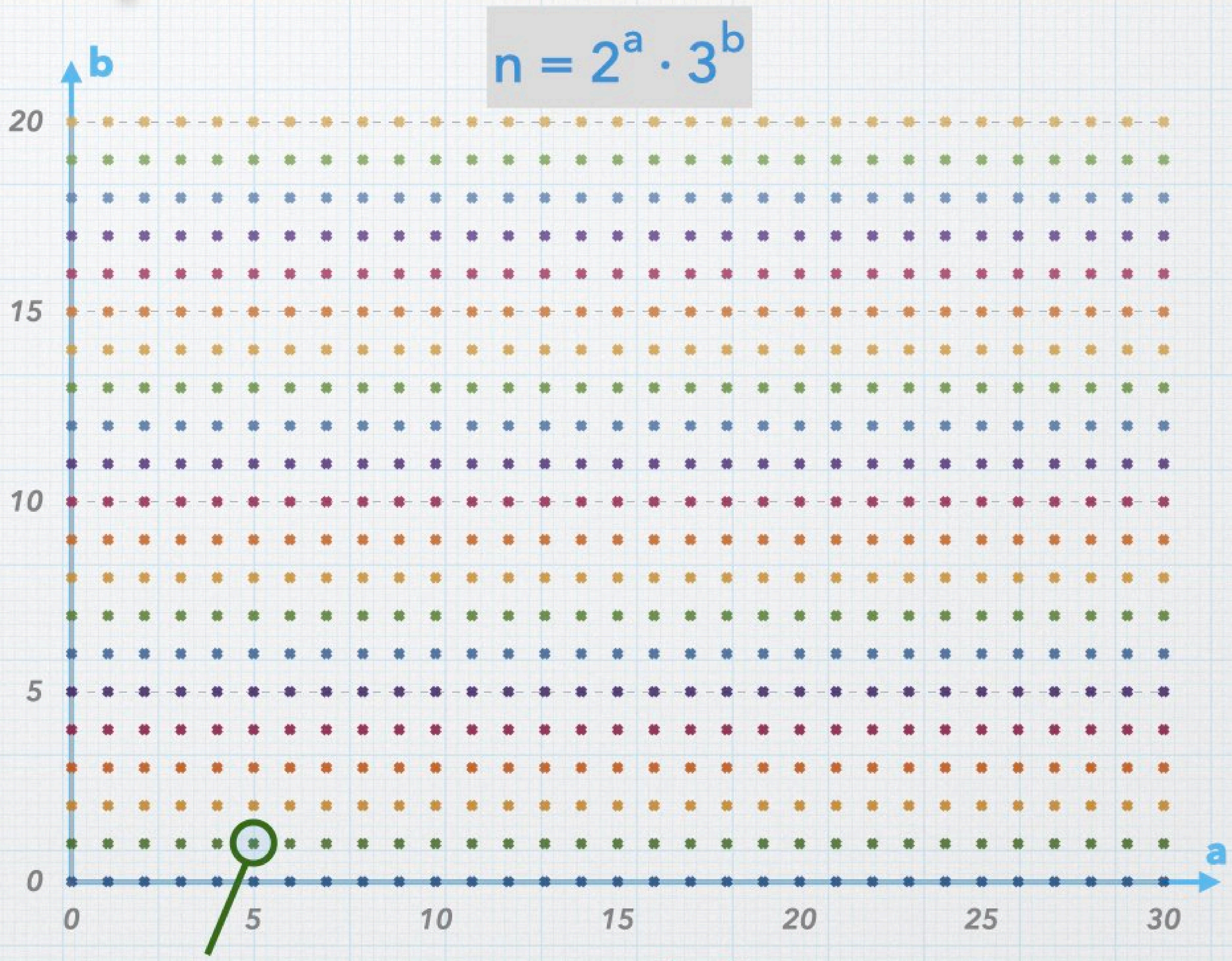
Exponent Lattice Rules and Inequalities

$n = 2^a \cdot 3^b$



Divisor Chart (without numbers)

Exponent Lattice Rules and Inequalities



This is $(a, b) = (5,1) = 2^5\ 3^1 = 96$.

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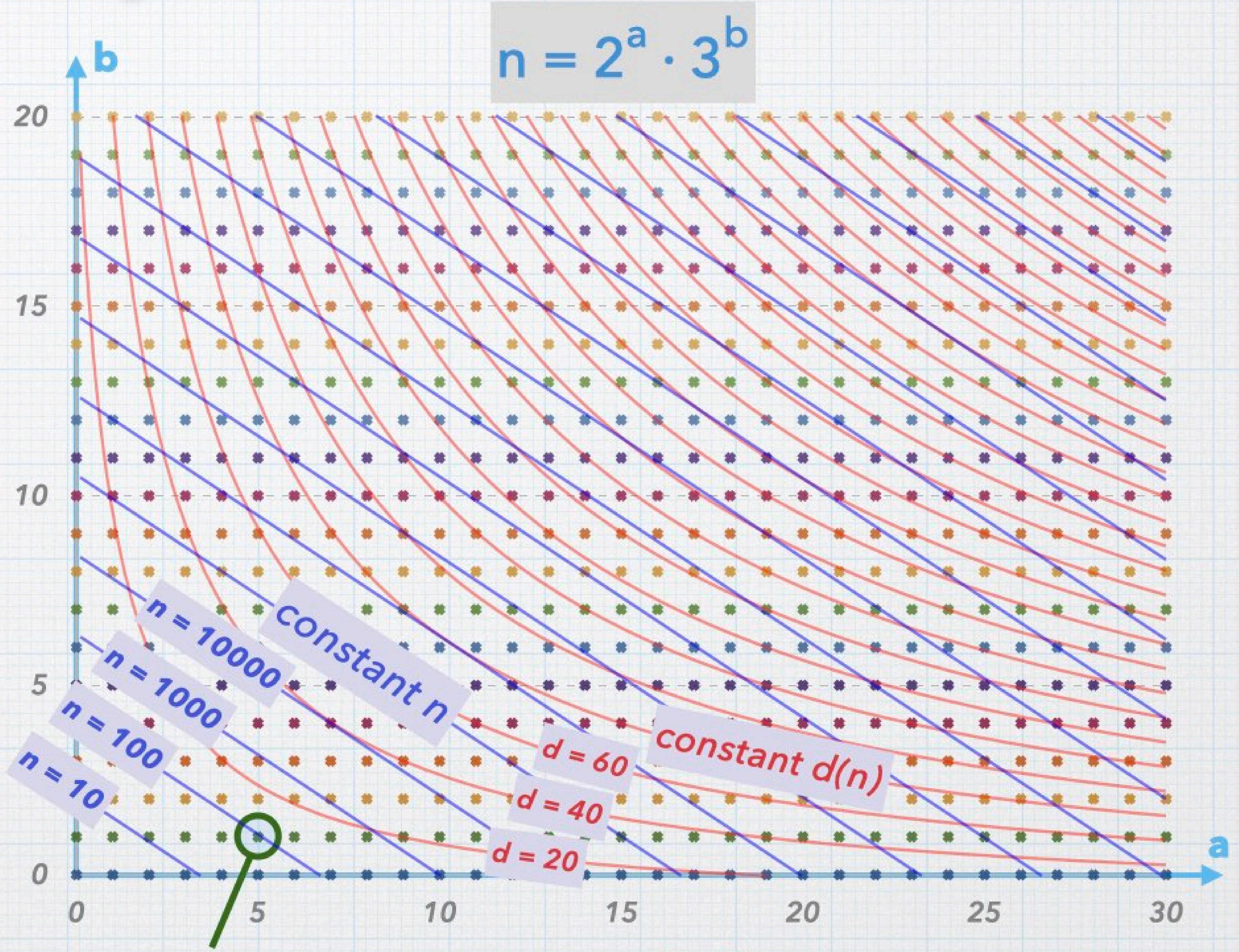

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to be supercomposite, we need:

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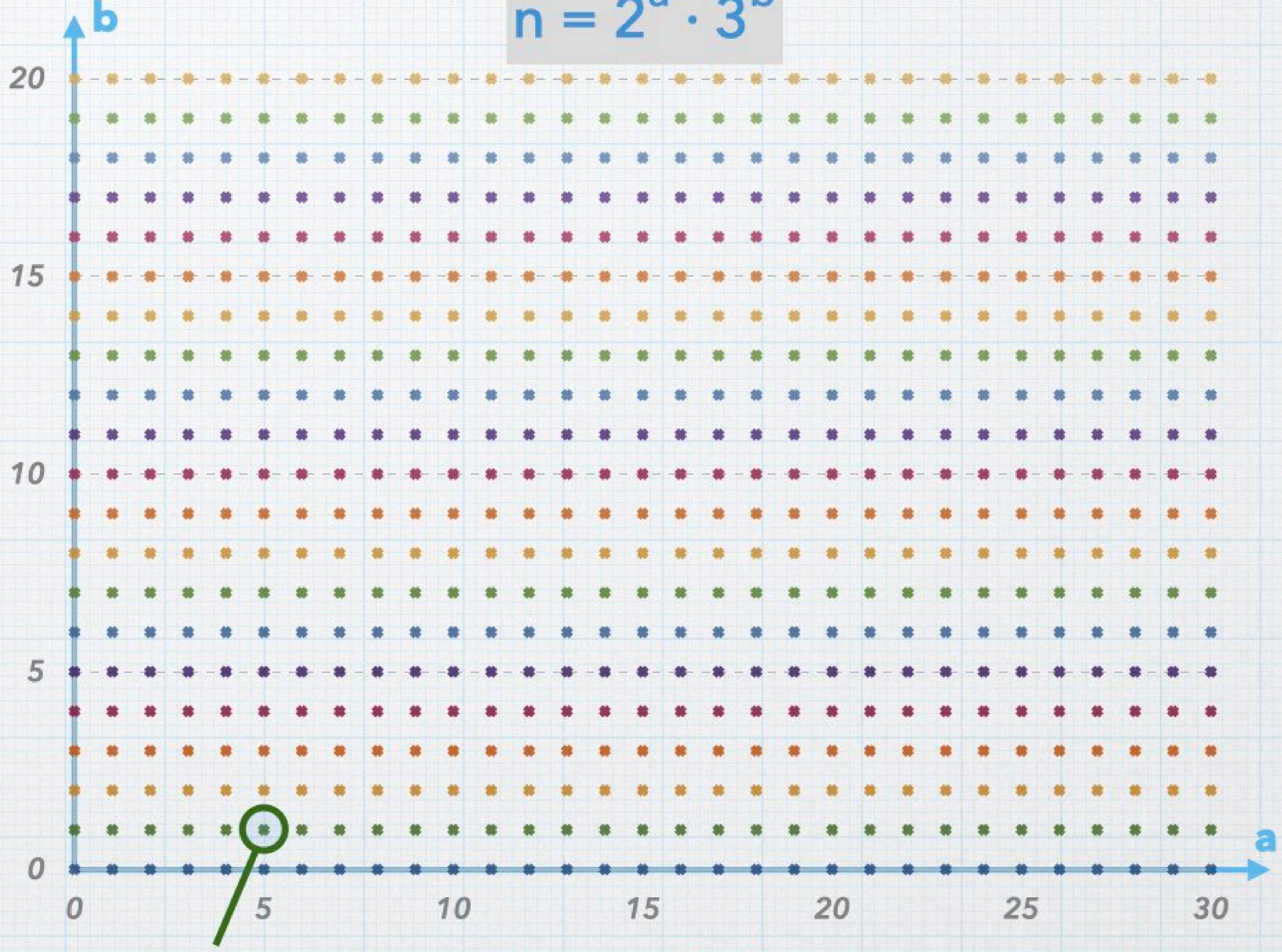

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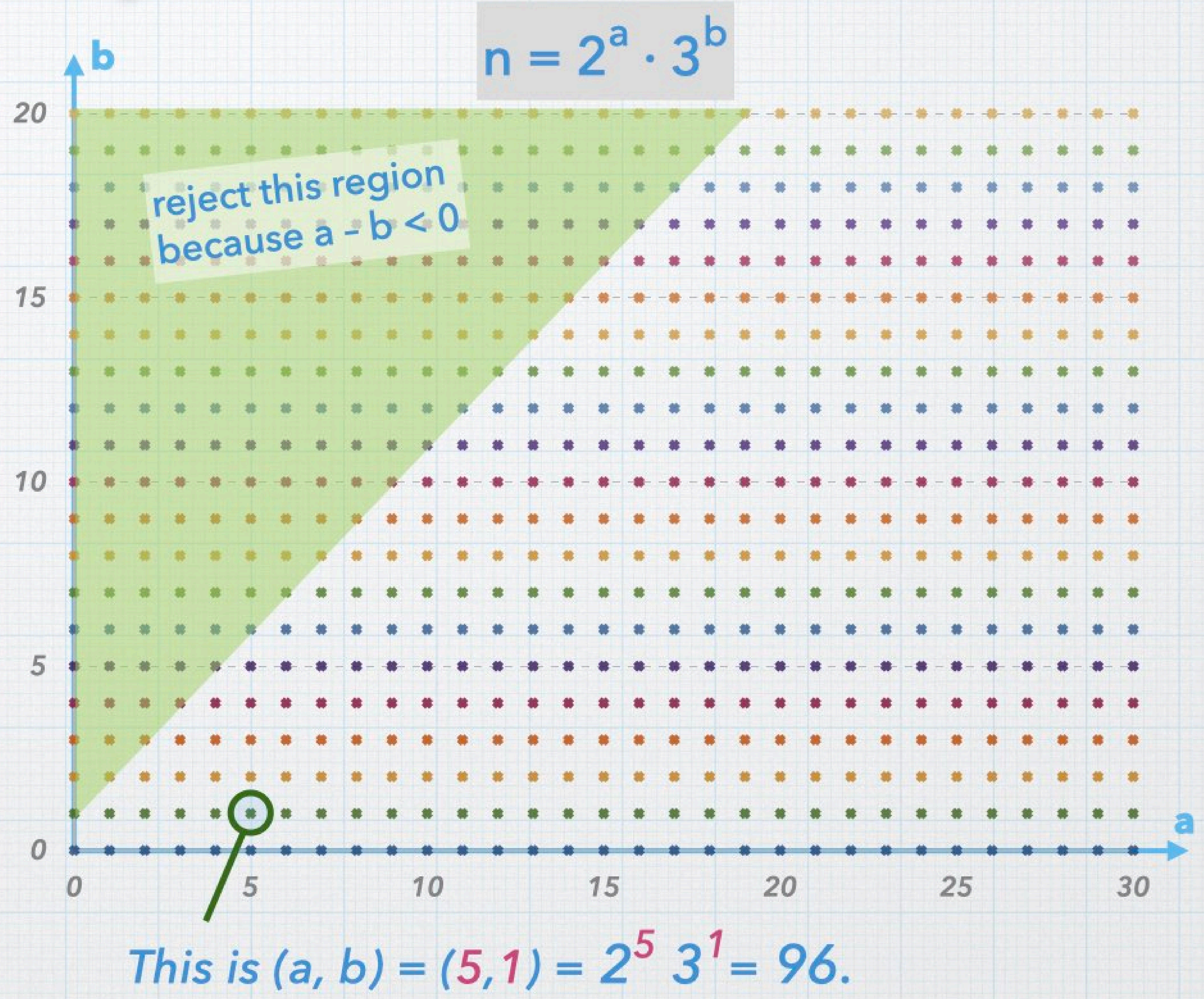

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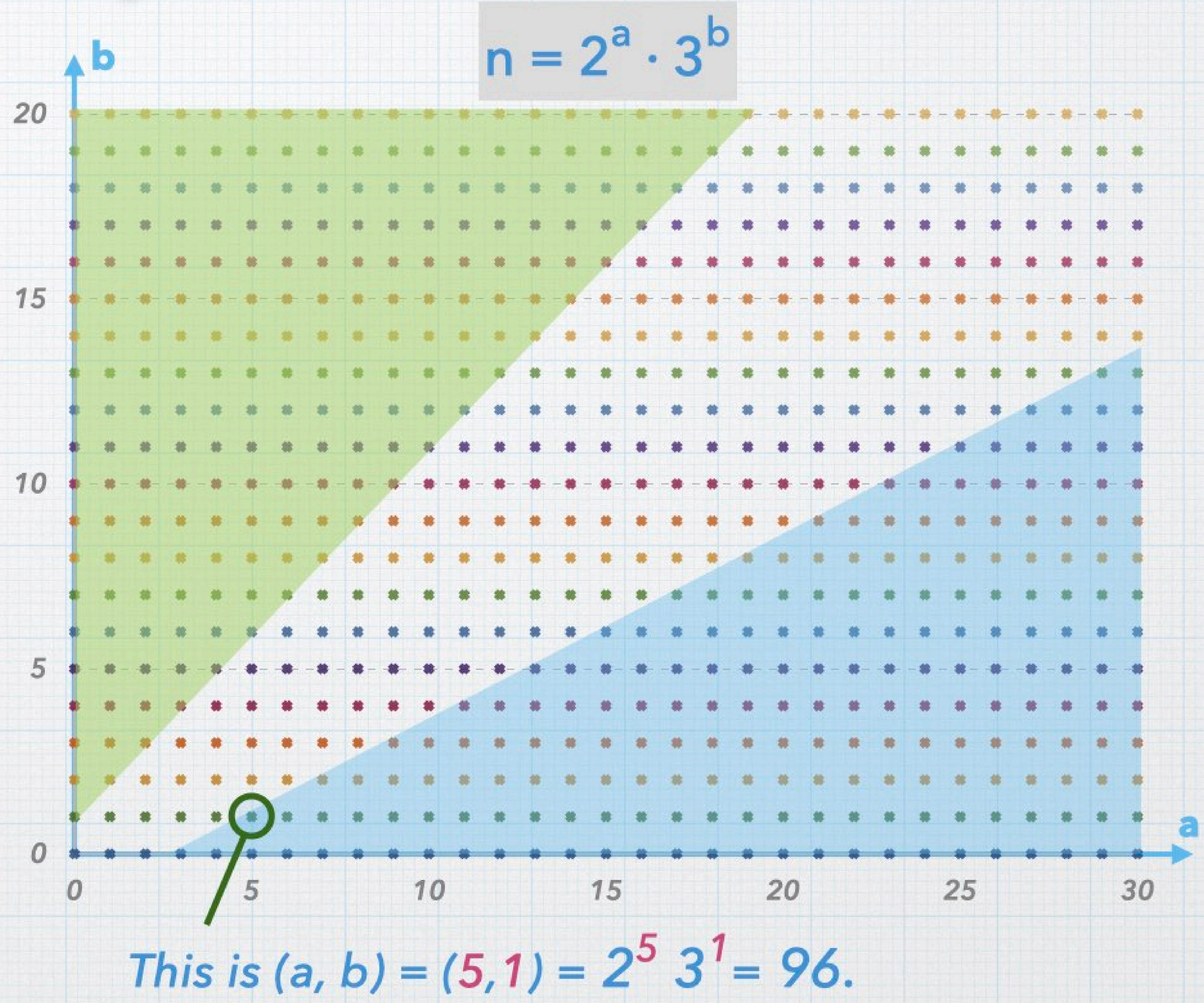

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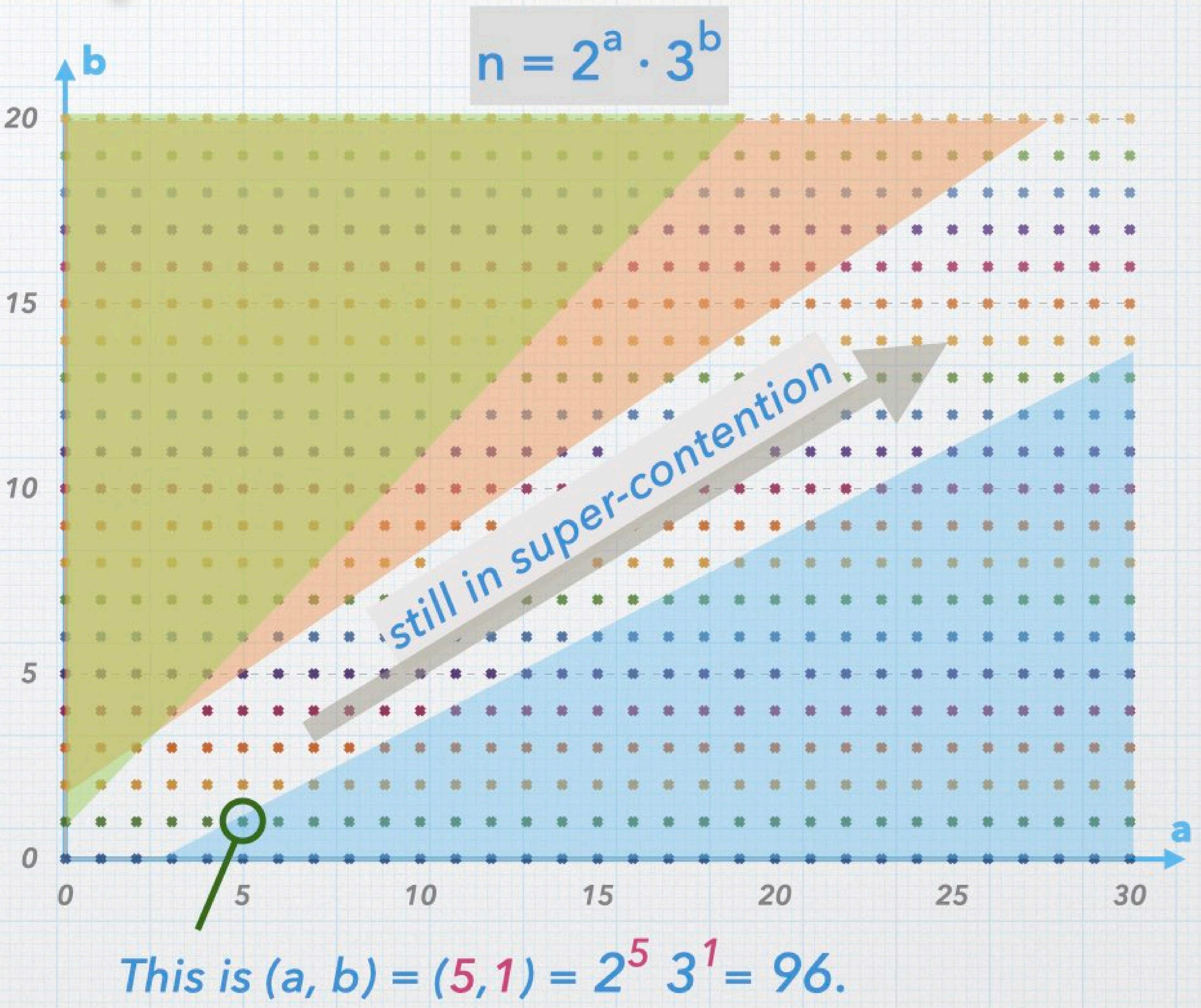

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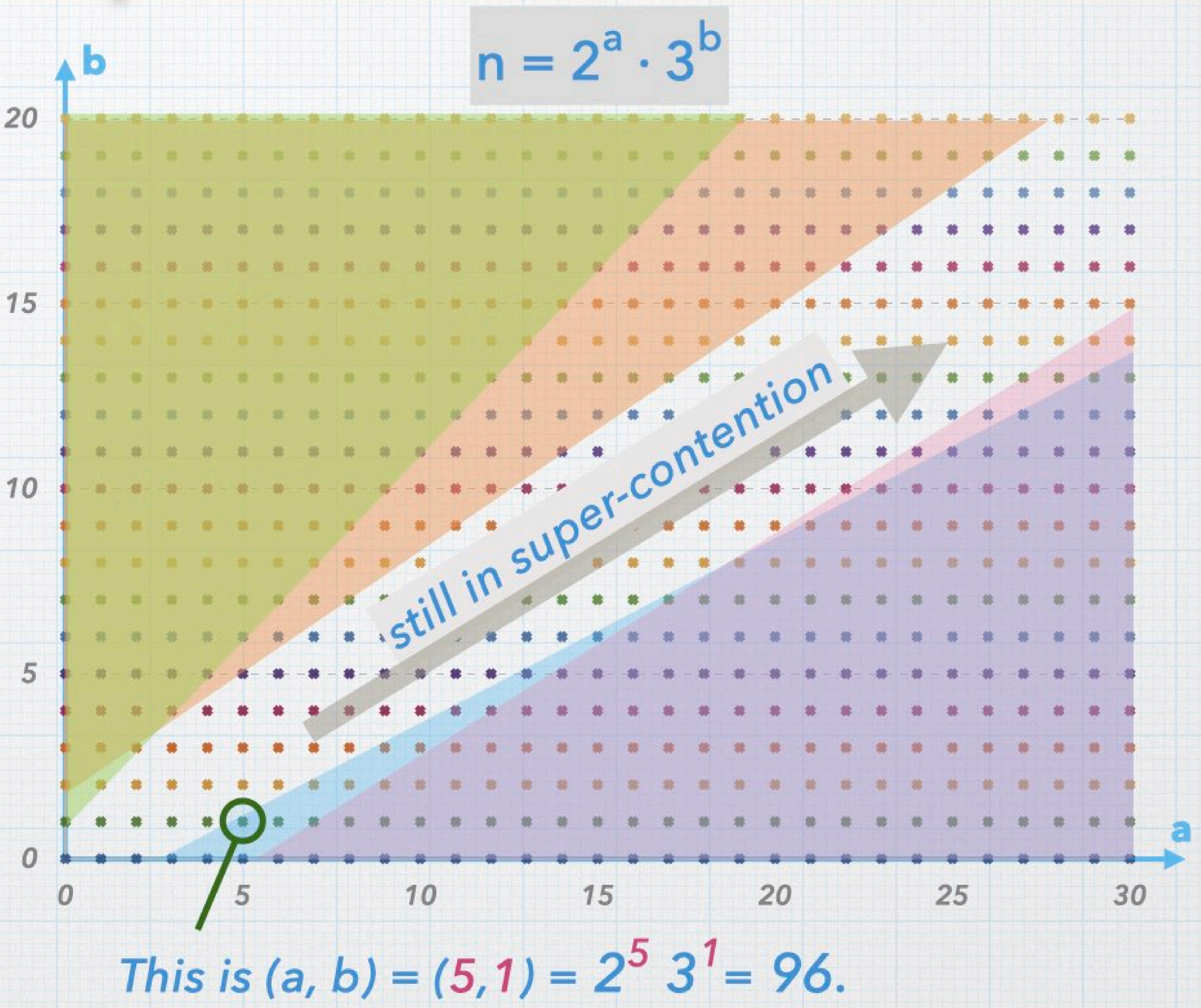
- $3a - 5b \leq 16$ *(rule of 27/32)*



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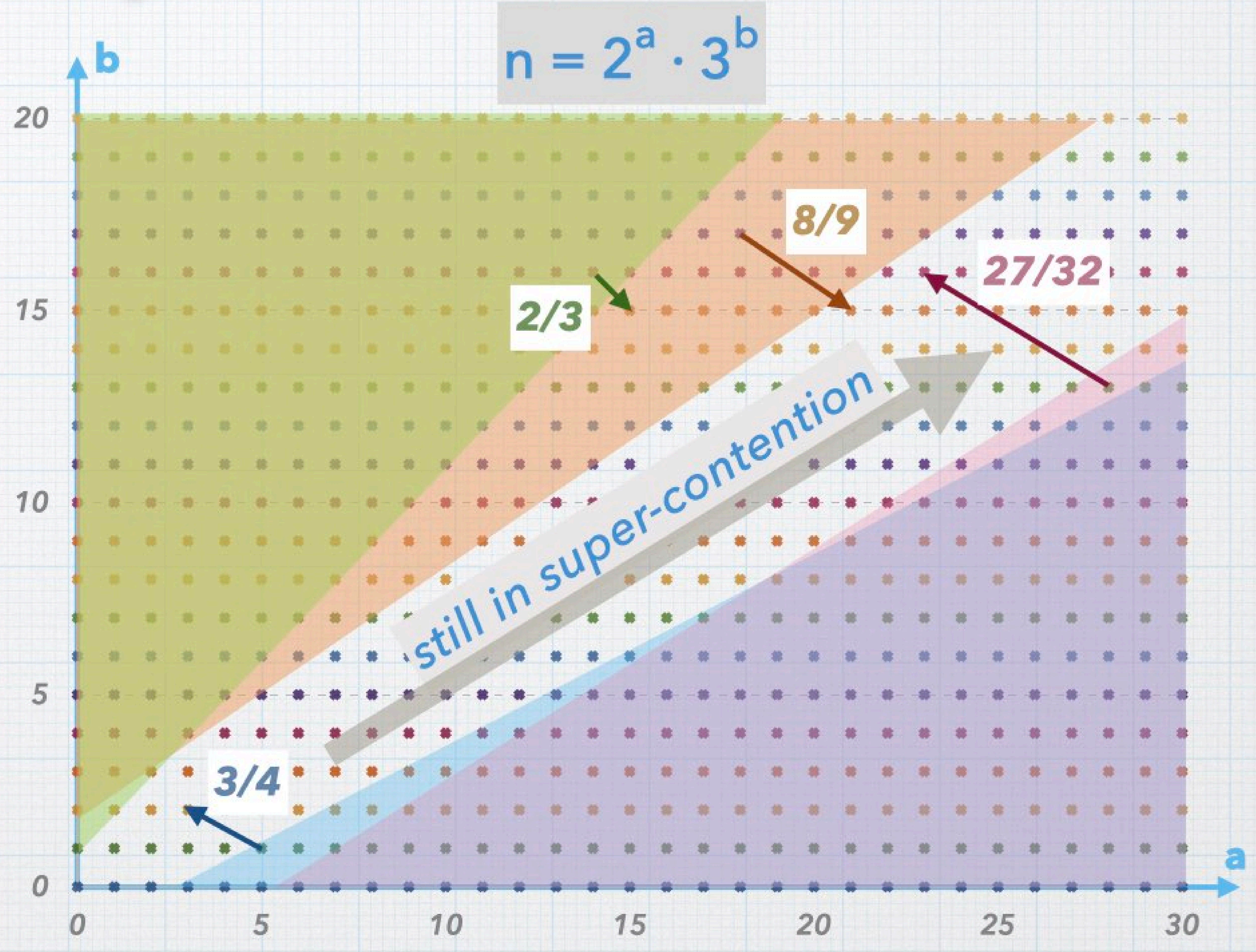
- $2a - 3b \geq -4$ *(rule of 8/9)*
- $3a - 5b \leq 16$ *(rule of 27/32)*

Other rules can be applied to eliminate more $n = (a, b)$ from contention.

(243/256, 2048/2187, etc.)

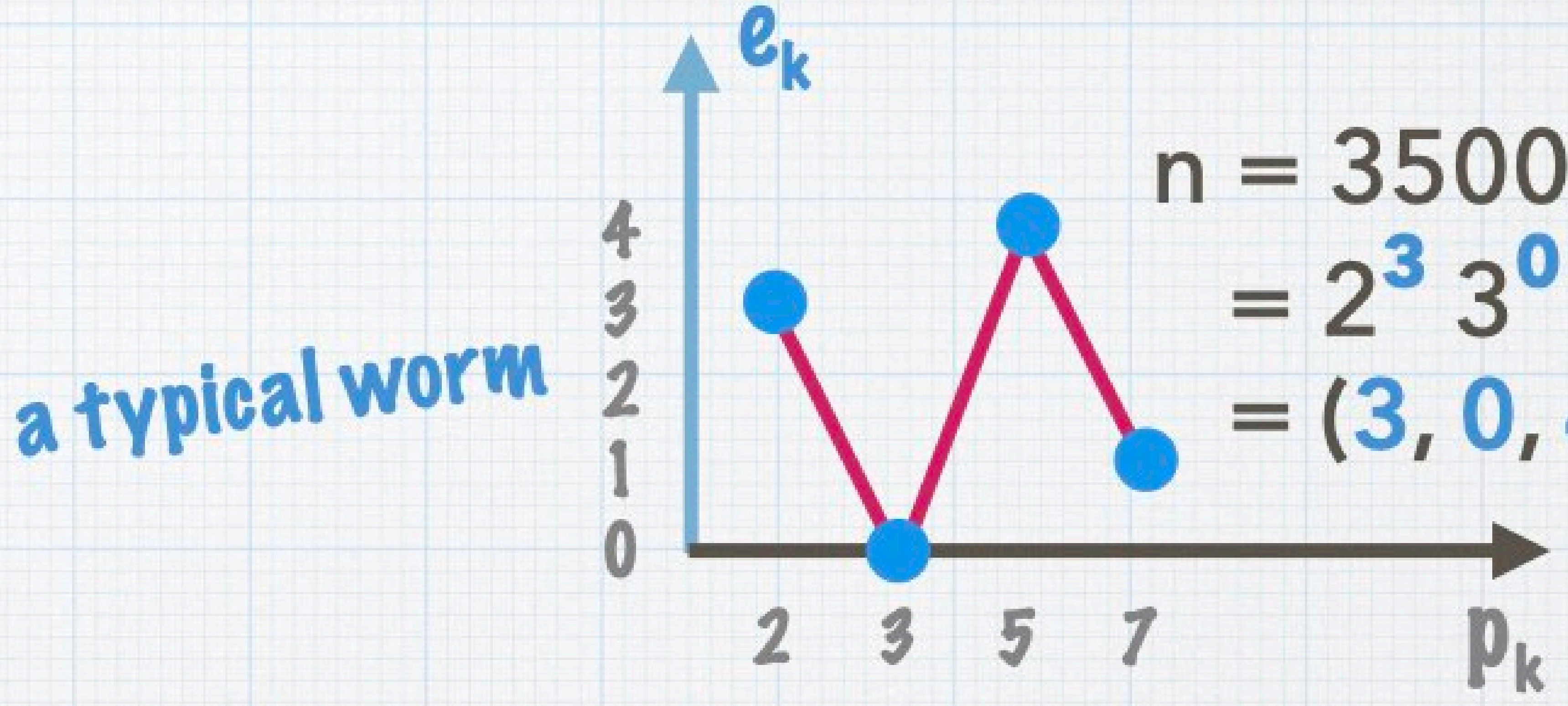


The slopes approach log 2 / log 3 as a limit.

Exponent Worms in Action

Each p (x-axis) is raised to its power (y-axis)

$n = 2^a\, 3^b\, 5^c \ldots = p_1^{e_1}\, p_2^{e_2} \ldots$



$n = 35000$
$= 2^3\, 3^0\, 5^4\, 7^1$
$= (3, 0, 4, 1)$

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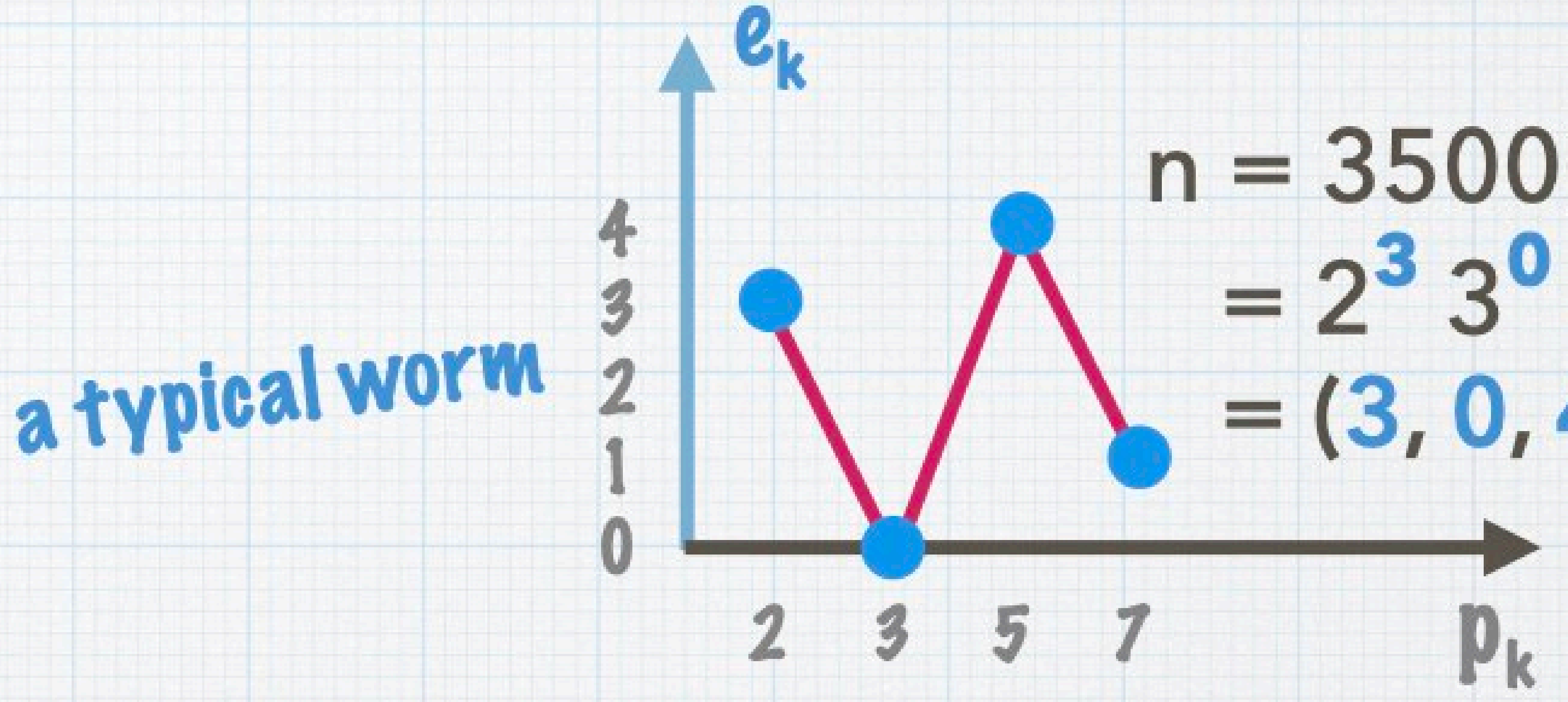
Let's look at more exponent worms...



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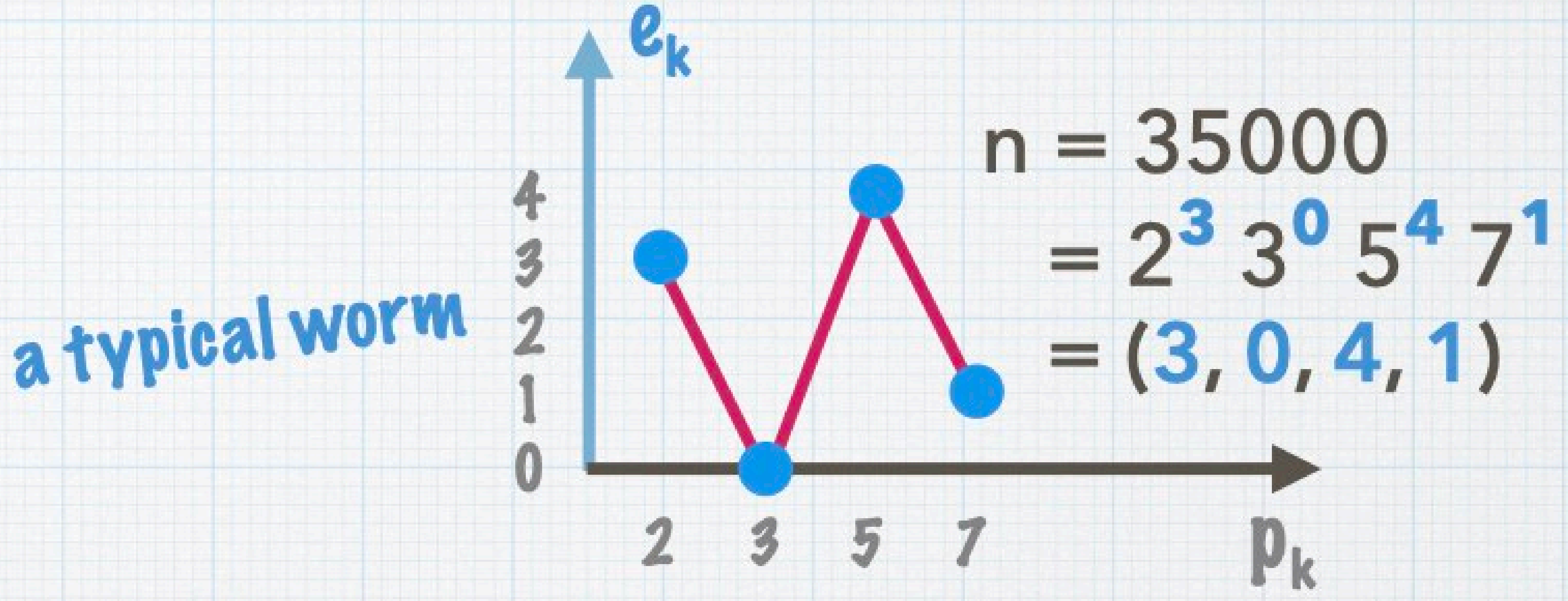
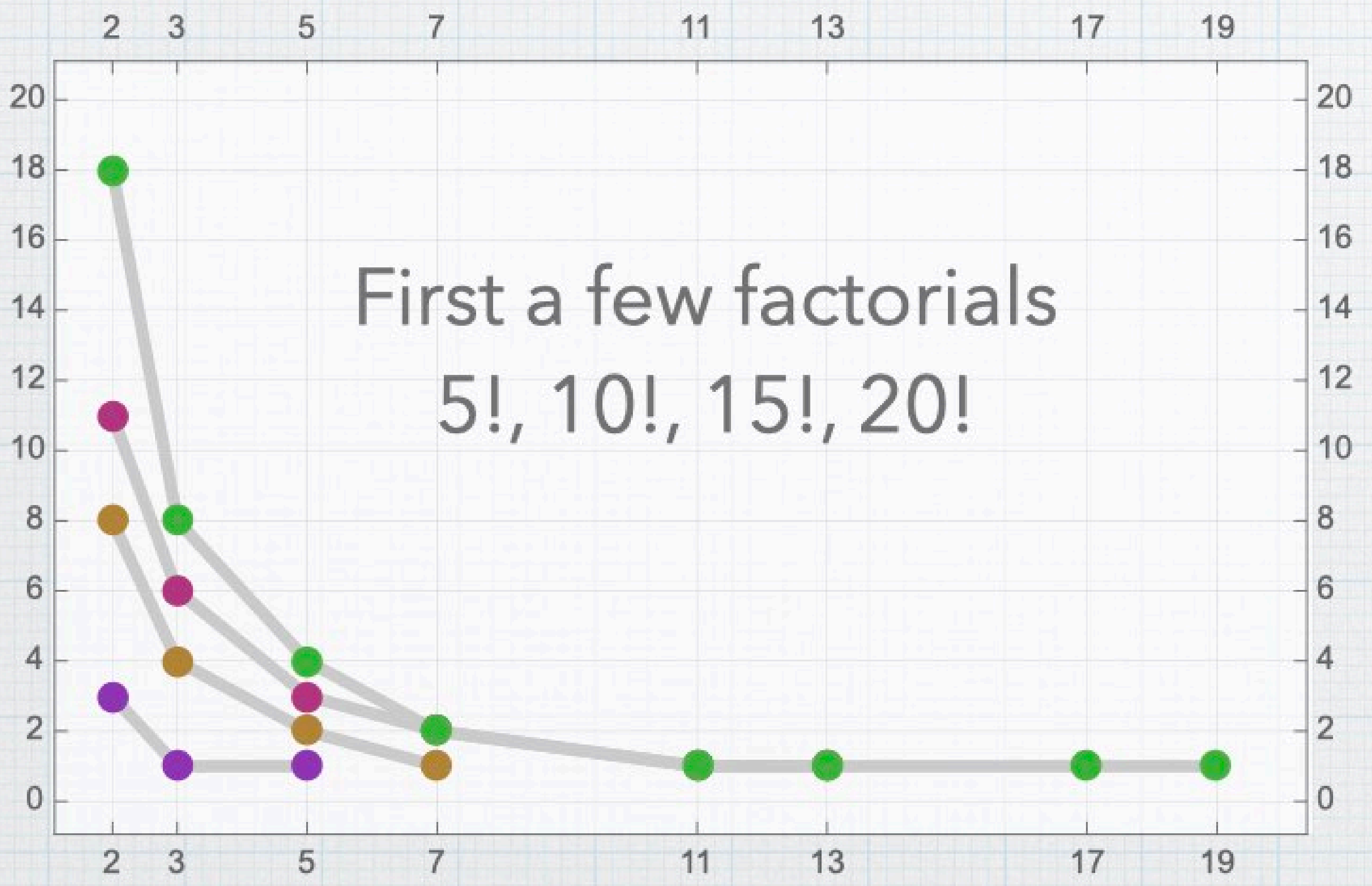


Let's look at more exponent worms...

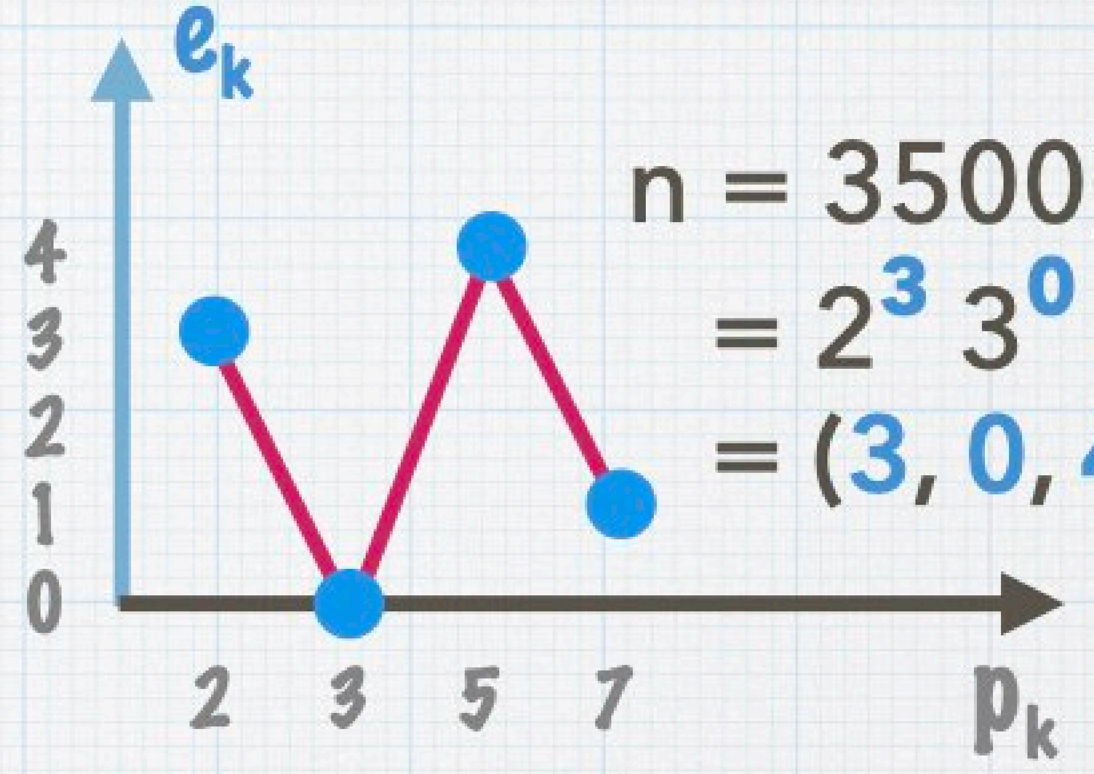


Exponent Worms in Action

Each p (x-axis) is raised to its power (y-axis)

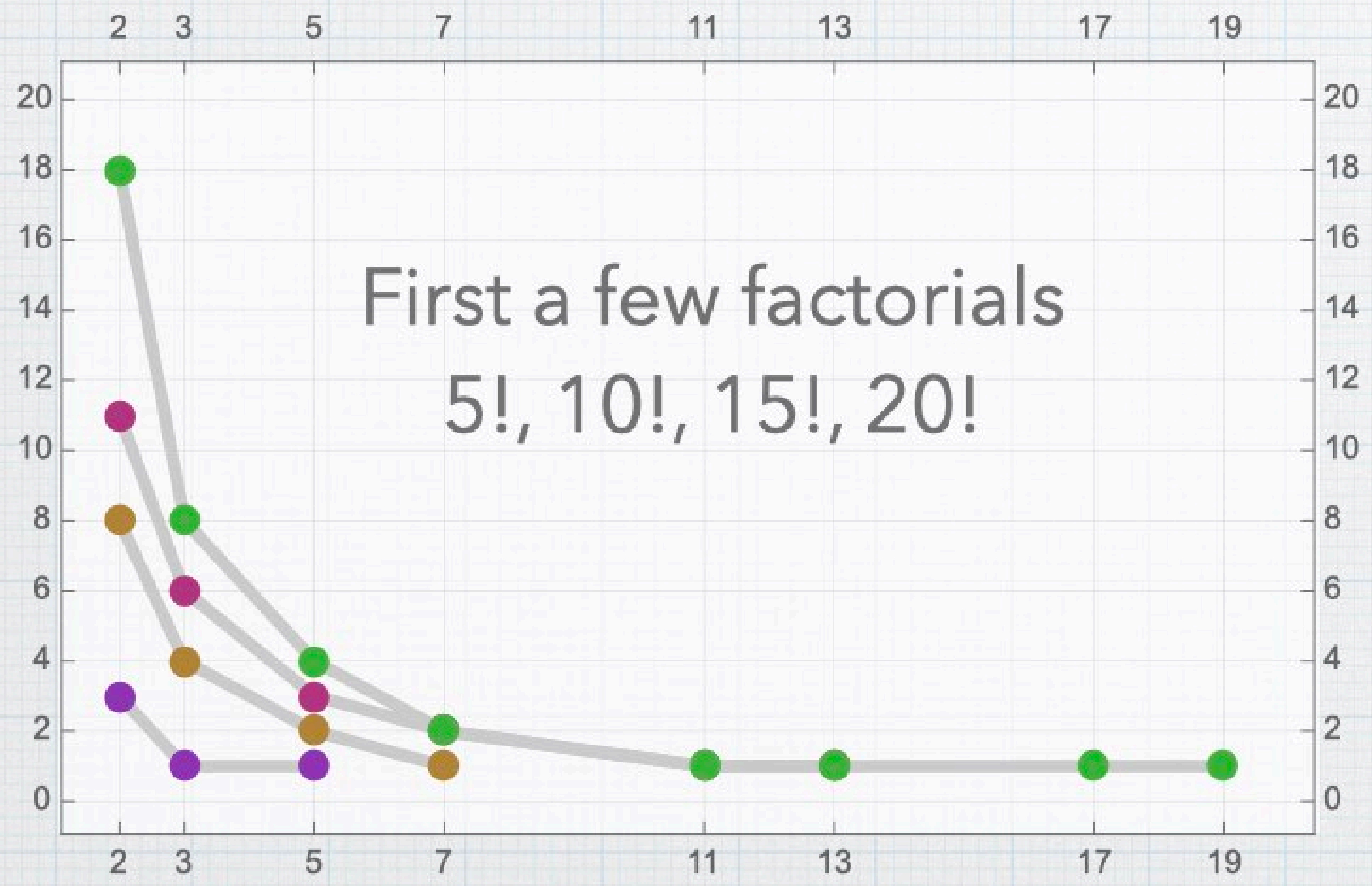
$$n = 2^a\, 3^b\, 5^c \ldots = p_1^{e_1}\, p_2^{e_2} \ldots$$

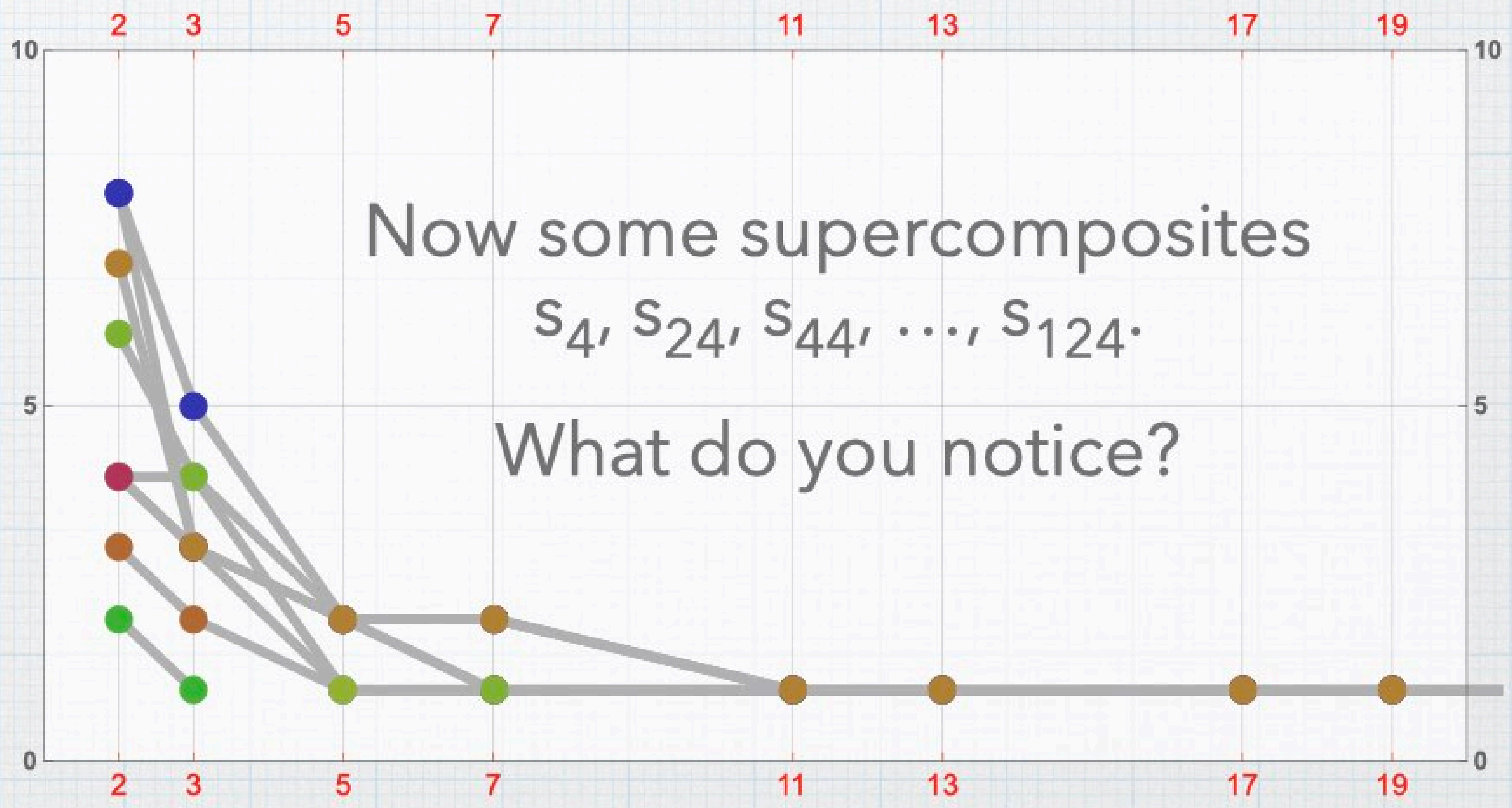
a typical worm



$$n = 35000$$
$$= 2^3\, 3^0\, 5^4\, 7^1$$
$$= (3, 0, 4, 1)$$

Let's look at more exponent worms...



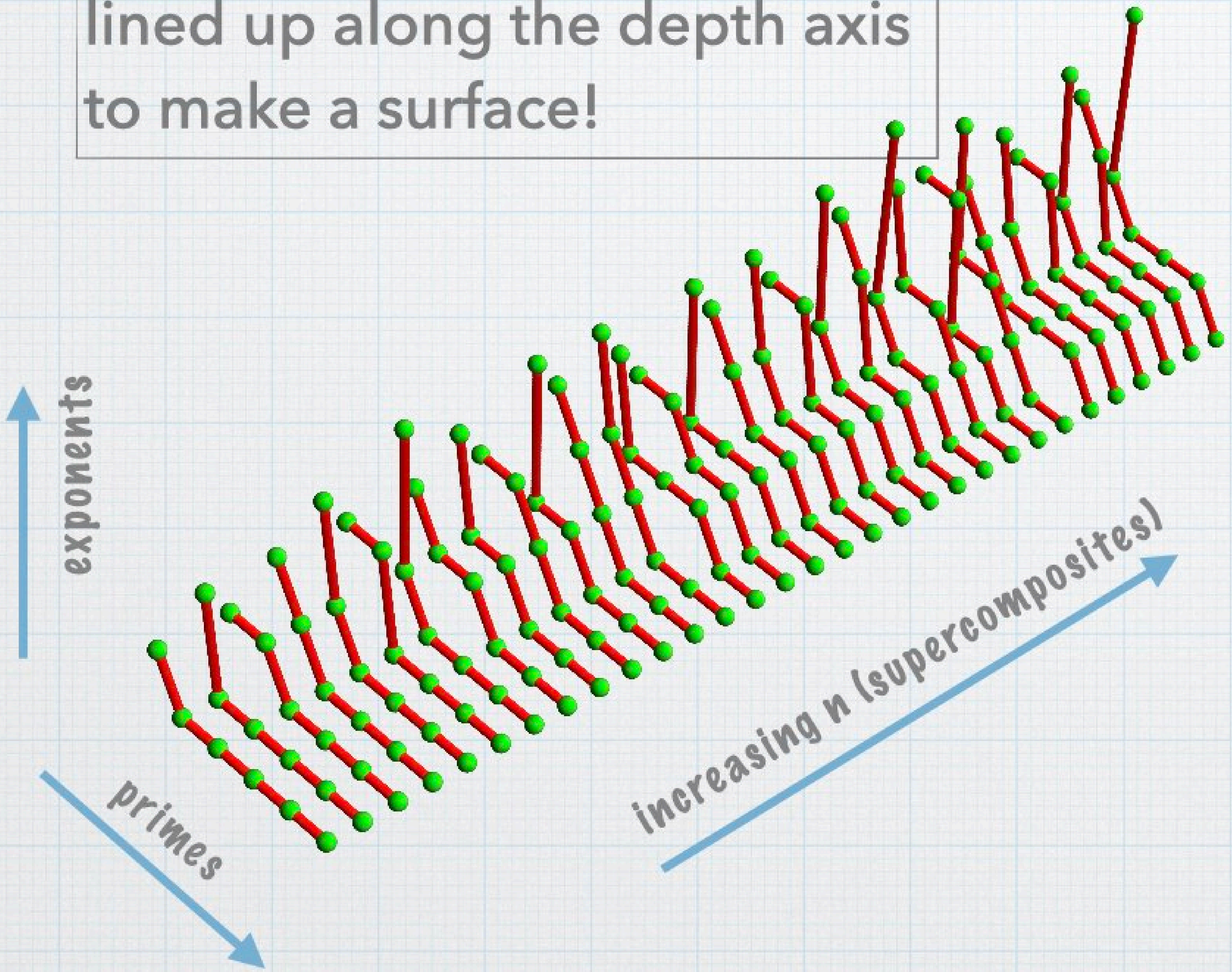


The Worms Crawl into 3D

The exponent worms can be lined up along the depth axis to make a surface!

The Worms Crawl into 3D

The exponent worms can be lined up along the depth axis to make a surface!



The Worms Crawl into 3D

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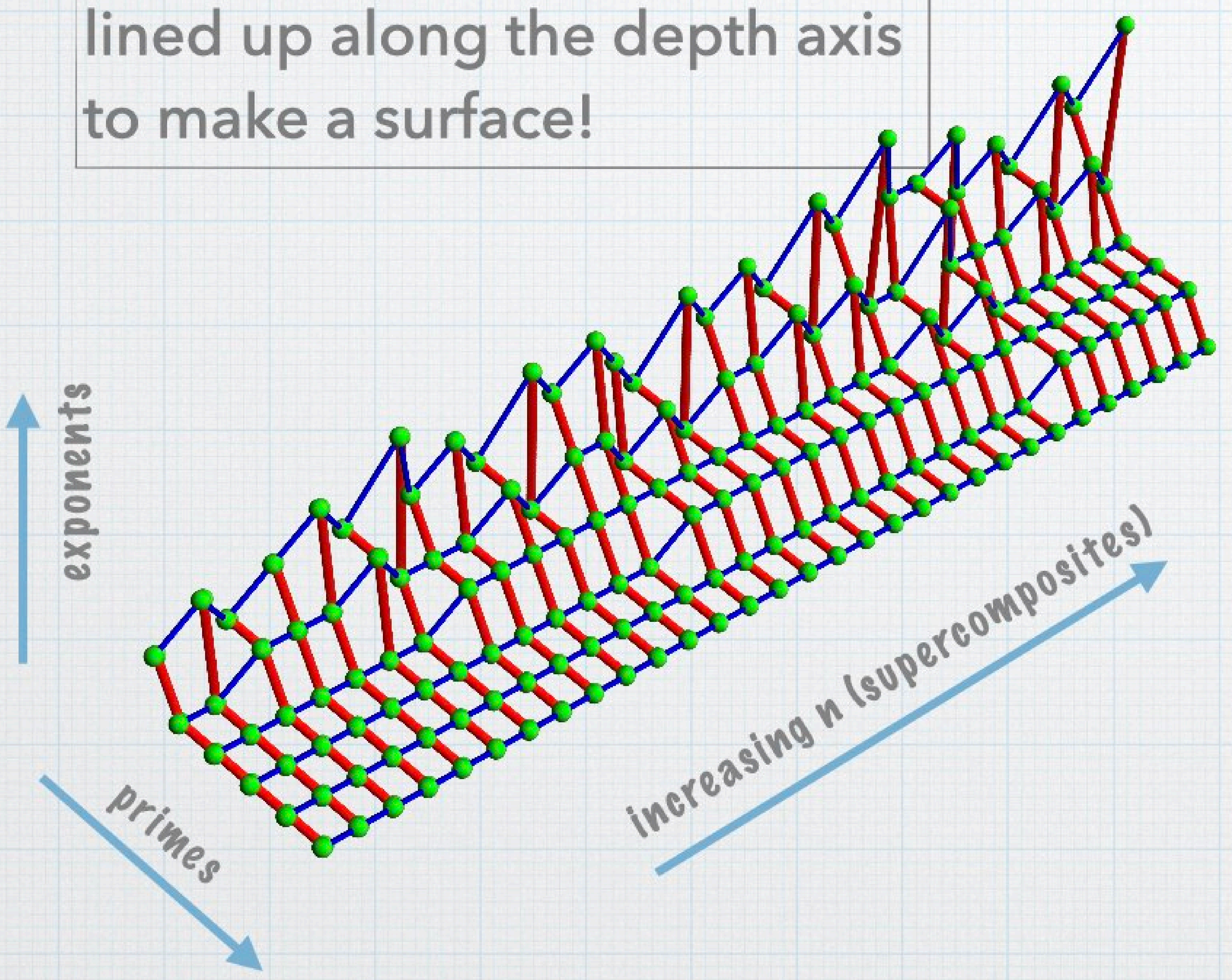

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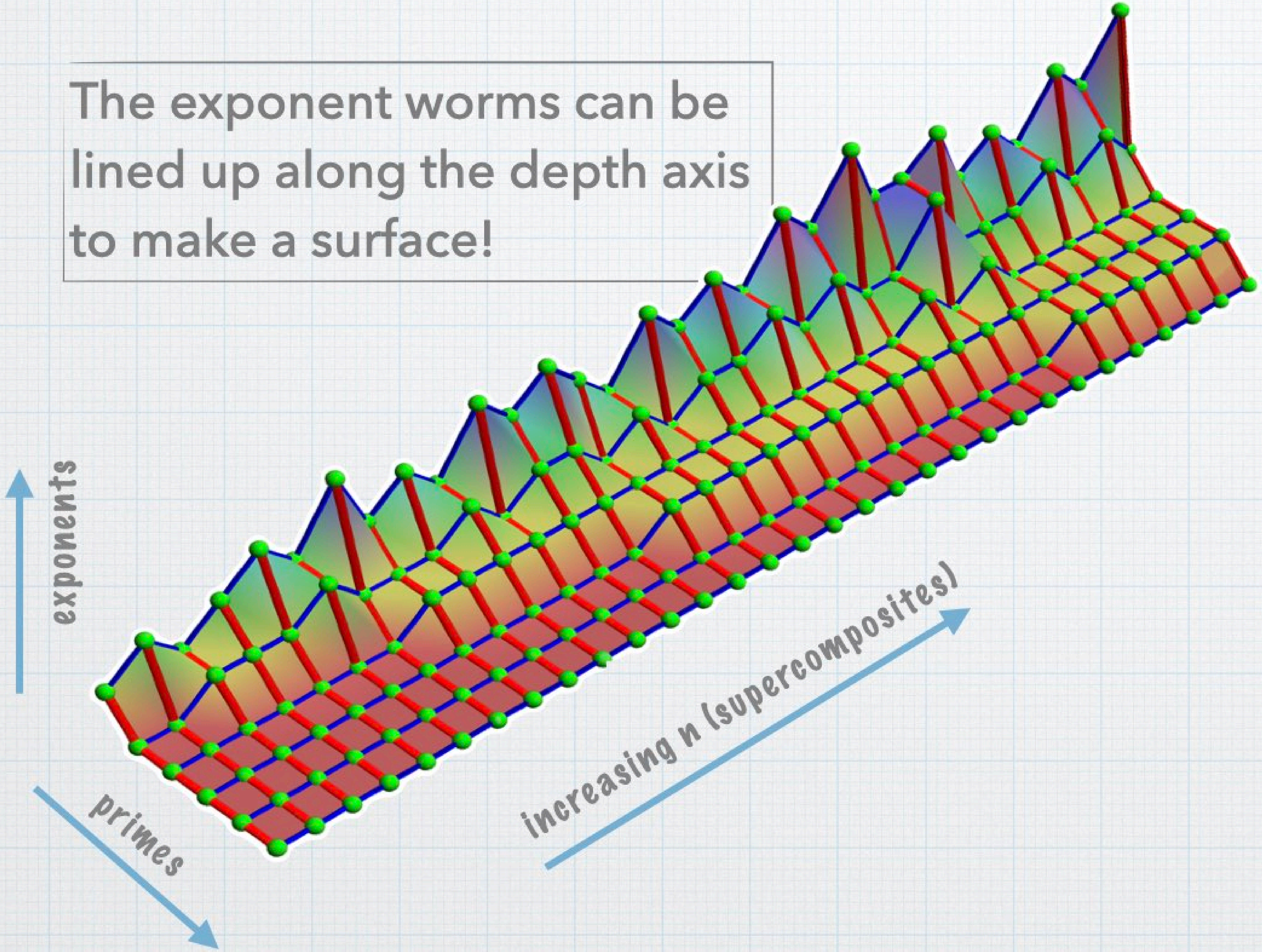

The Worms Crawl into 3D

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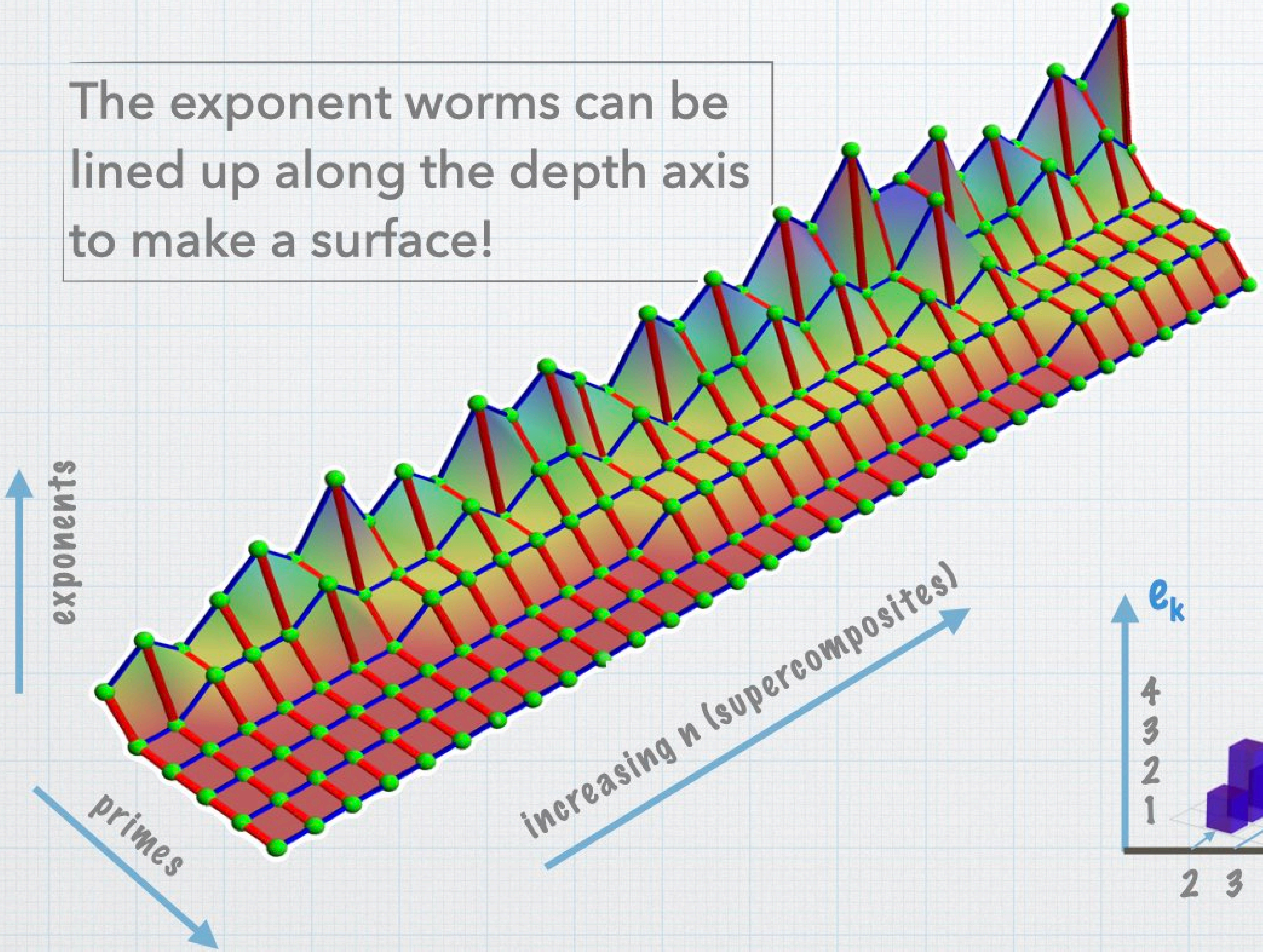
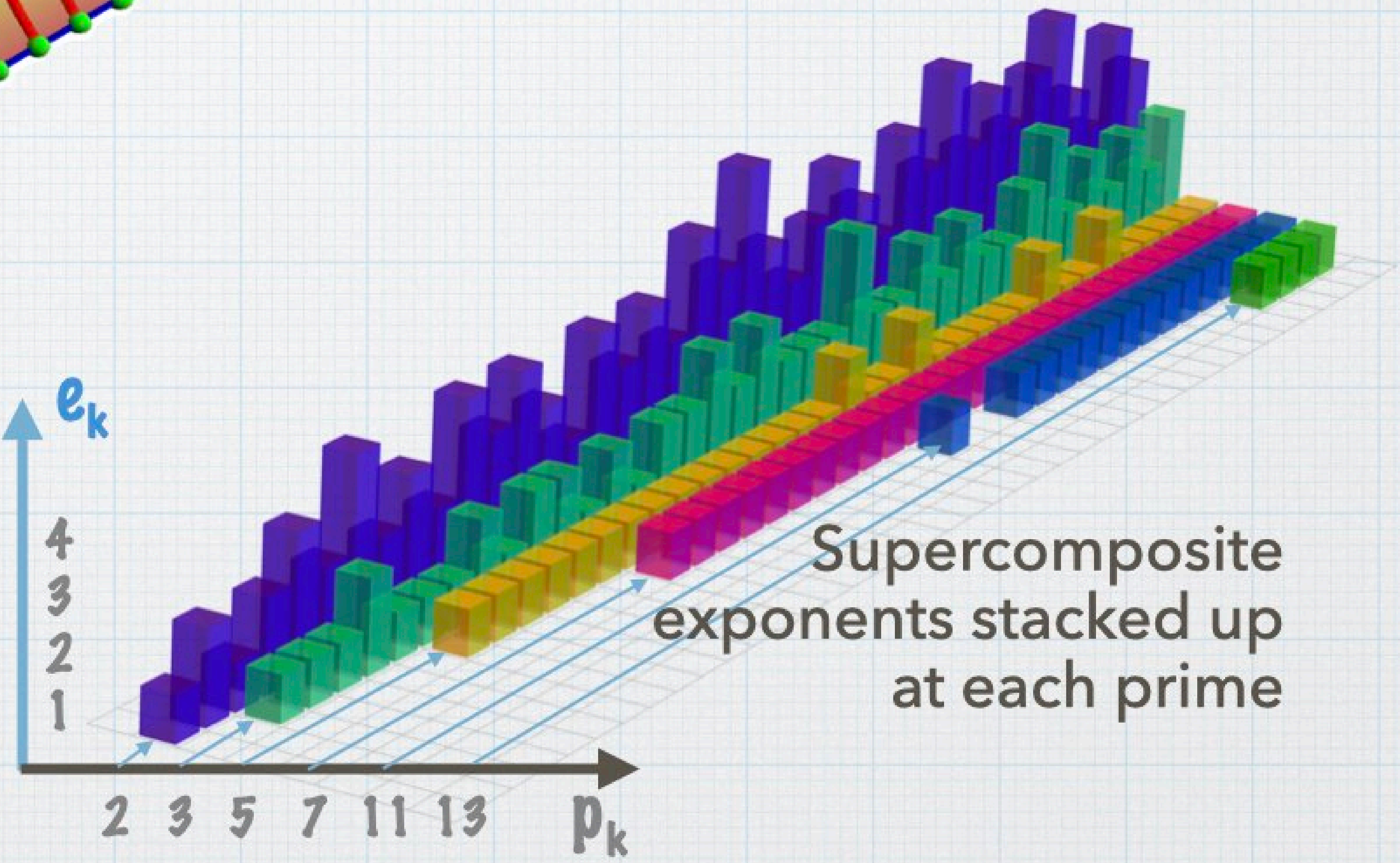
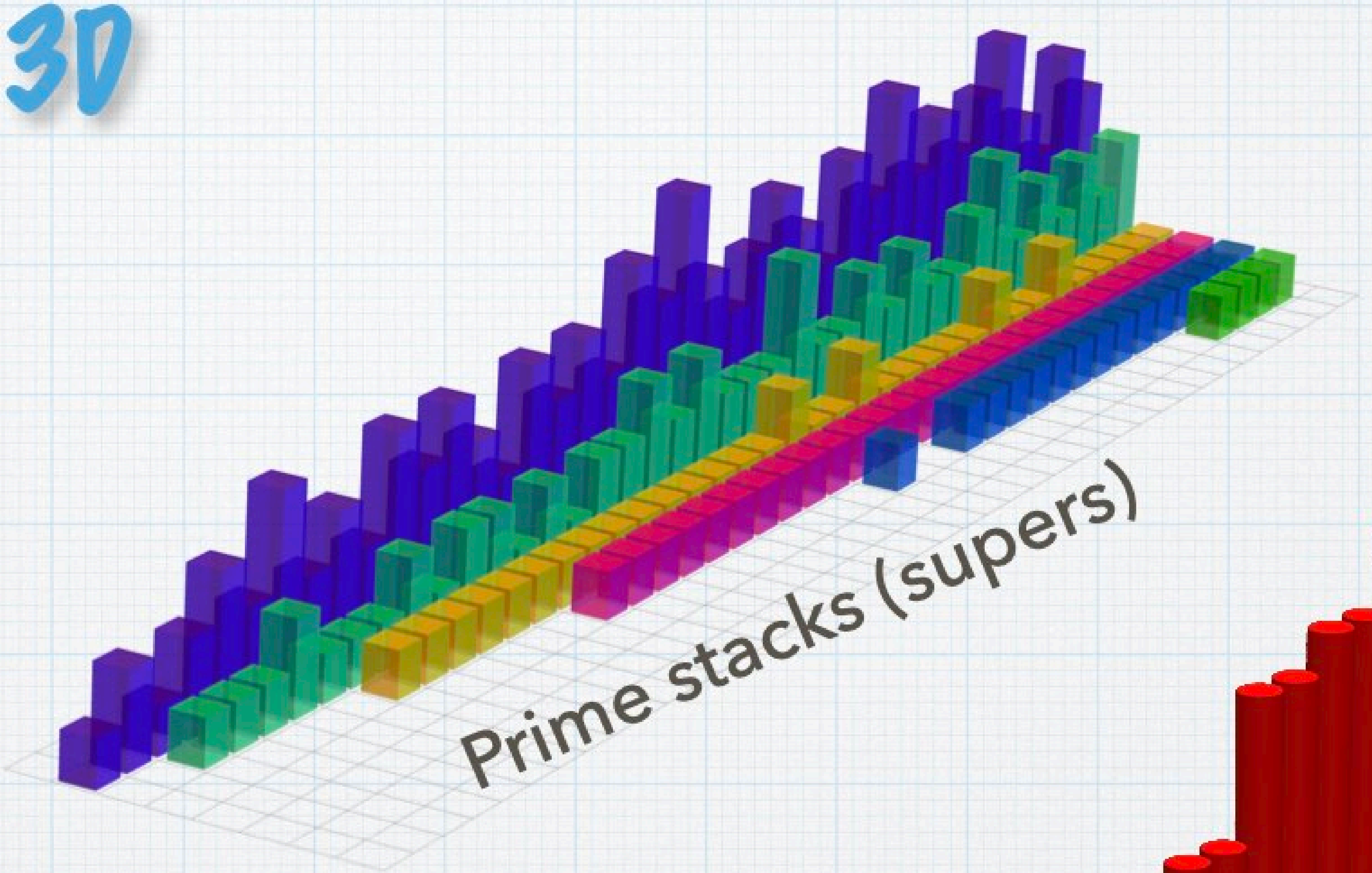
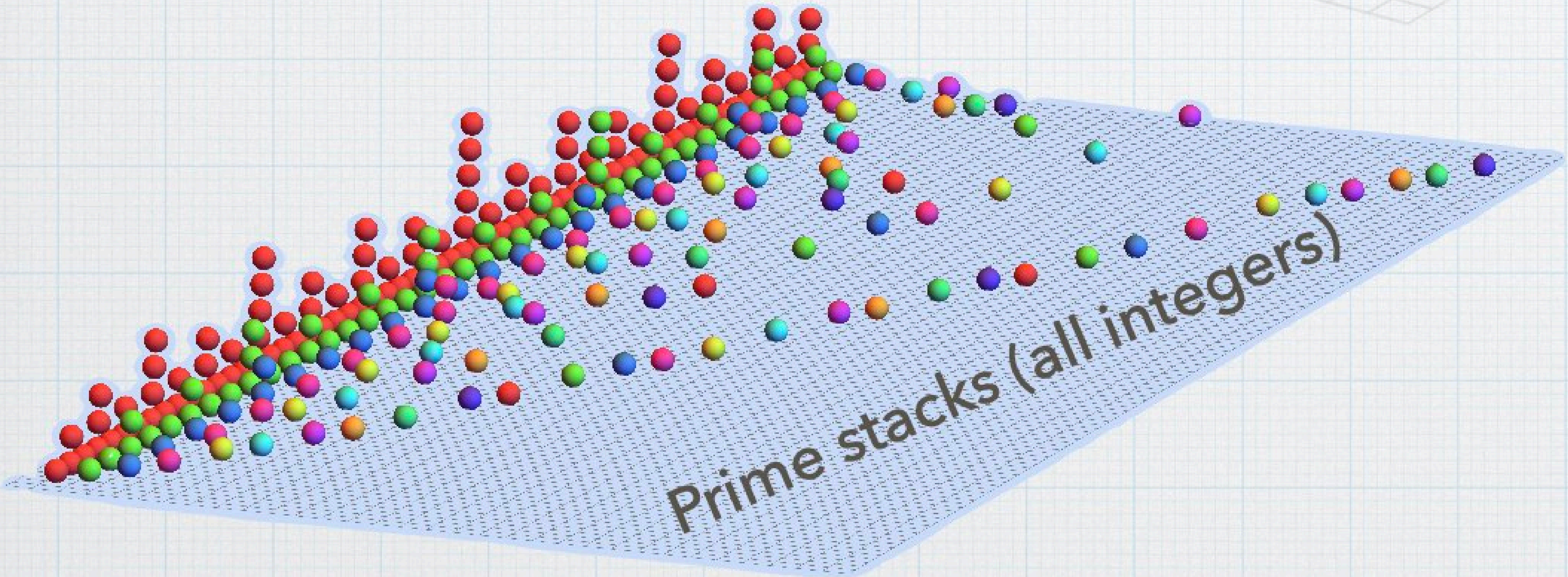


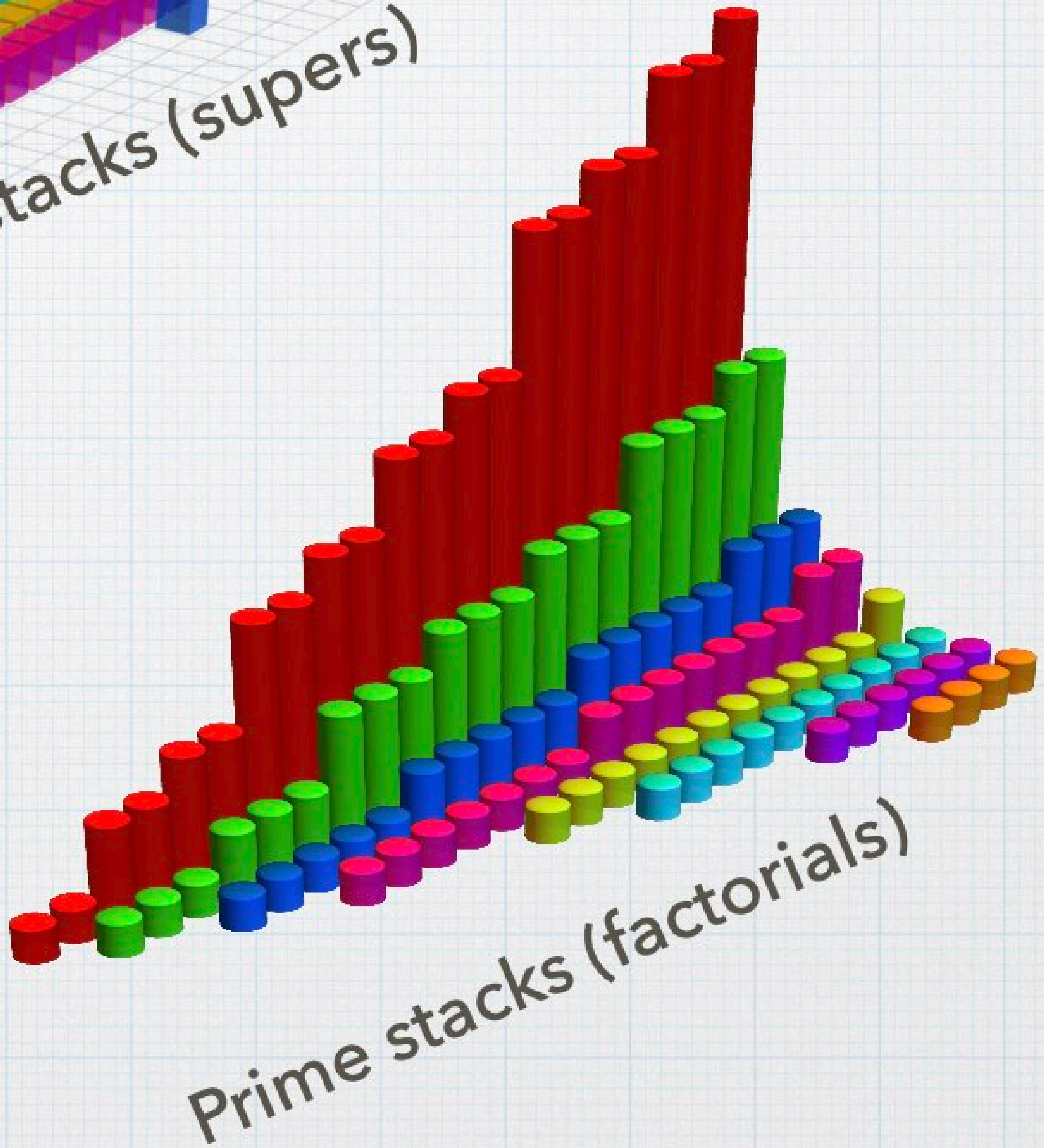


The Worms Crawl into 3D

The exponent worms can be lined up along the depth axis to make a surface!



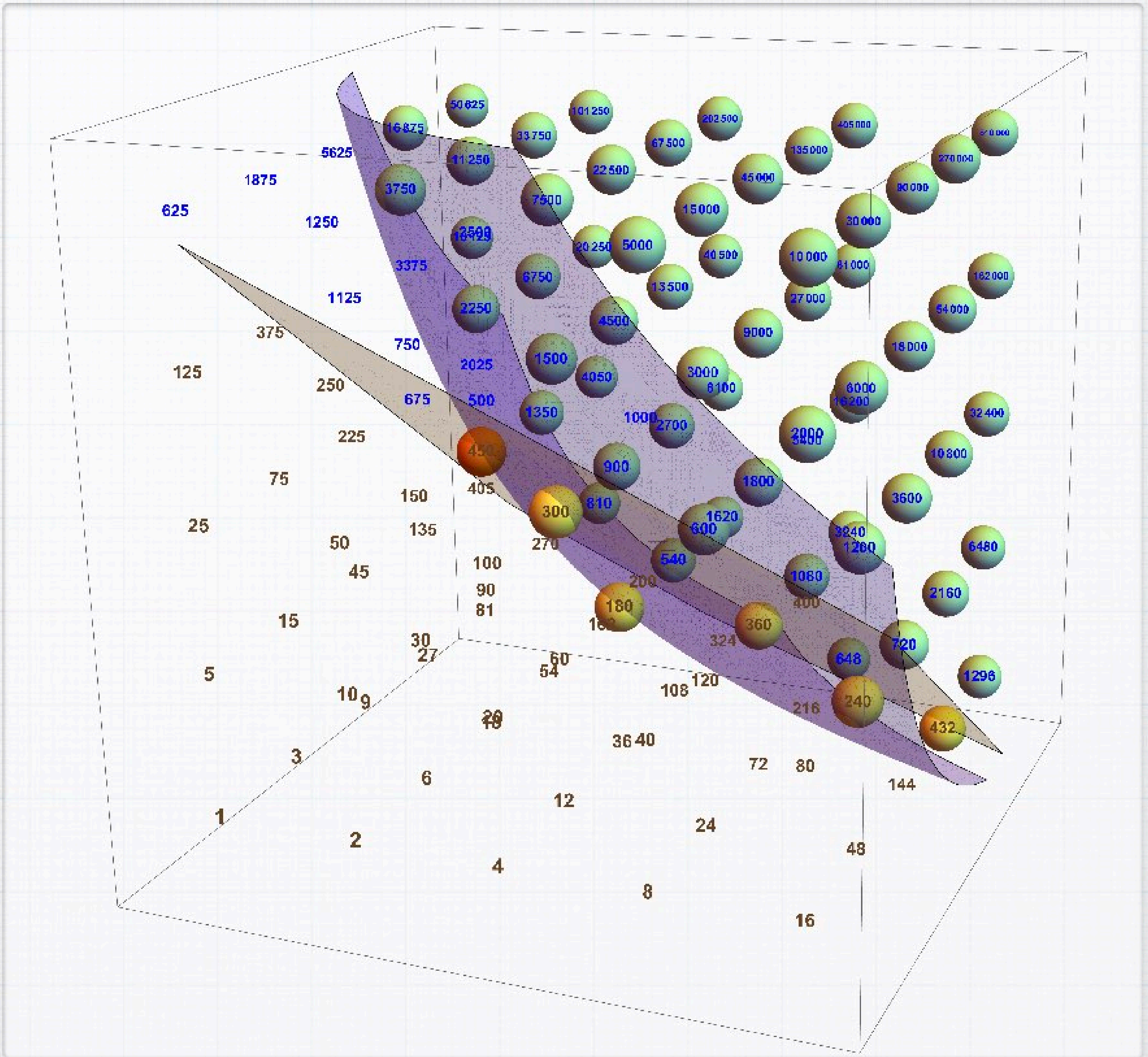




3D Exponent Lattices and Contours

Think of $n = 2^a 3^b 5^c \ldots$ plotted as ***(a, b, c)***, an "*exponentuple.*"

(*Recall: $d(n) = (a+1)(b+1)(c+1)\ldots$*)

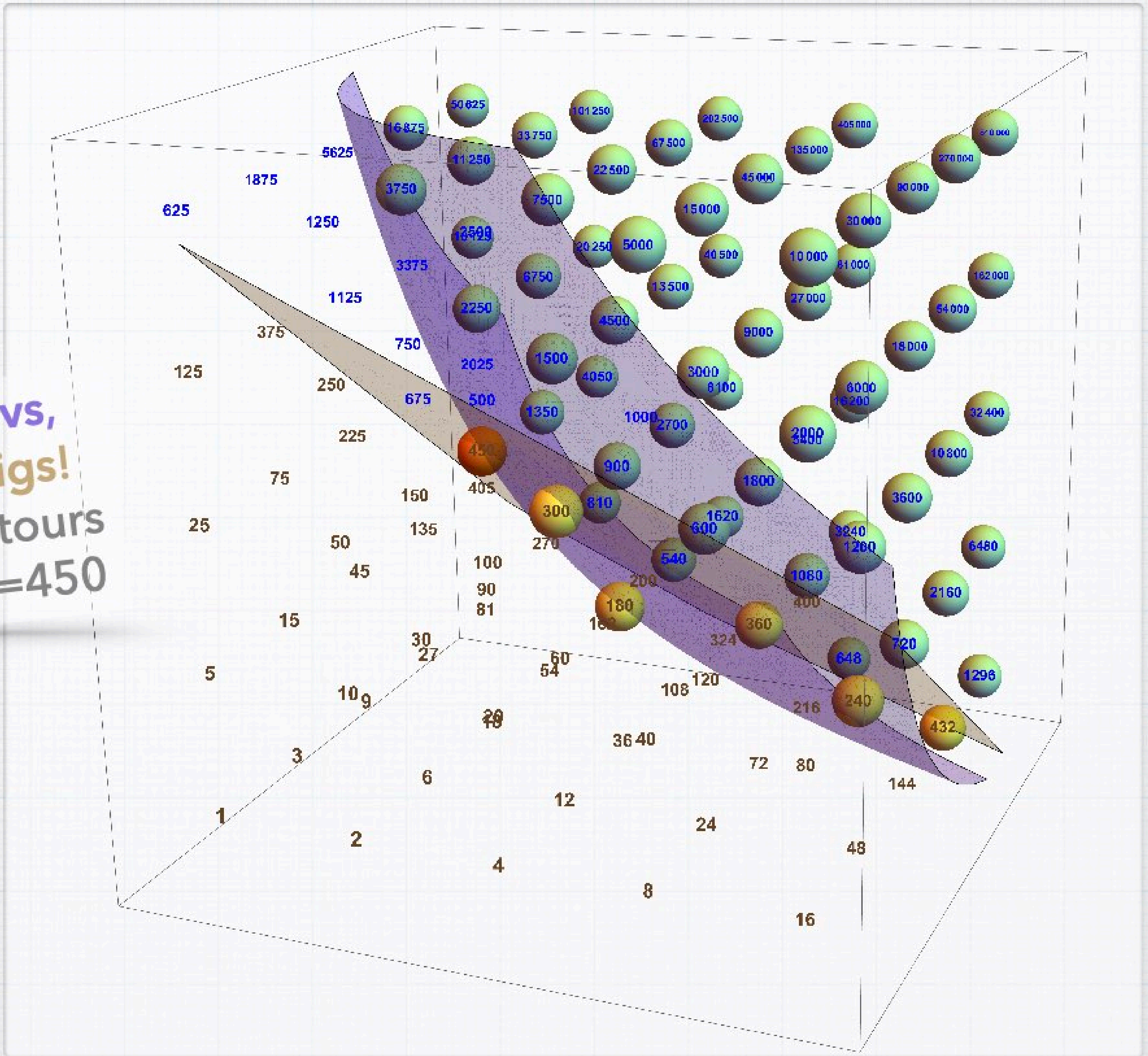


3D Exponent Lattices and Contours

Think of $n = 2^a 3^b 5^c \ldots$ plotted as ***(a, b, c)***, an "*exponentuple.*"
(*Recall:* $d(n) = (a+1)(b+1)(c+1)\ldots$)

Contours *(n = const)* are **planes**
$a \log 2 + b \log 3 + c \log 5 = \log n$

Contours *(d(n) = const)* are
hyperbolic sheets
$(a + 1)(b + 1)(c + 1) = d(n)$



3D Exponent Lattices and Contours

Think of $n = 2^a 3^b 5^c \ldots$ plotted as ***(a, b, c)***, an "*exponentuple.*"

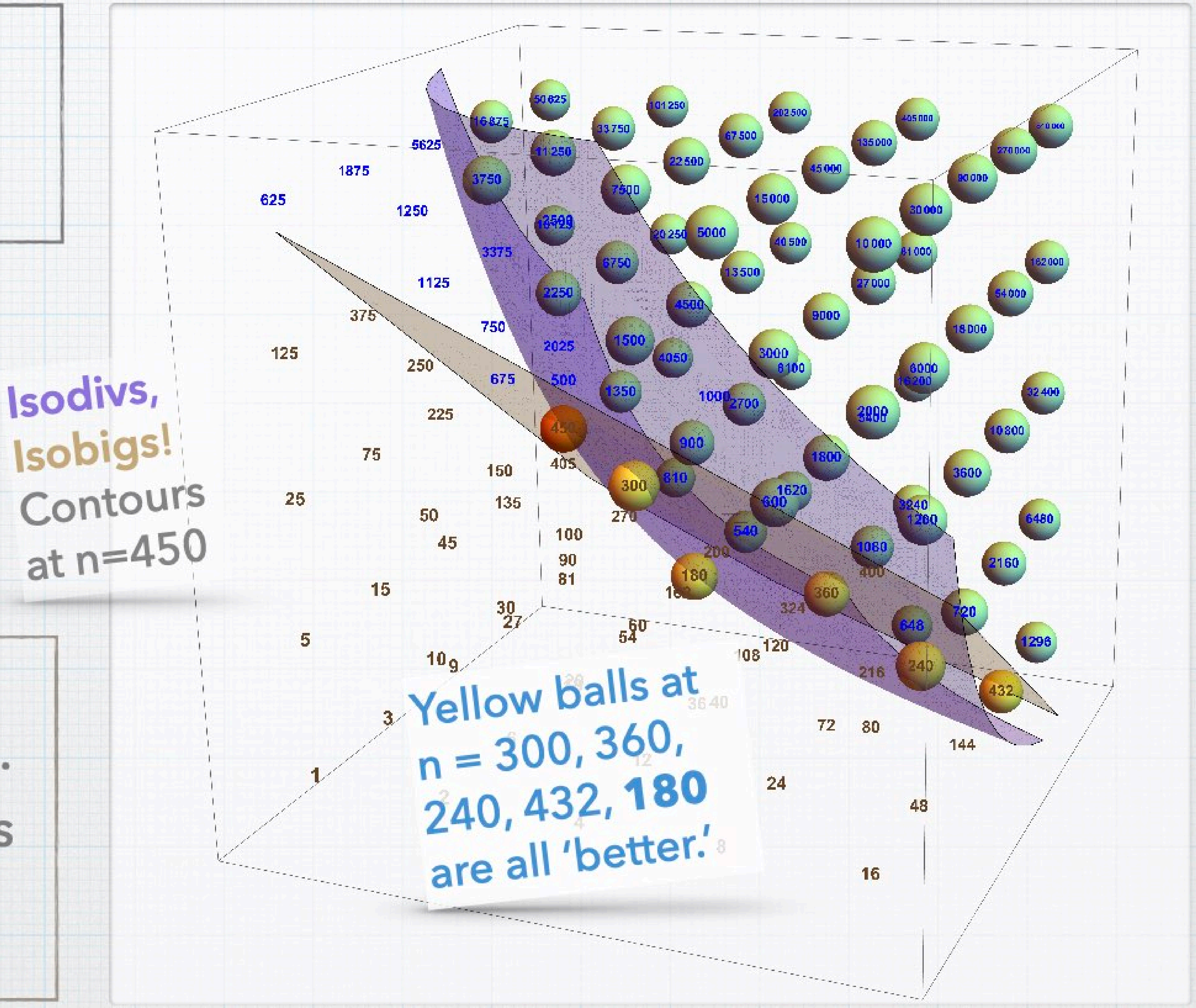
(*Recall:* $d(n) = (a+1)(b+1)(c+1)\ldots$)

Contours *(n = const)* are **planes**

$a \log 2 + b \log 3 + c \log 5 = \log n$

Contours *(d(n) = const)* are **hyperbolic sheets**

$(a + 1)(b + 1)(c + 1) = d(n)$



The **orange ball** is at ***(1, 2, 2)***, that means $n = 2^1 3^2 5^2 = 450$.

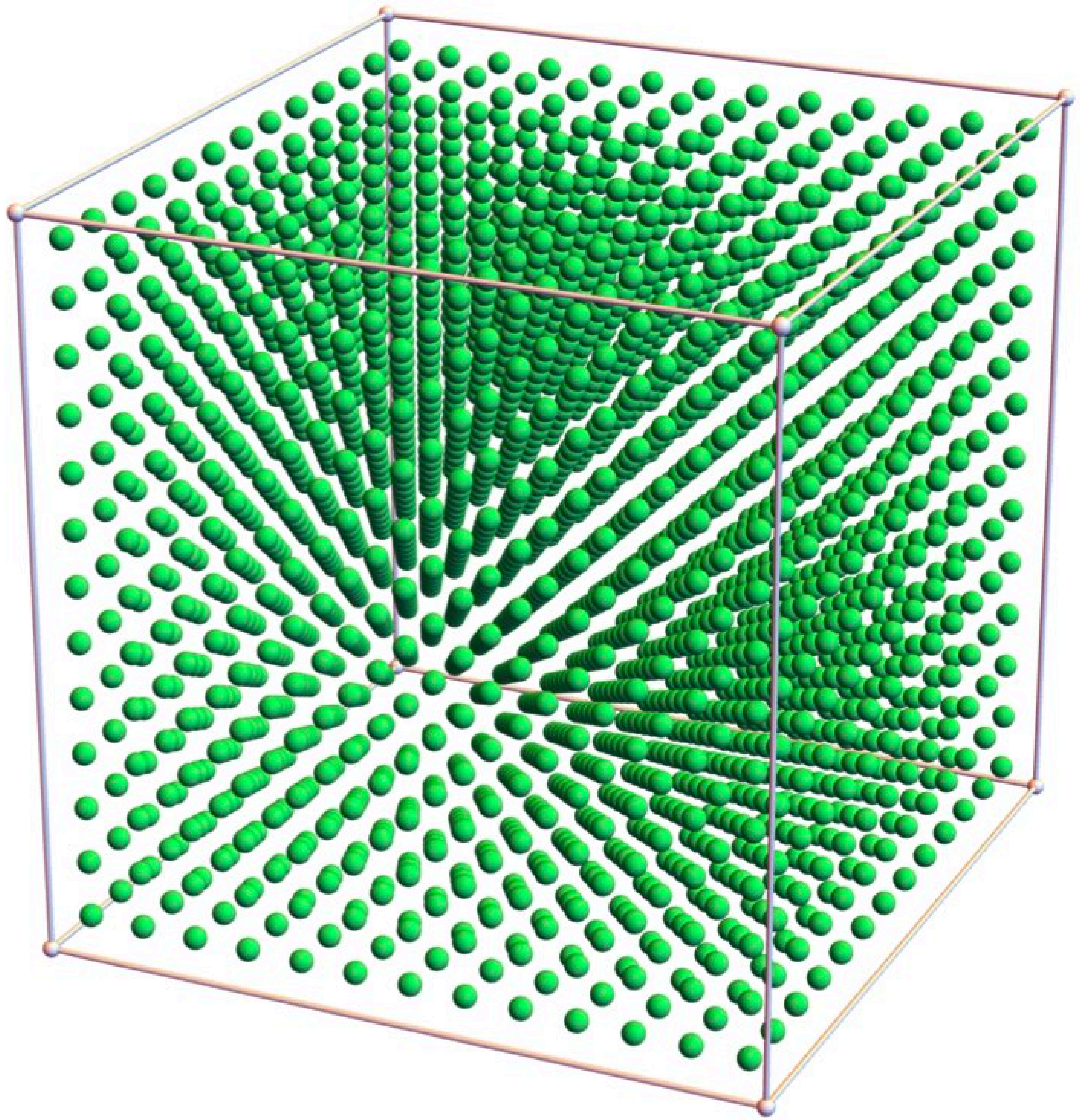
The yellow balls are smaller n's with at least as many divisors, so 450 is ***not*** supercomposite.

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with $0 \le a, b, c \le 12$, and get rid of any with $a < b$, $b < c$, $a < c$.

 # left

2197



All 13^3 = **2197** (a,b,c) : $0 \le a,b,c \le 12$

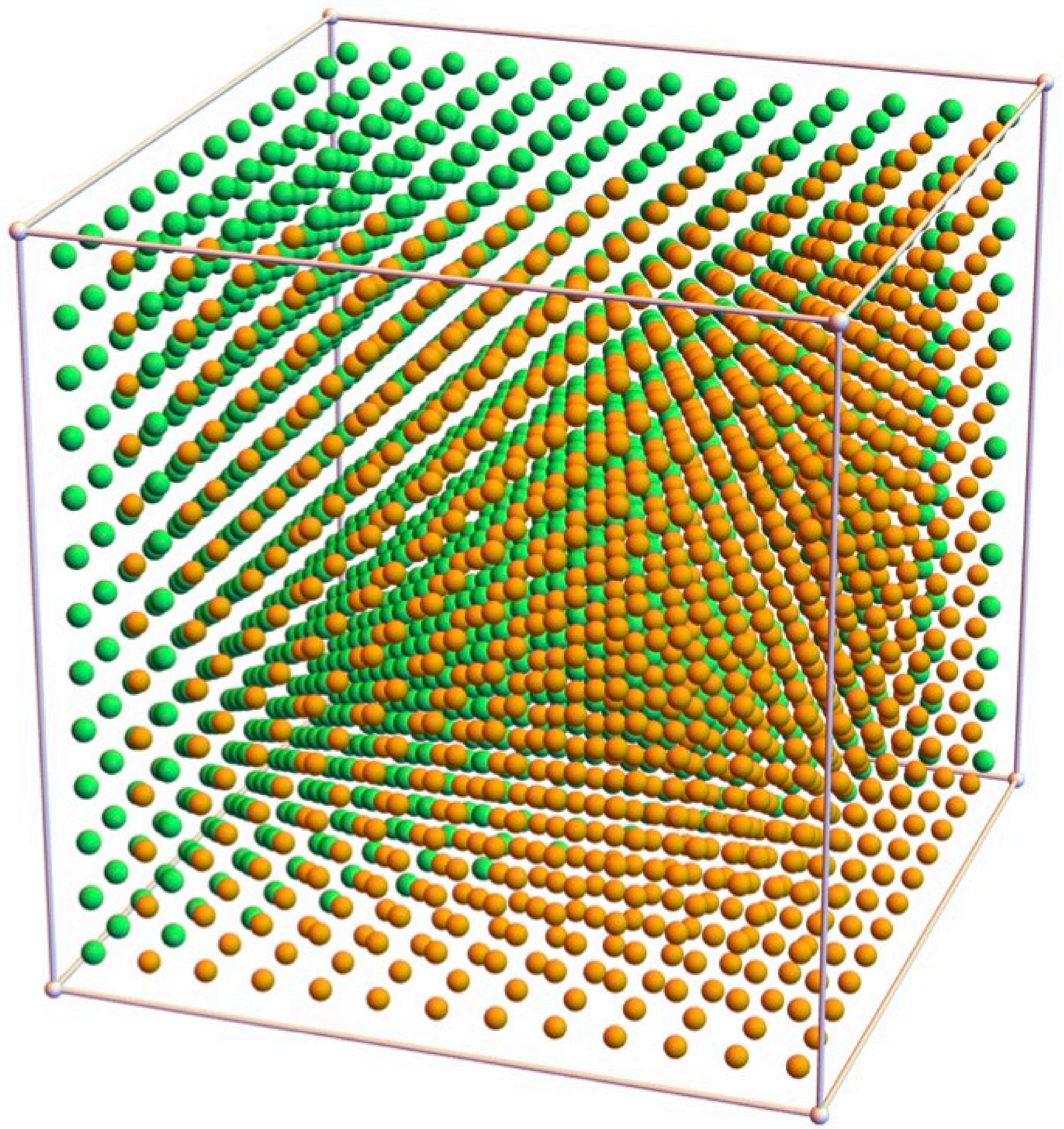
Exponent Lattices and the “Super-Sector”

 # left

2197

Start with all (a, b, c) in a cube with $0 \le a, b, c \le 12$, and get rid of any with $a < b$, $b < c$, $a < c$.

Let’s watch these bad lattice points get shaved off and expose the super-sector:



First get rid of all (a, b, c) with $a < b$

Exponent Lattices and the "Super-Sector"

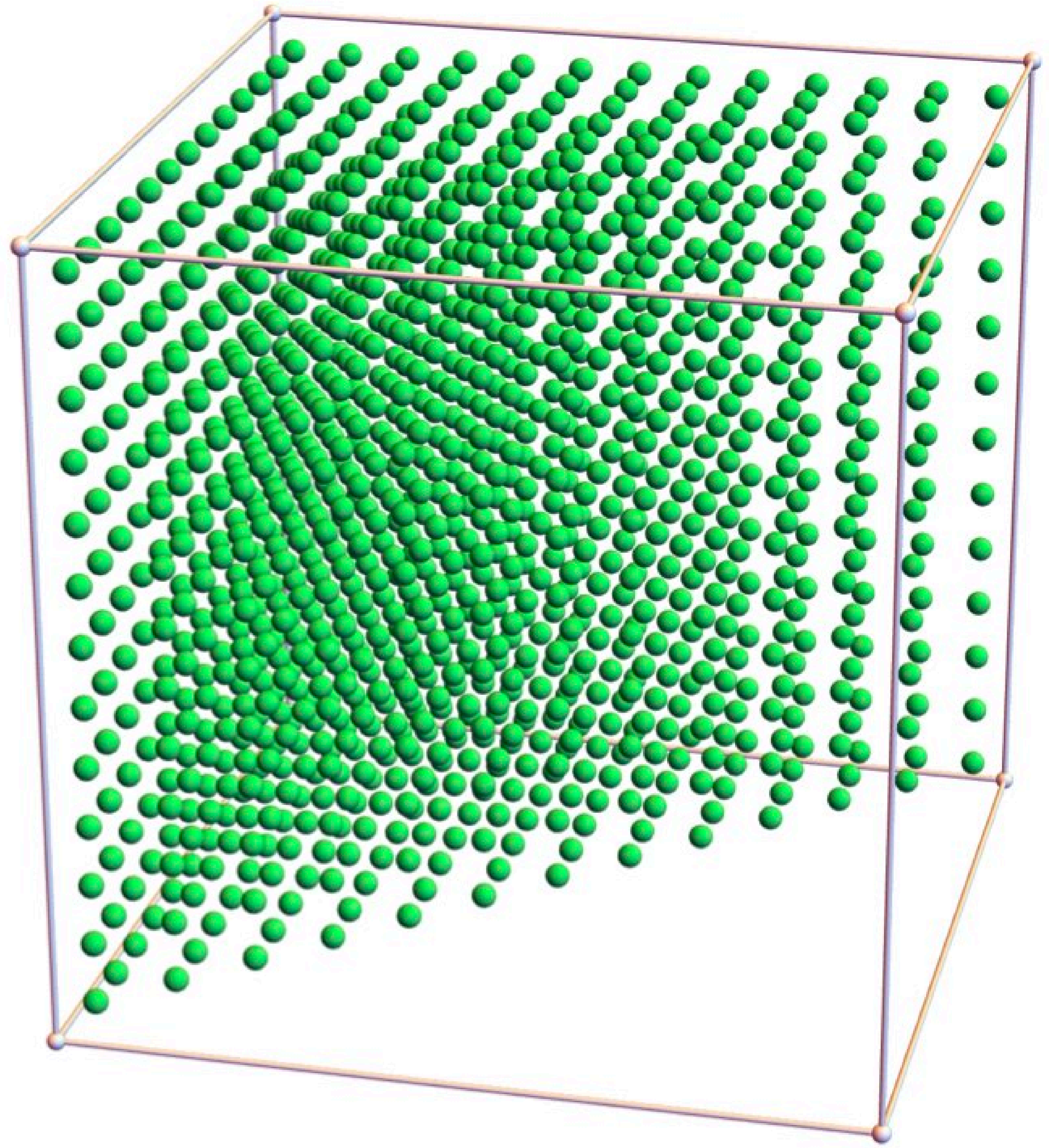
 # left

~~2197~~

1183

Start with all (a, b, c) in a cube with $0 \leq a, b, c \leq 12$, and get rid of any with $a < b$, $b < c$, $a < c$.

Let's watch these bad lattice points get shaved off and expose the super-sector:



This leaves **1183** triples with $a \geq b$

Exponent Lattices and the "Super-Sector"

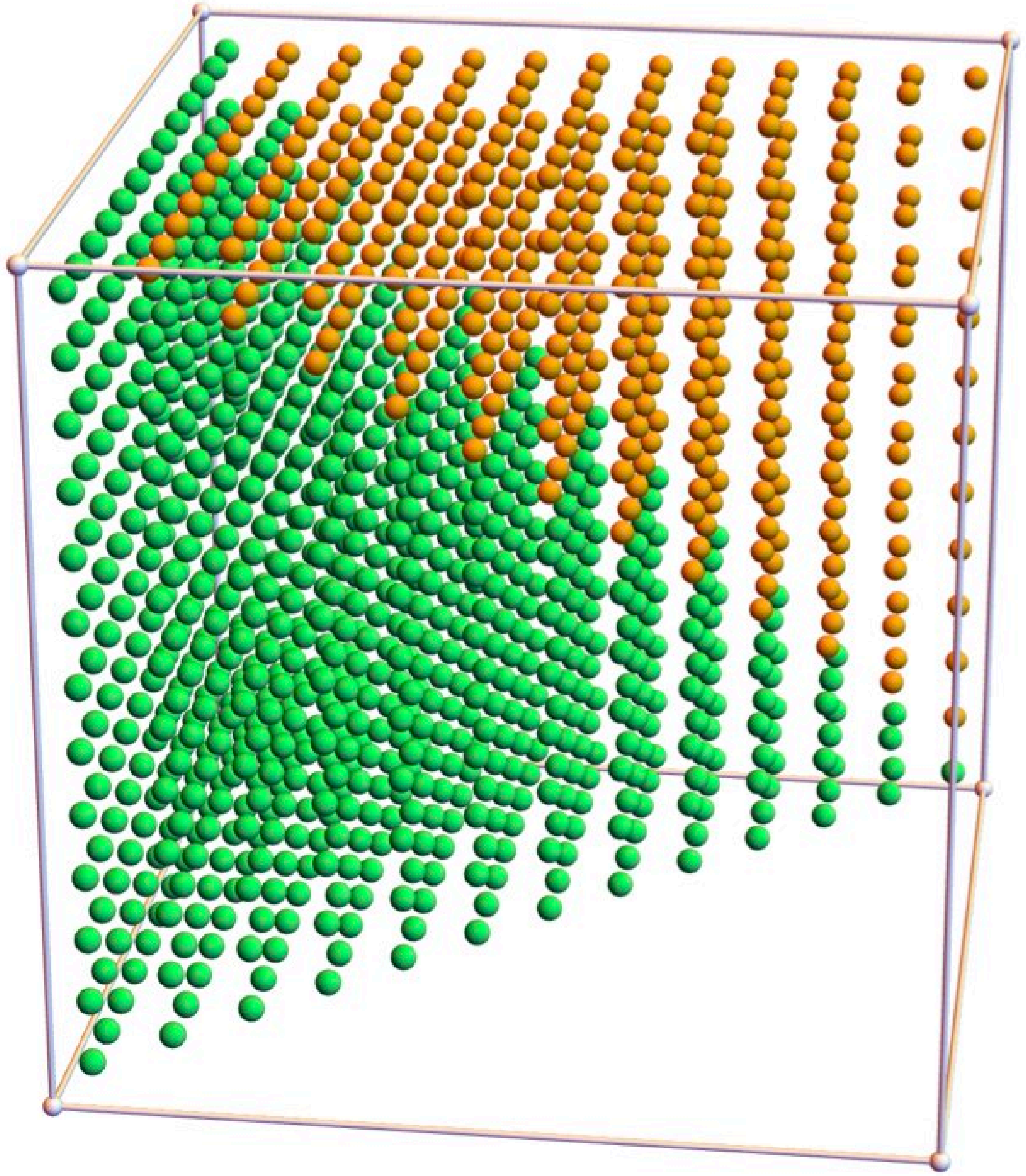
Start with all (a, b, c) in a cube with 0 ≤ a, b, c ≤ 12, and get rid of any with a < b, b < c, a < c.

 # left

~~2197~~

1183

Let's watch these bad lattice points get shaved off and expose the super-sector:



Now get rid of the (a, b, c) with a < c

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with 0 ≤ a, b, c ≤ 12, and get rid of any with a < b, b < c, a < c.

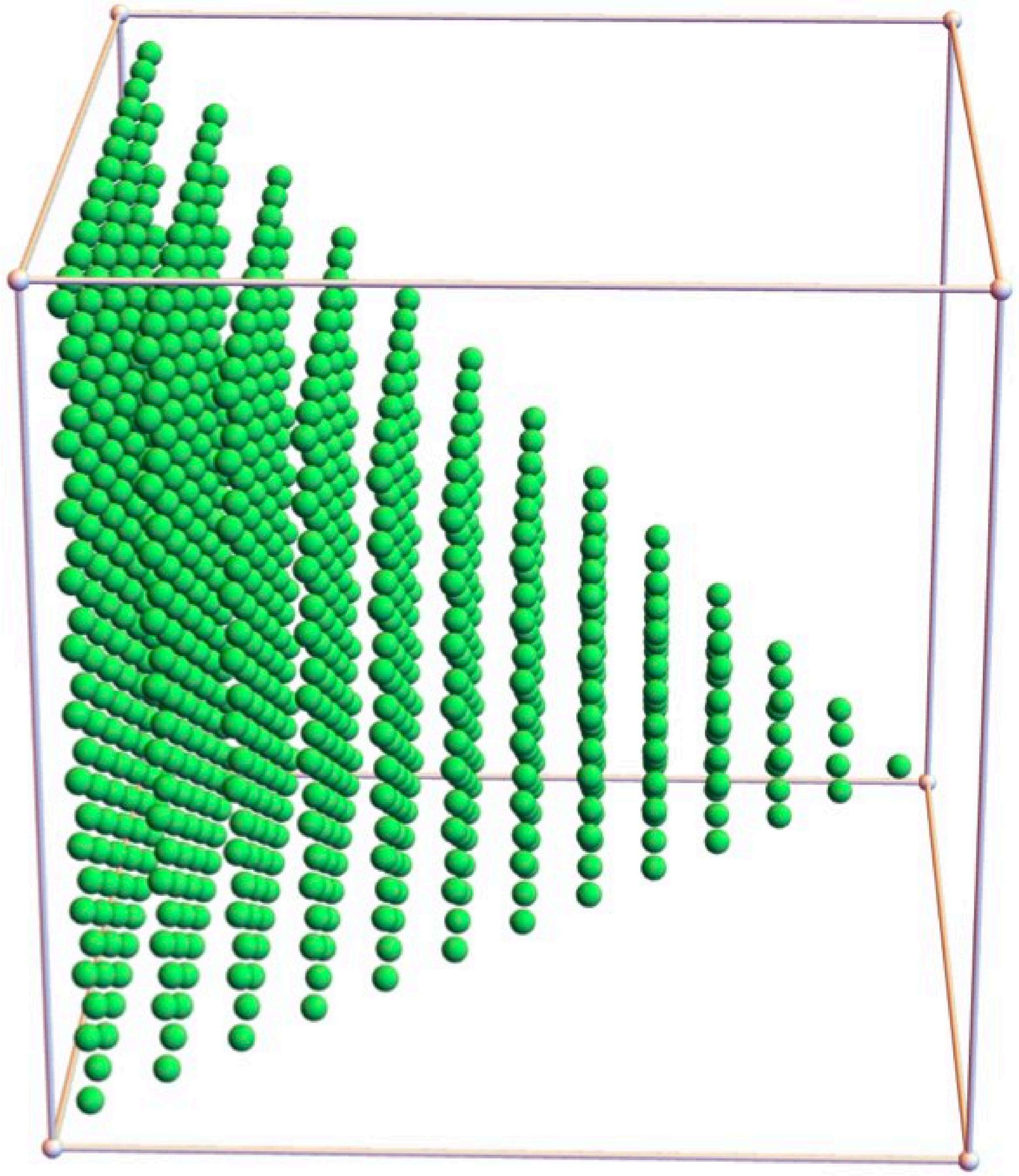
 # left

~~*2197*~~

~~*1183*~~

819

Let's watch these bad lattice points get shaved off and expose the super-sector:



There are **819** triples: a ≥ b and a ≥ c

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with $0 \le a, b, c \le 12$, and get rid of any with $a < b$, $b < c$, $a < c$.

 # left

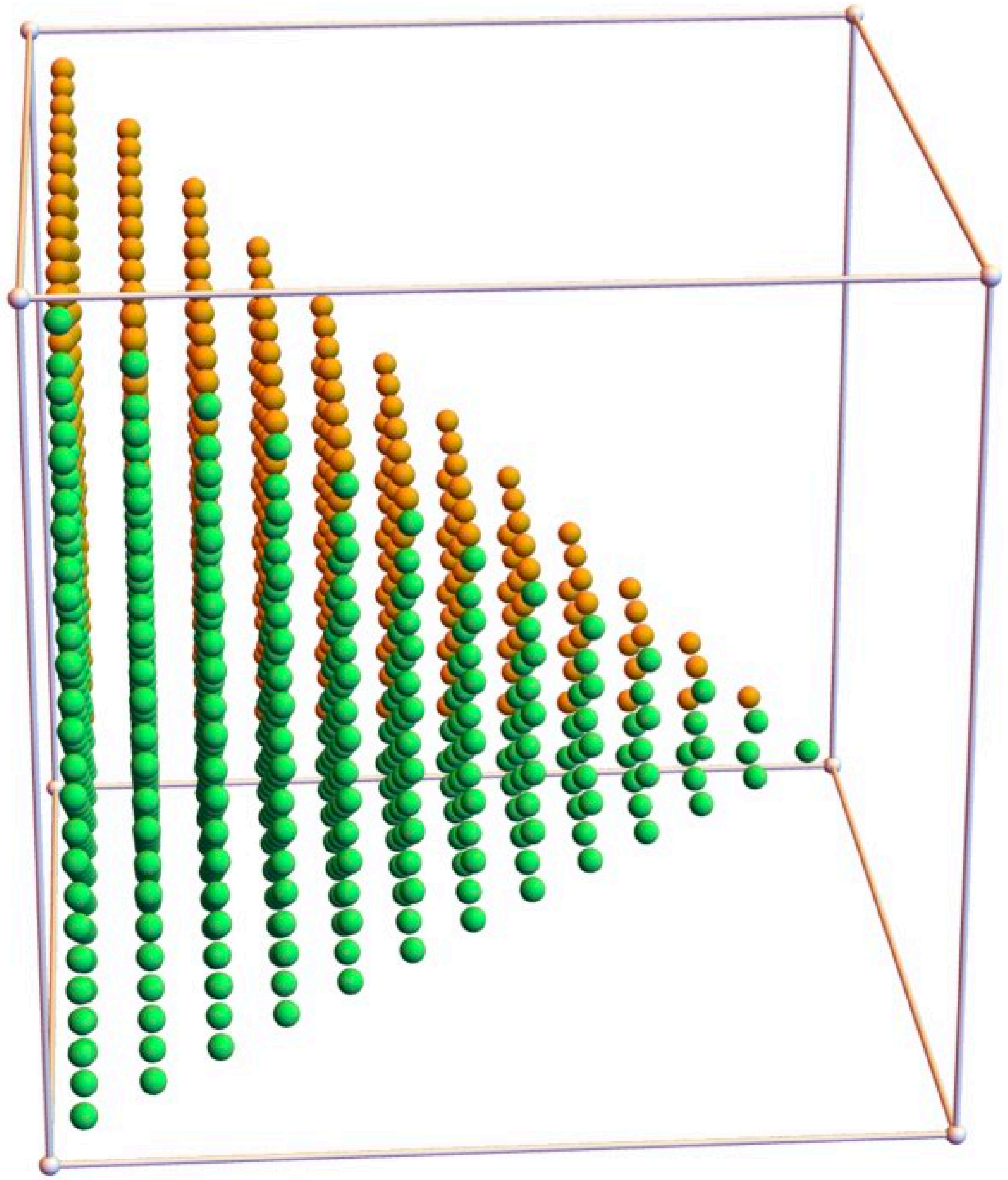
~~*2197*~~

~~*1183*~~

~~*819*~~

455

Let's watch these bad lattice points get shaved off and expose the super-sector:



Now lose those (a, b, c) with $b < c$

Exponent Lattices and the "Super-Sector"

\# left

~~2197~~

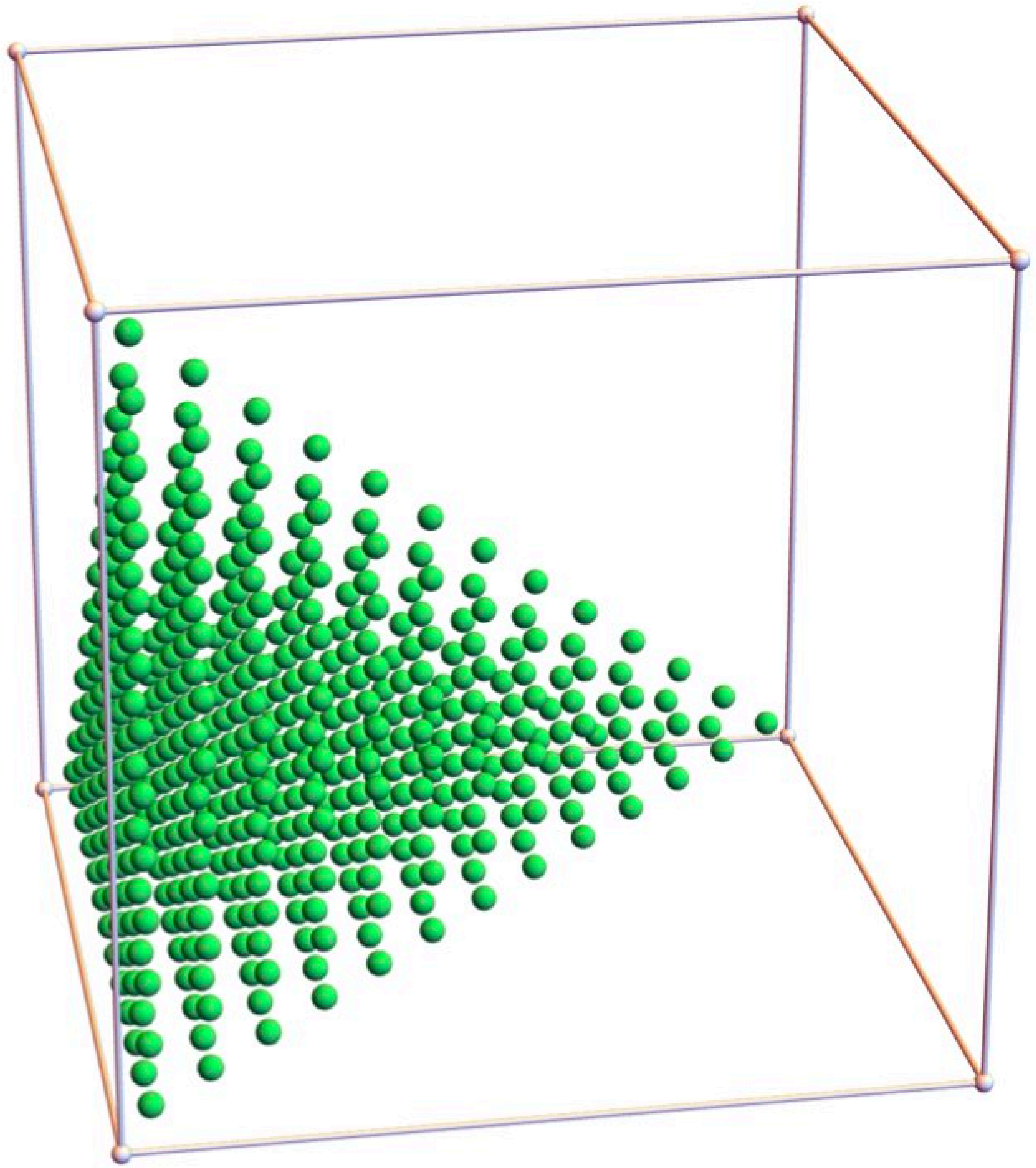
~~1183~~

~~819~~

455

Start with all (a, b, c) in a cube with 0 ≤ a, b, c ≤ 12, and get rid of any with a < b, b < c, a < c.

Let's watch these bad lattice points get shaved off and expose the super-sector:



Surviving **455** contenders, a ≥ b ≥ c

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with 0 ≤ a, b, c ≤ 12, and get rid of any with a < b, b < c, a < c.

left
~~2197~~
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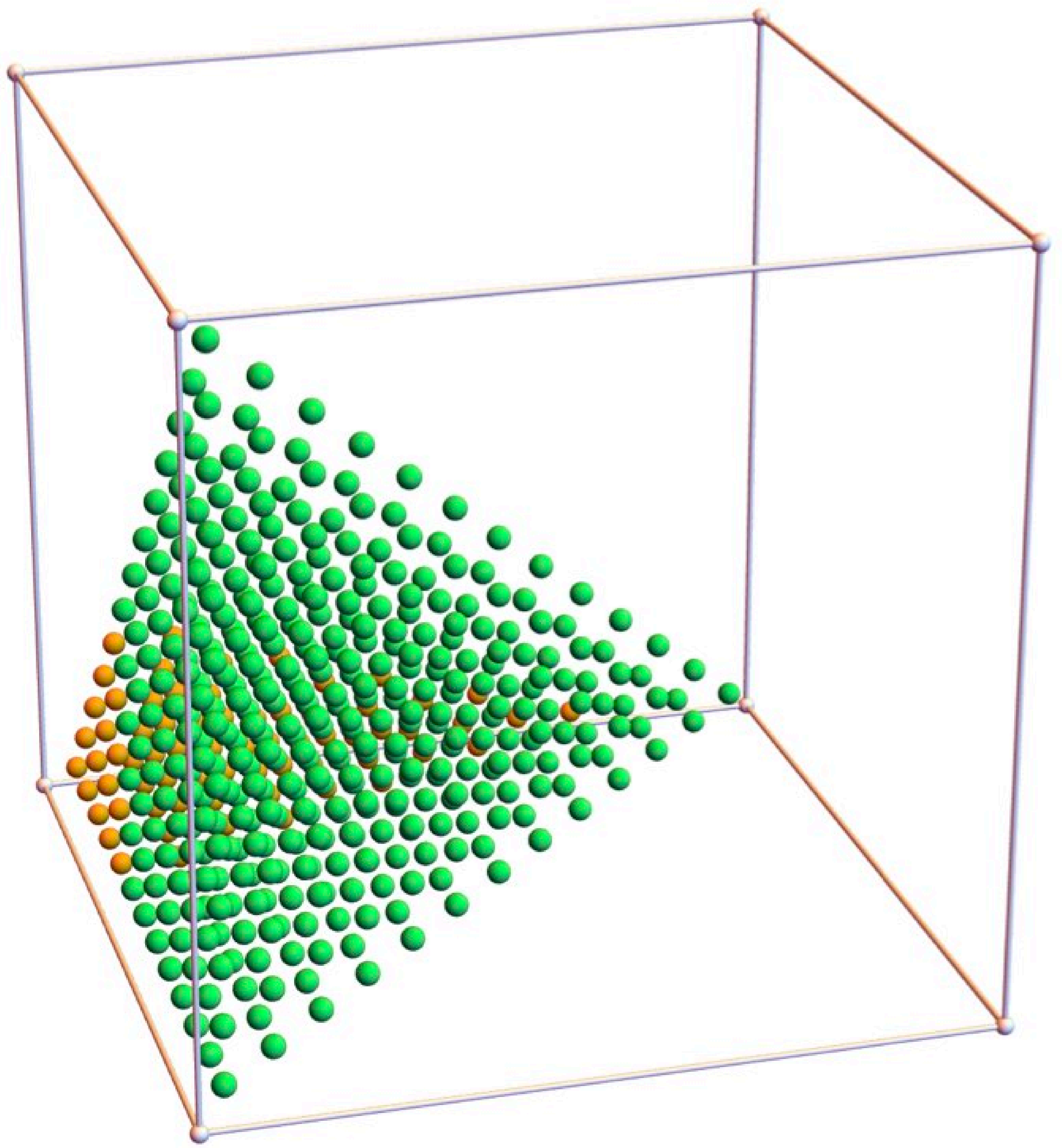
455

Let's watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.



Start applying rules: 3/4 kills a > 2b + 2

Exponent Lattices and the “Super-Sector”

Start with all (a, b, c) in a cube with 0 ≤ a, b, c ≤ 12, and get rid of any with a < b, b < c, a < c.

Let’s watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.

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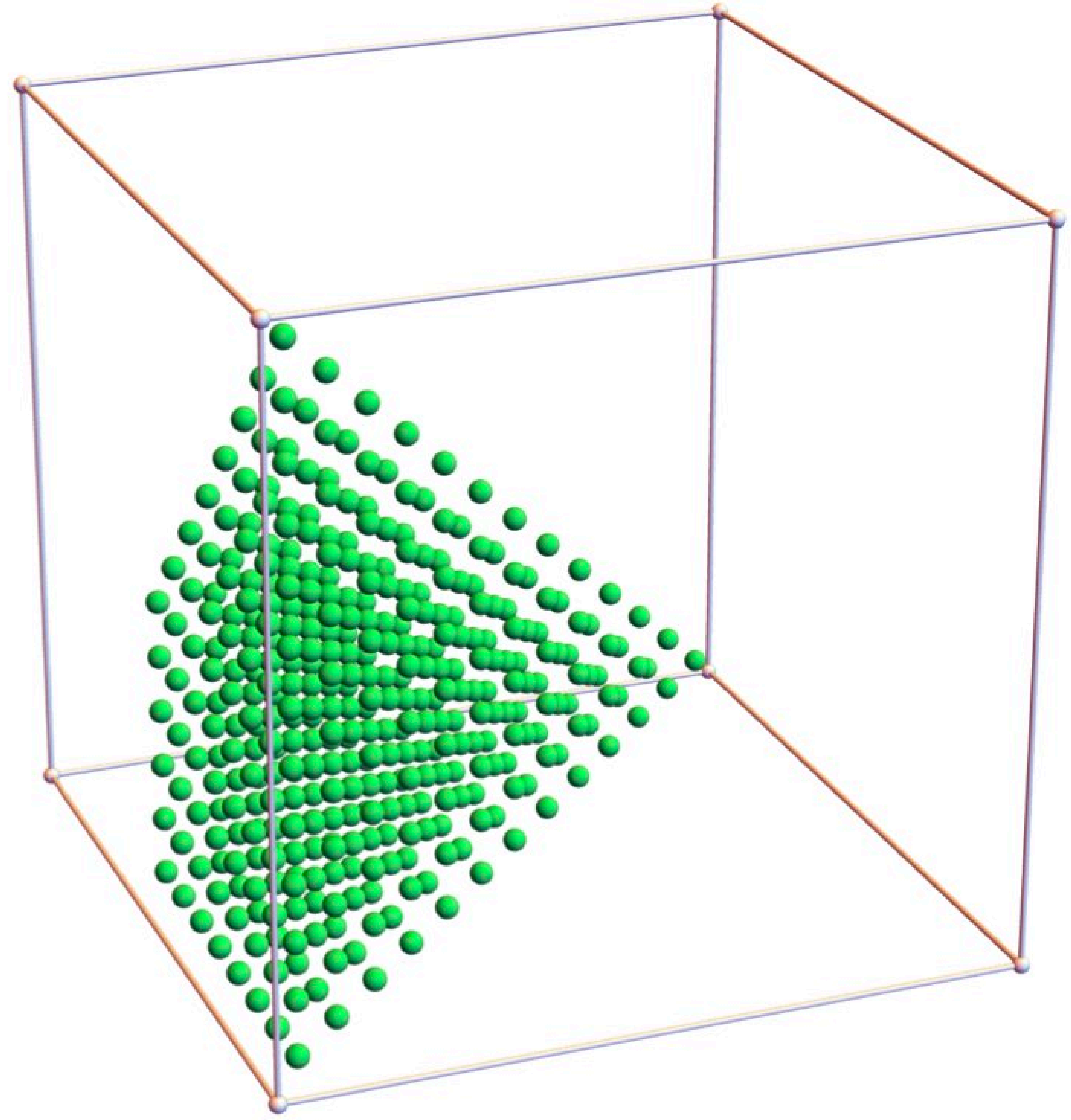
~~*455*~~
385



Down to **385** contenders, a ≤ 2b + 2

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with 0 ≤ a, b, c ≤ 12, and get rid of any with a < b, b < c, a < c.

Let's watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.

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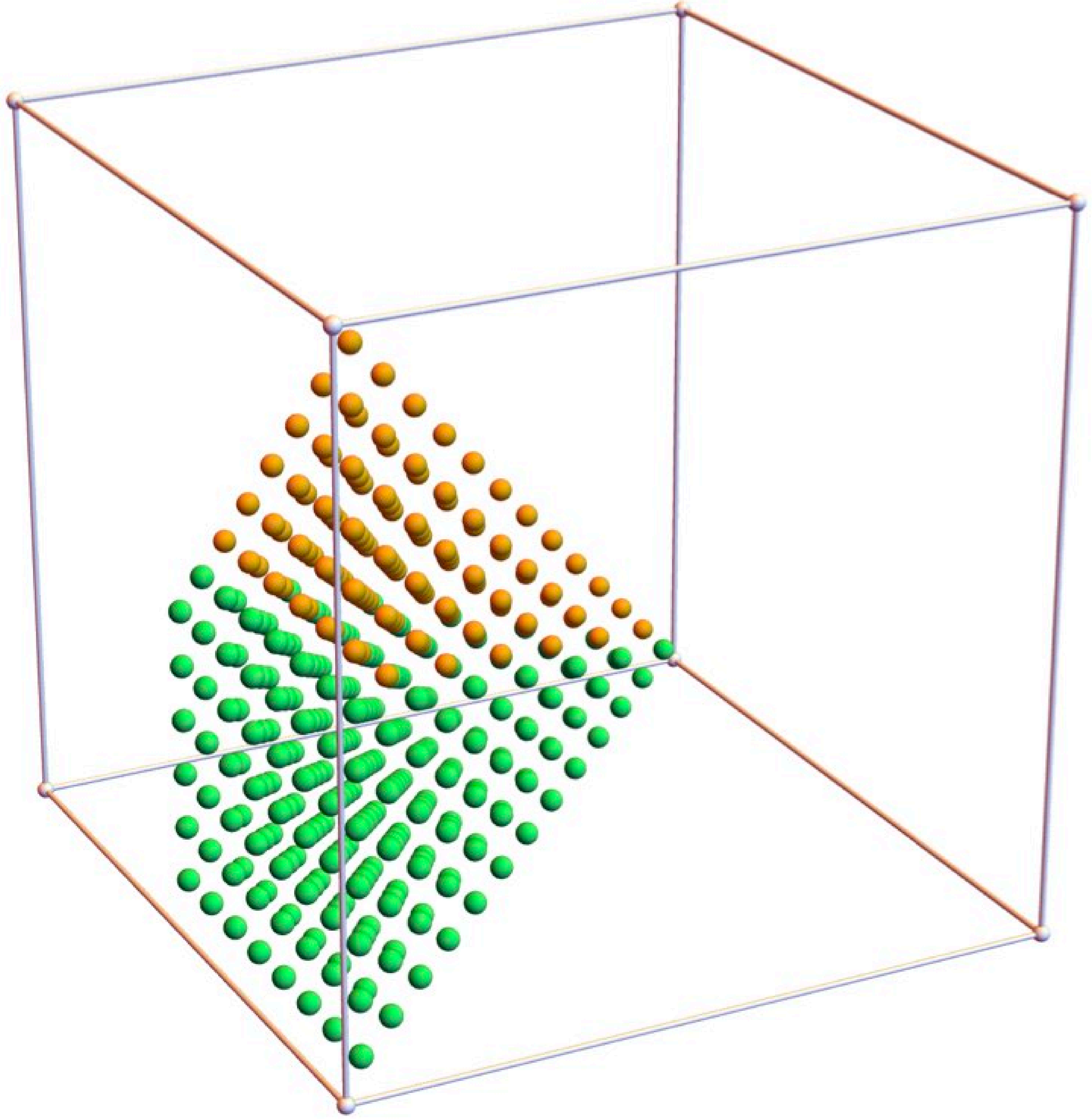
385



The 4/5 rule makes all a < 2c go away

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with $0 \le a, b, c \le 12$, and get rid of any with $a < b$, $b < c$, $a < c$.

Let's watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.

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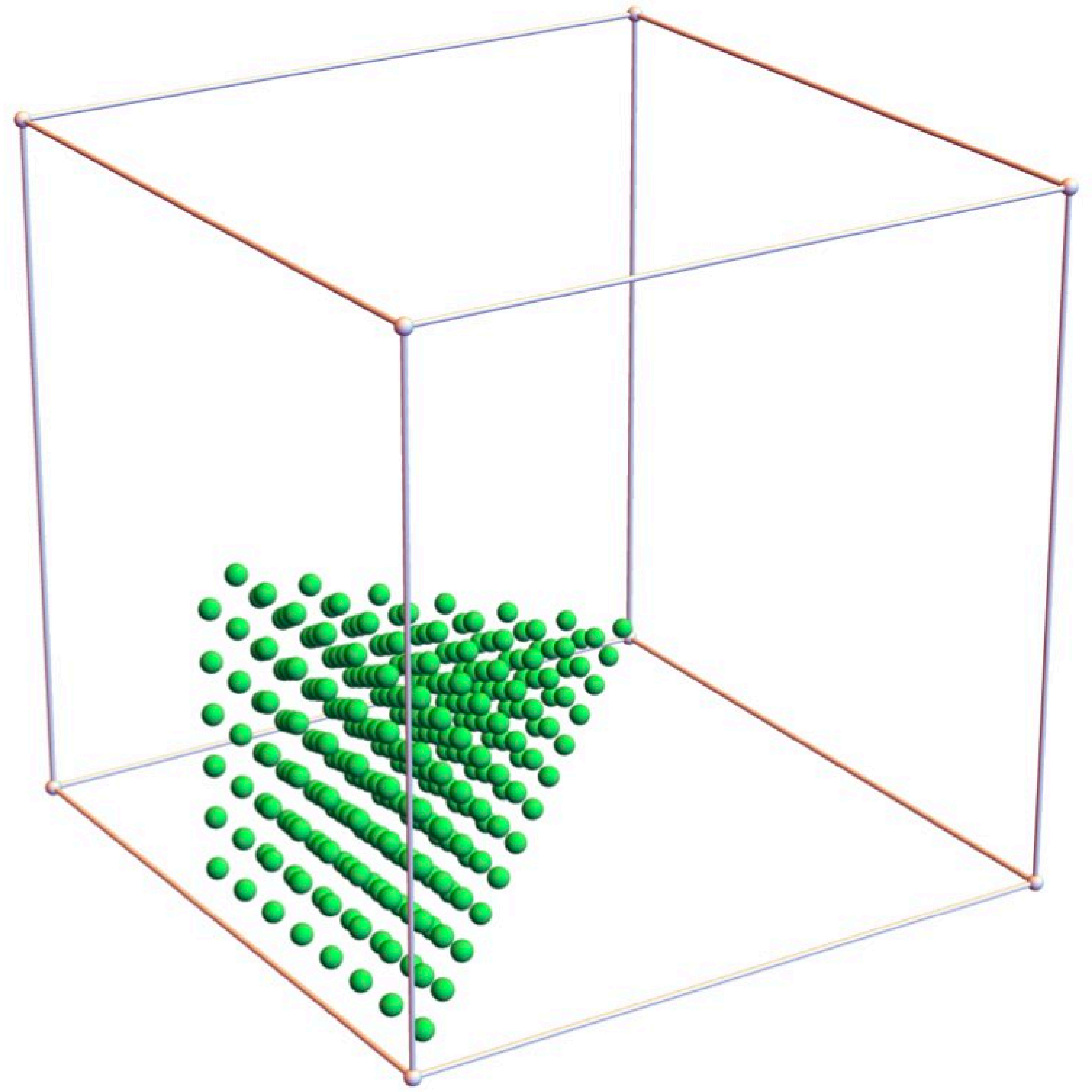
~~*385*~~
273



Down to **273** (a, b, c) with $a \ge 2c$

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with $0 \le a, b, c \le 12$, and get rid of any with $a < b$, $b < c$, $a < c$.

Let's watch these bad lattice points get shaved off and expose the super-sector:

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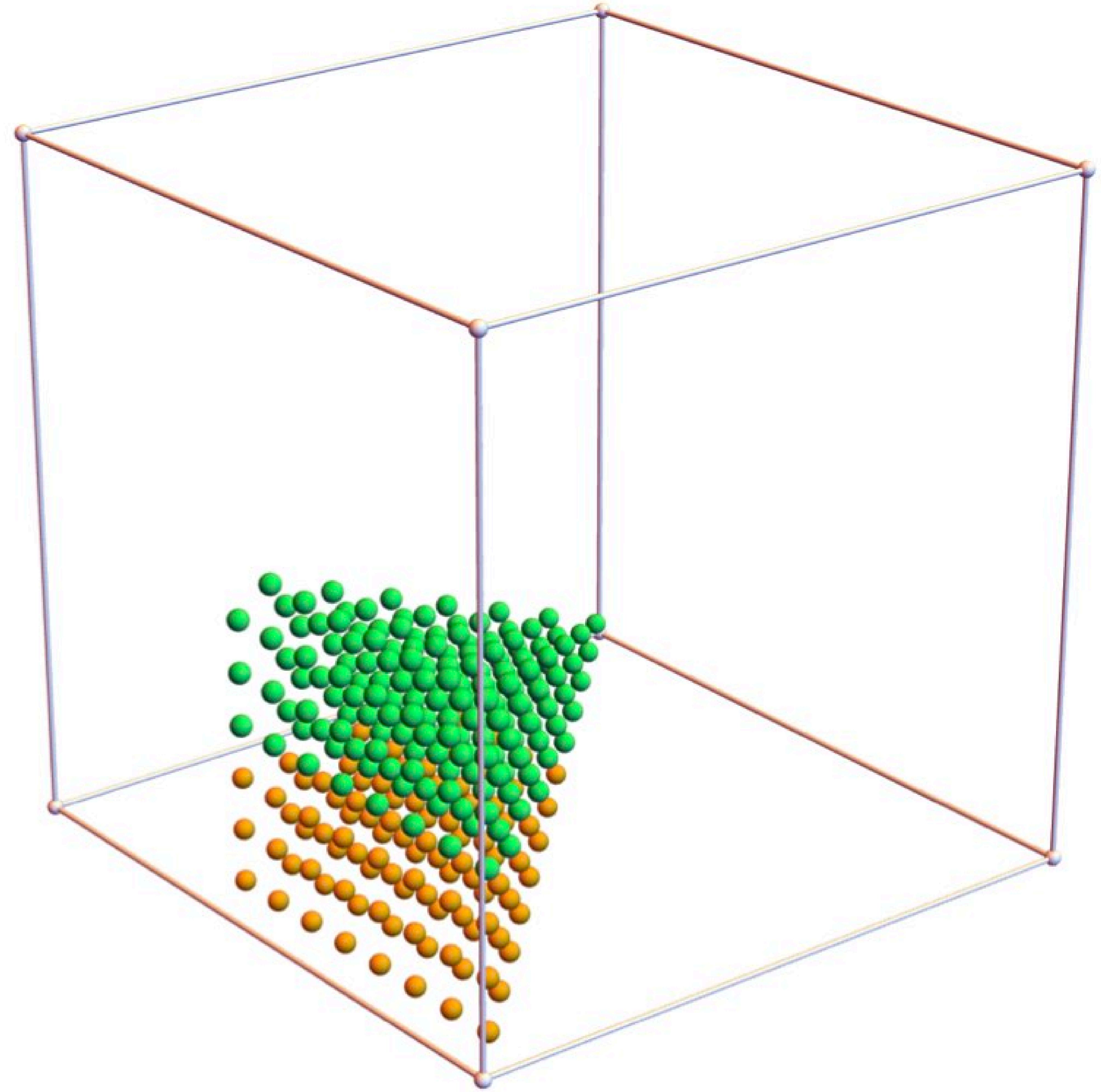

The 5/8 rule requires $a \le 3c + 4$

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with 0 ≤ a, b, c ≤ 12, and get rid of any with a < b, b < c, a < c.

Let's watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.

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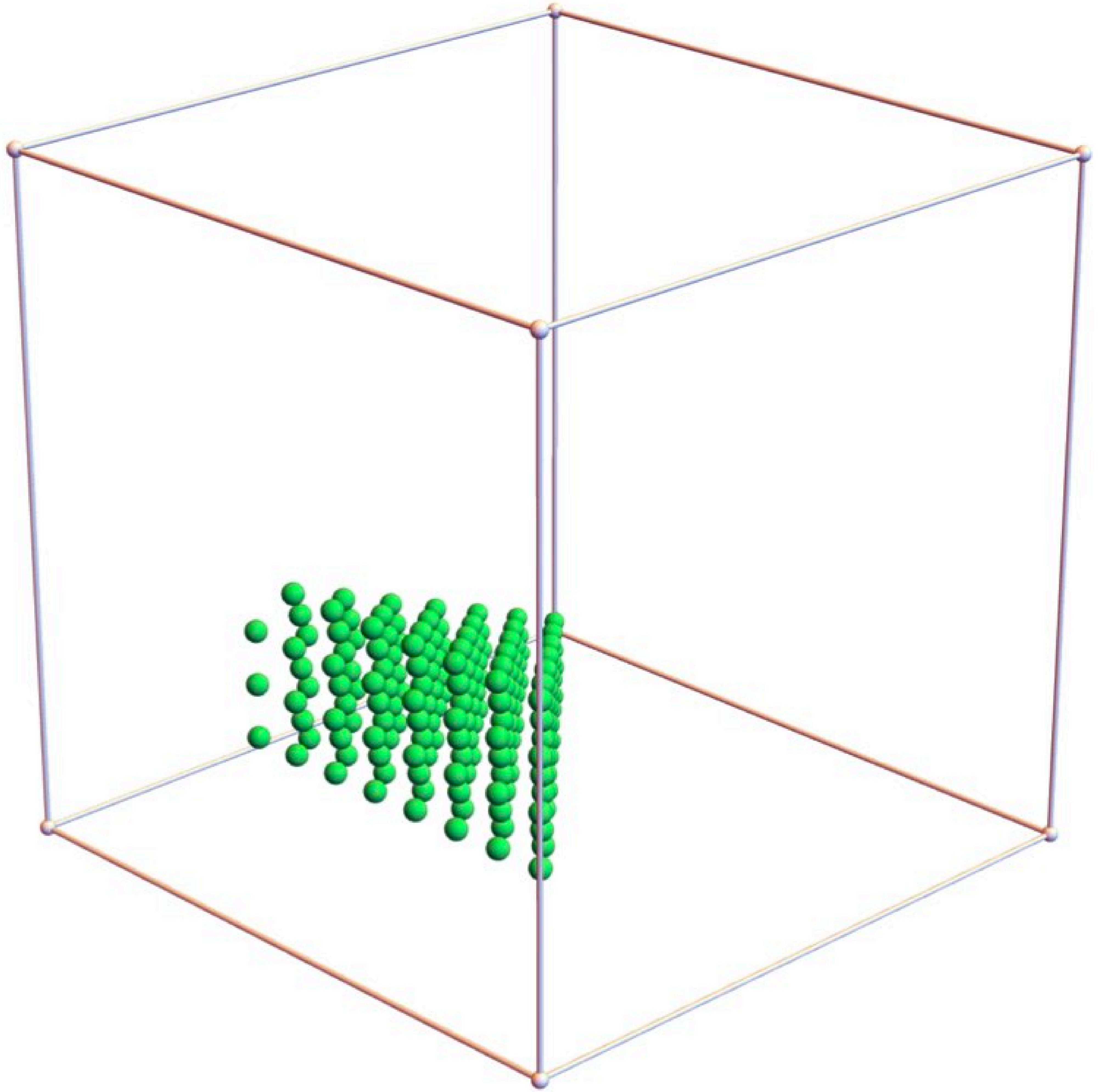
176



Just **176** (a, b, c) with a ≤ 3c + 4

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with $0 \le a, b, c \le 12$, and get rid of any with $a < b$, $b < c$, $a < c$.

Let's watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.

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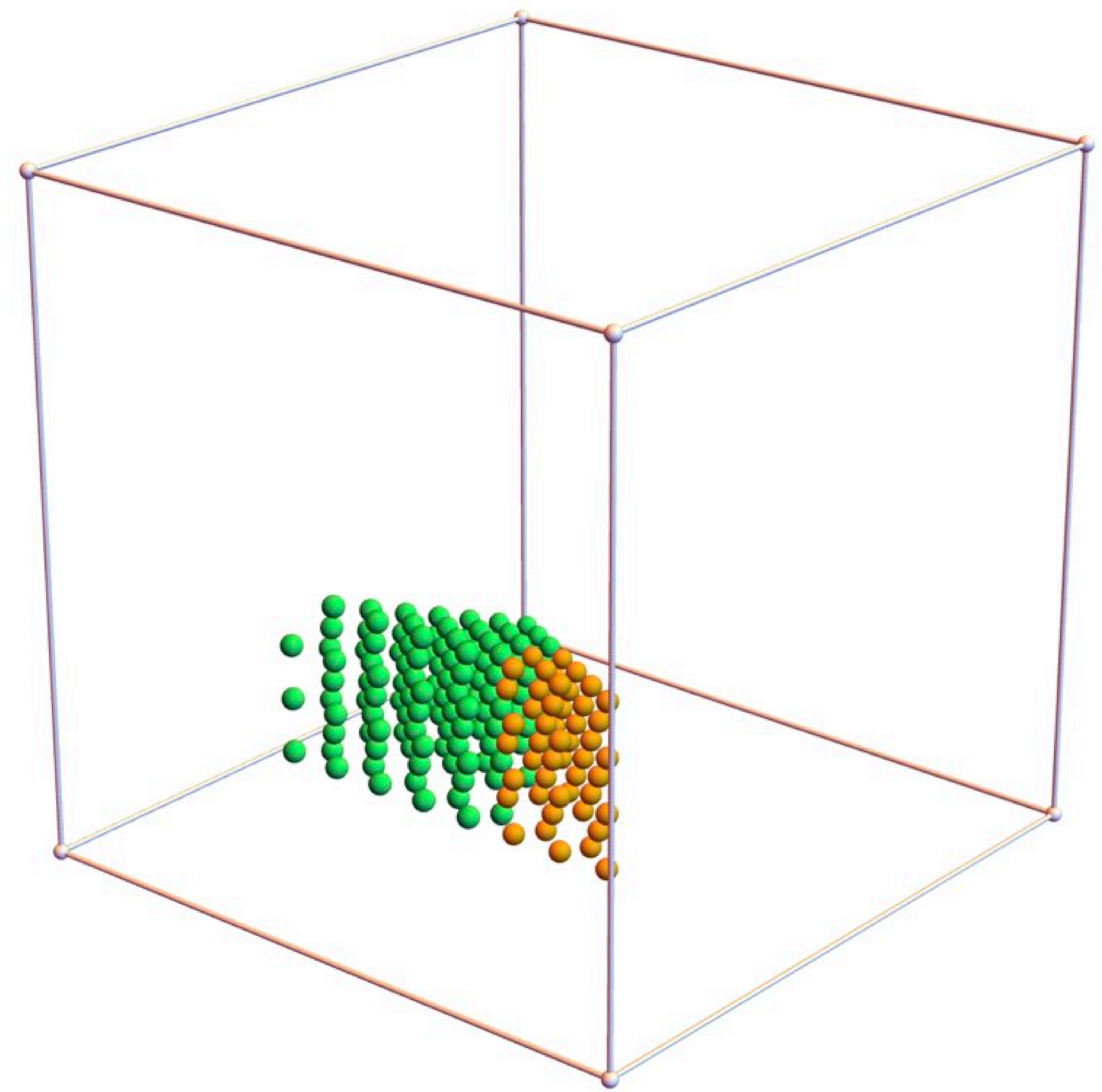

On to the 8/9 rule, need $2a \ge 3b - 4$

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with $0 \le a, b, c \le 12$, and get rid of any with $a < b$, $b < c$, $a < c$.

Let's watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.

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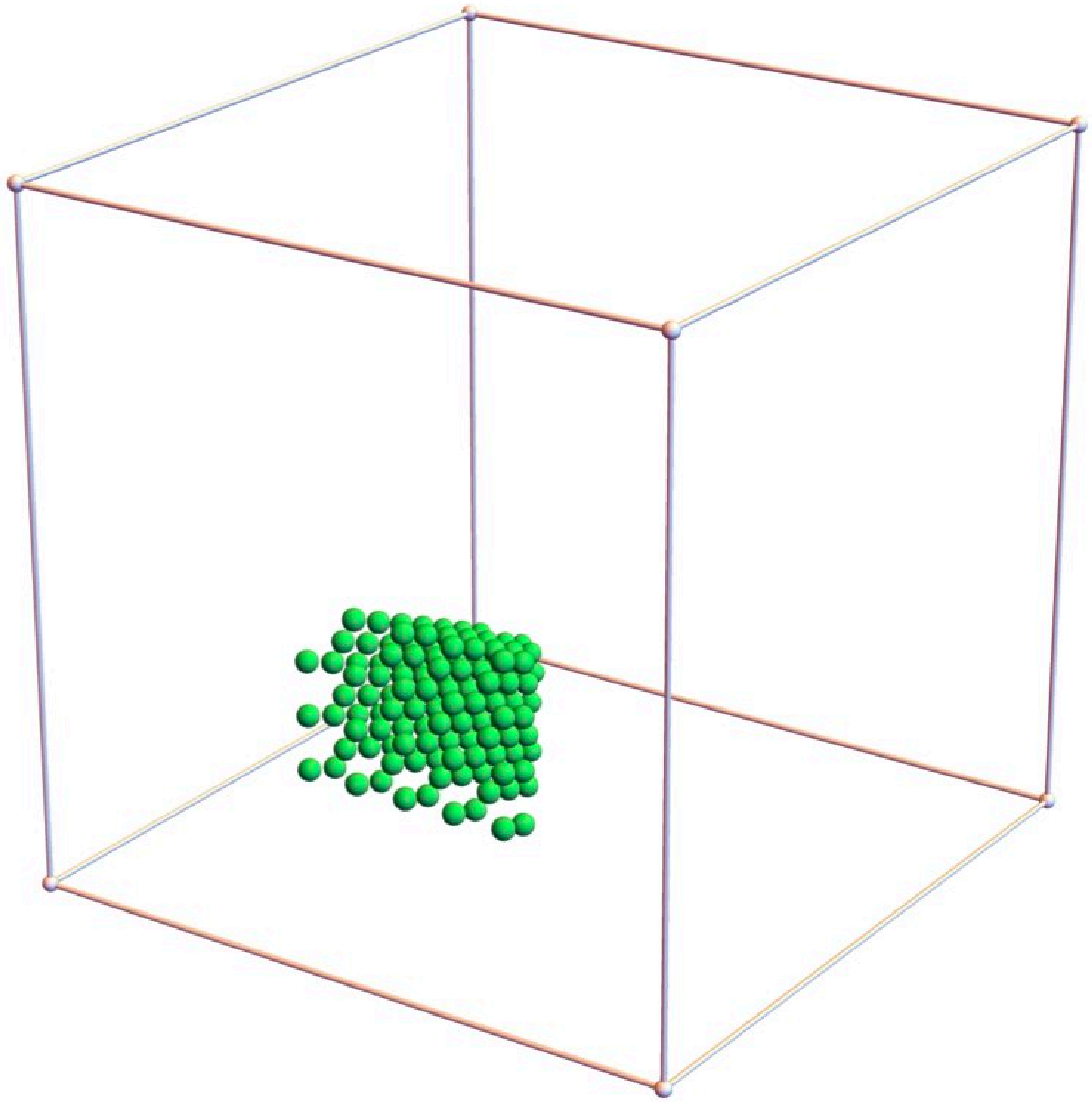

Down to **127** triples with $2a \le 3c + 4$

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with 0 ≤ a, b, c ≤ 12, and get rid of any with a < b, b < c, a < c.

Let's watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.

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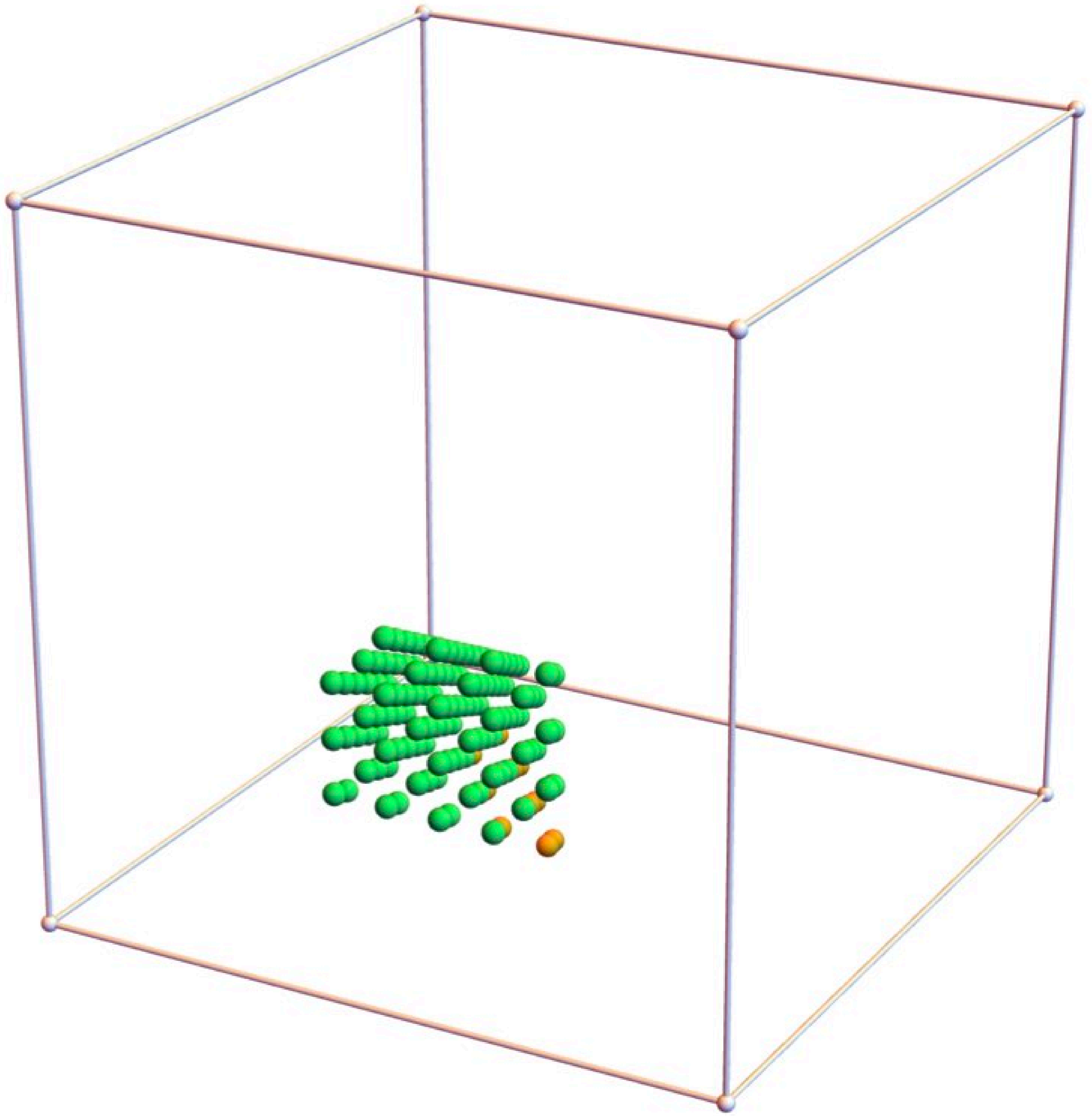

The 5/9 rule, a few b > 2c + 2 exit

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with $0 \le a, b, c \le 12$, and get rid of any with $a < b$, $b < c$, $a < c$.

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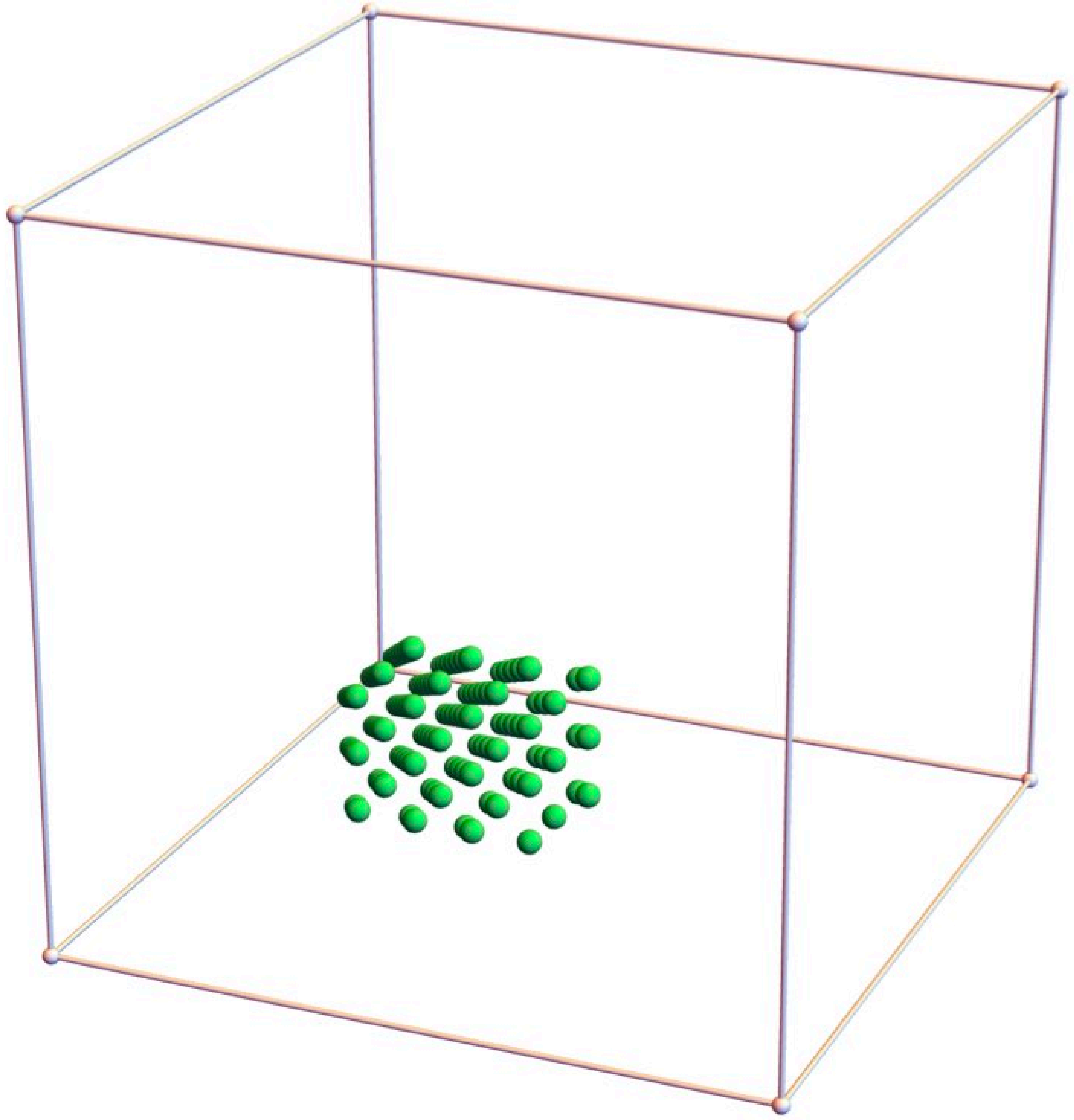

Down to **117** triples with $b \le 2c + 2$

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with 0 ≤ a, b, c ≤ 12, and get rid of any with a < b, b < c, a < c.

Let's watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.

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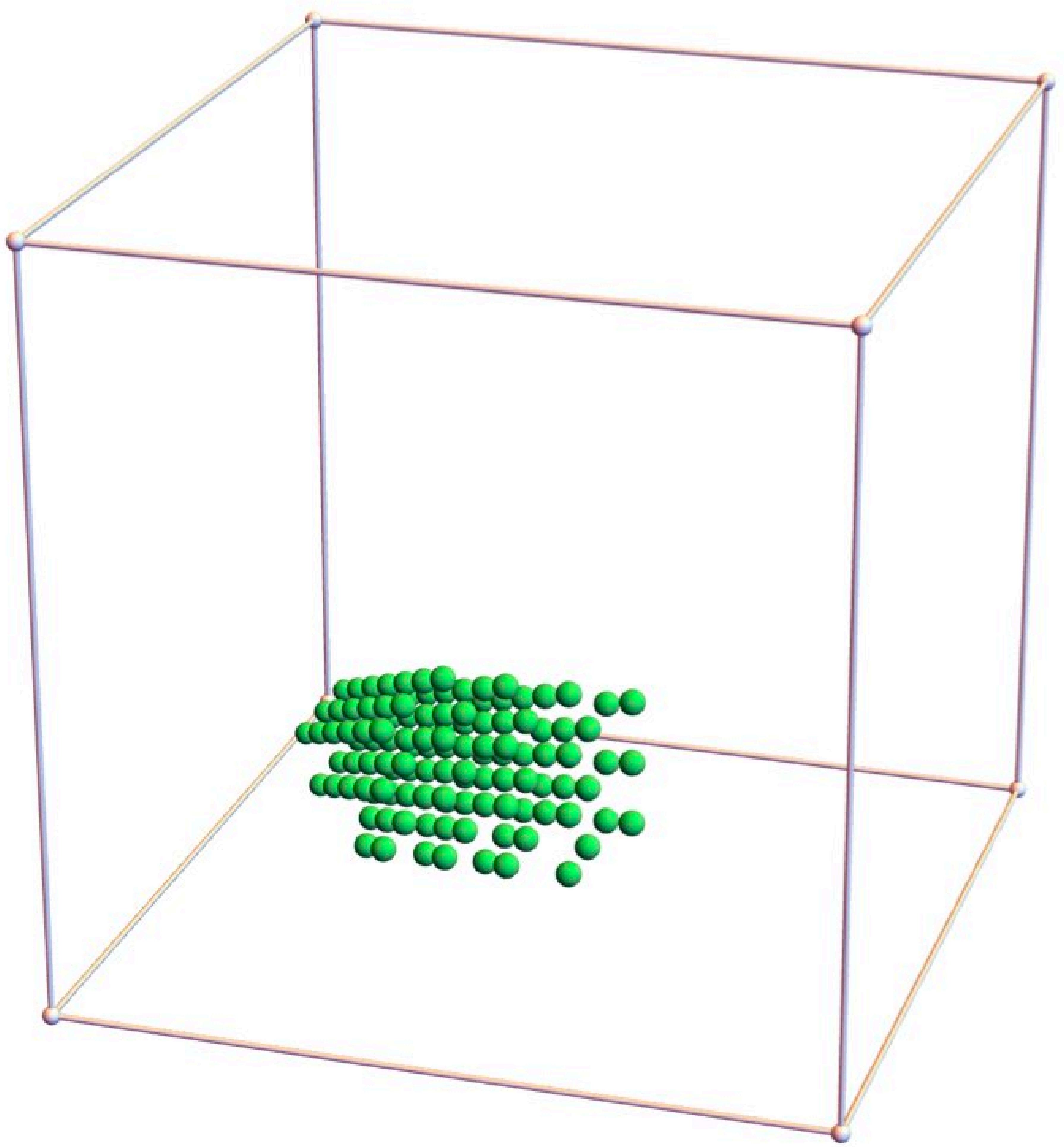

The 81/125 rule, no small triples leave.

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with 0 ≤ a, b, c ≤ 12, and get rid of any with a < b, b < c, a < c.

Let's watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.

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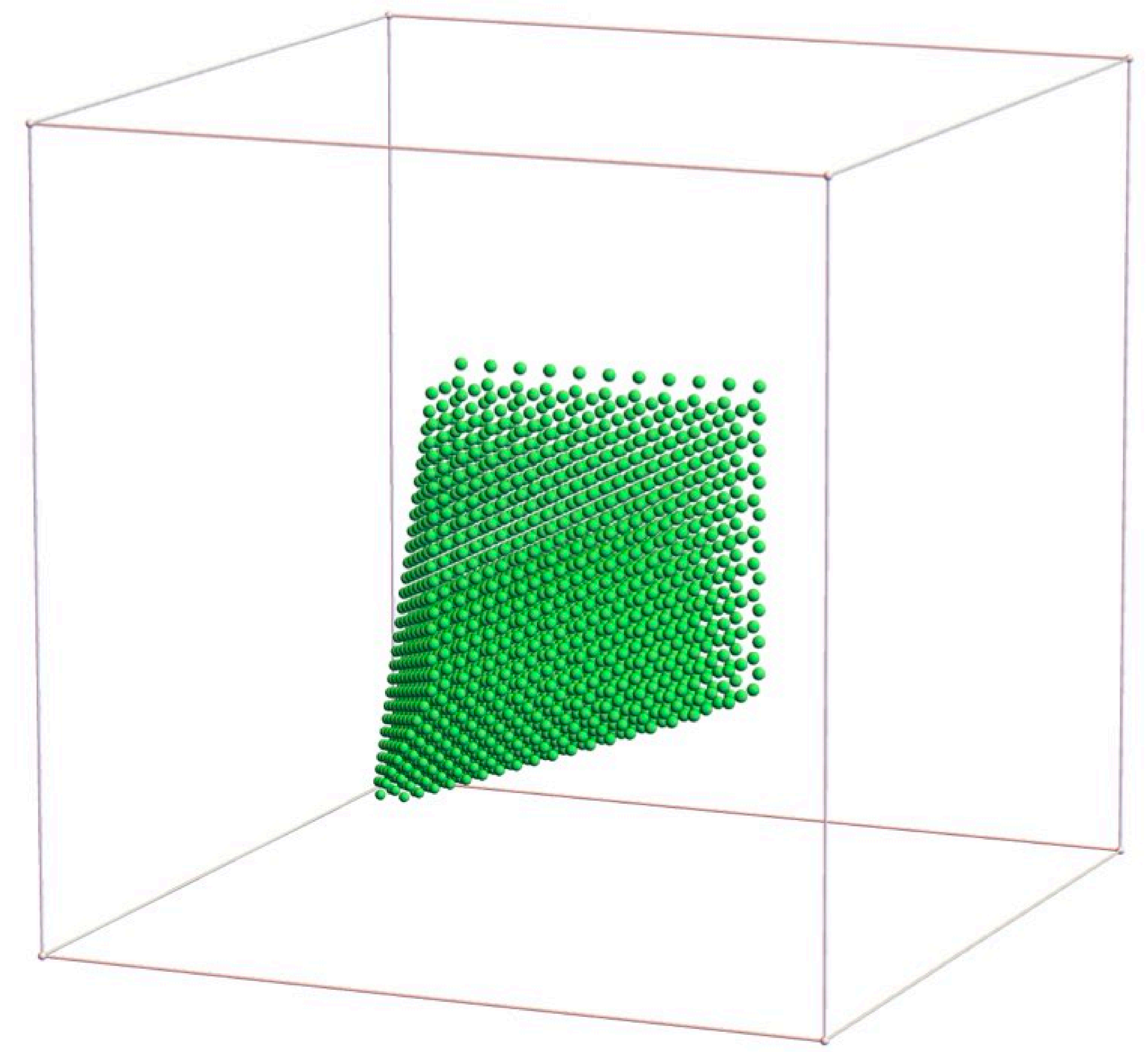

Let's use lots more dots, 55 each way

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with $0 \le a, b, c \le 12$, and get rid of any with $a < b$, $b < c$, $a < c$.

Let's watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.

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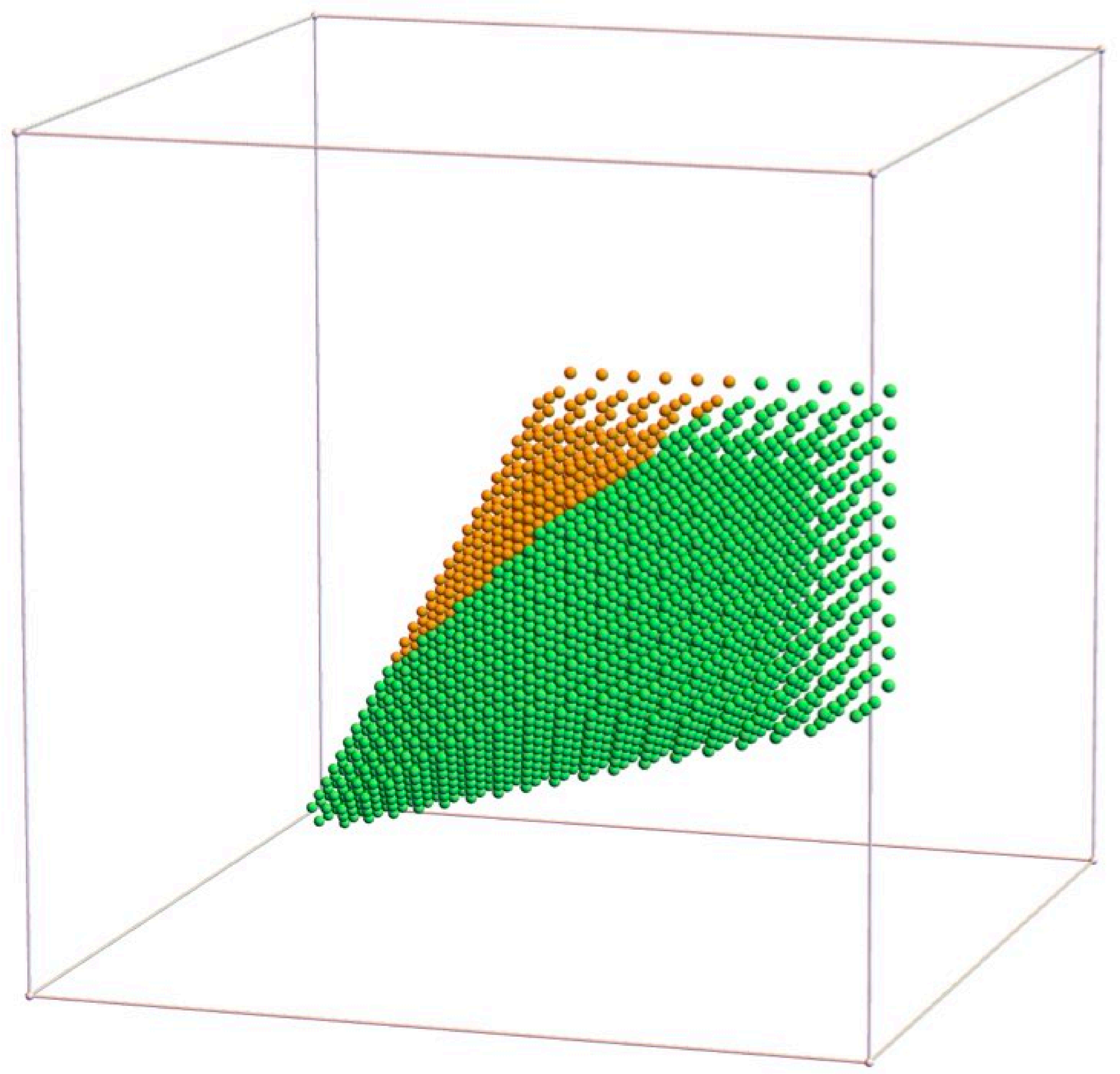

Now the 81/125 rules out a bunch

Exponent Lattices and the "Super-Sector"

Start with all (a, b, c) in a cube with 0 ≤ a, b, c ≤ 12, and get rid of any with a < b, b < c, a < c.

Let's watch these bad lattice points get shaved off and expose the super-sector:

Now apply some rules involving just two primes, such as 3/4 or 5/8, to eliminate certain (a, b, c) triples from super-contention.

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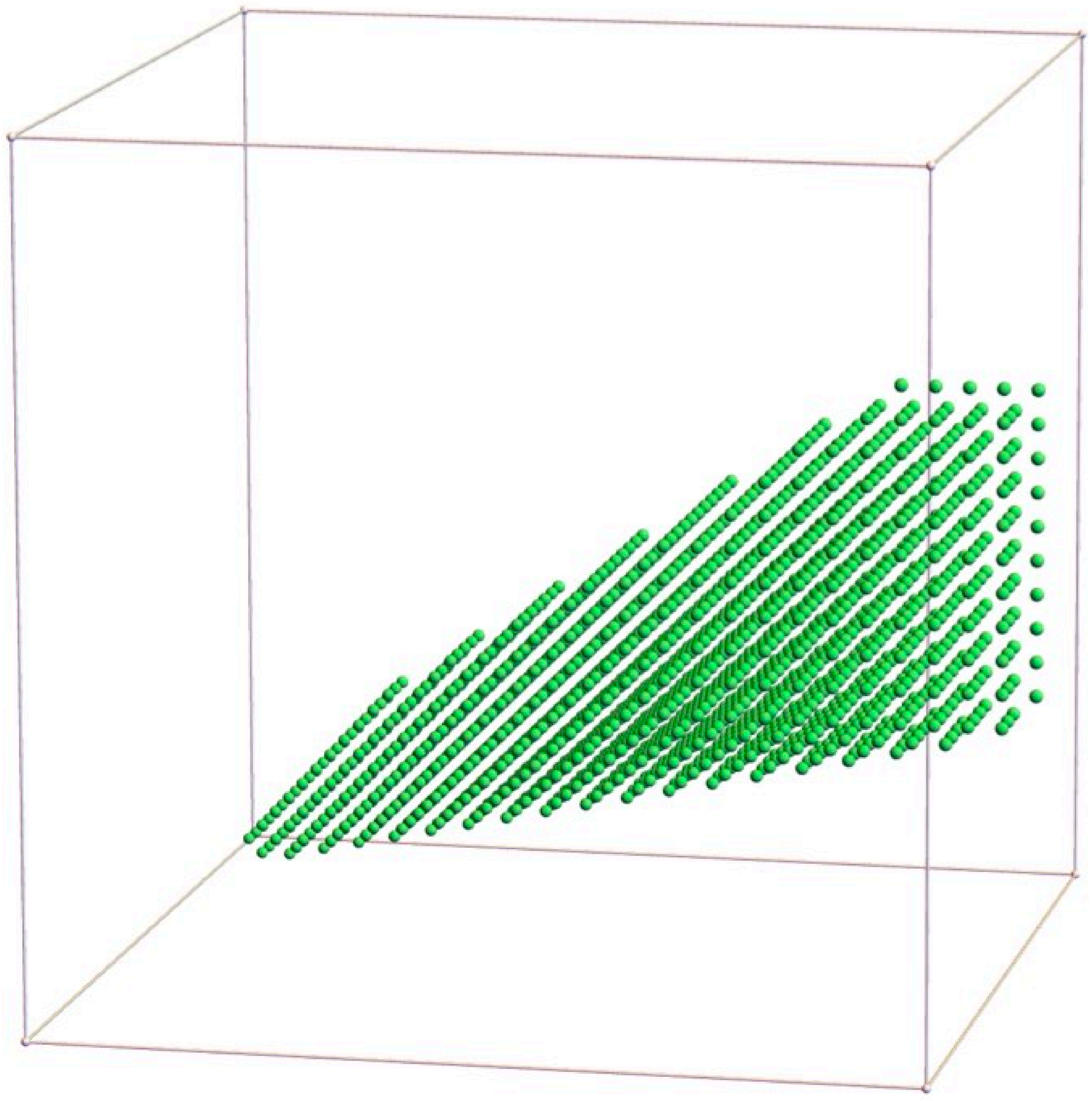


The survivors, after "linear shaving"

Exponent Lattices and the "Super-Sector"

For $n = 2^a 3^b 5^c \ldots$ to be super-composite, we have new rules that use all of a, b, and c:

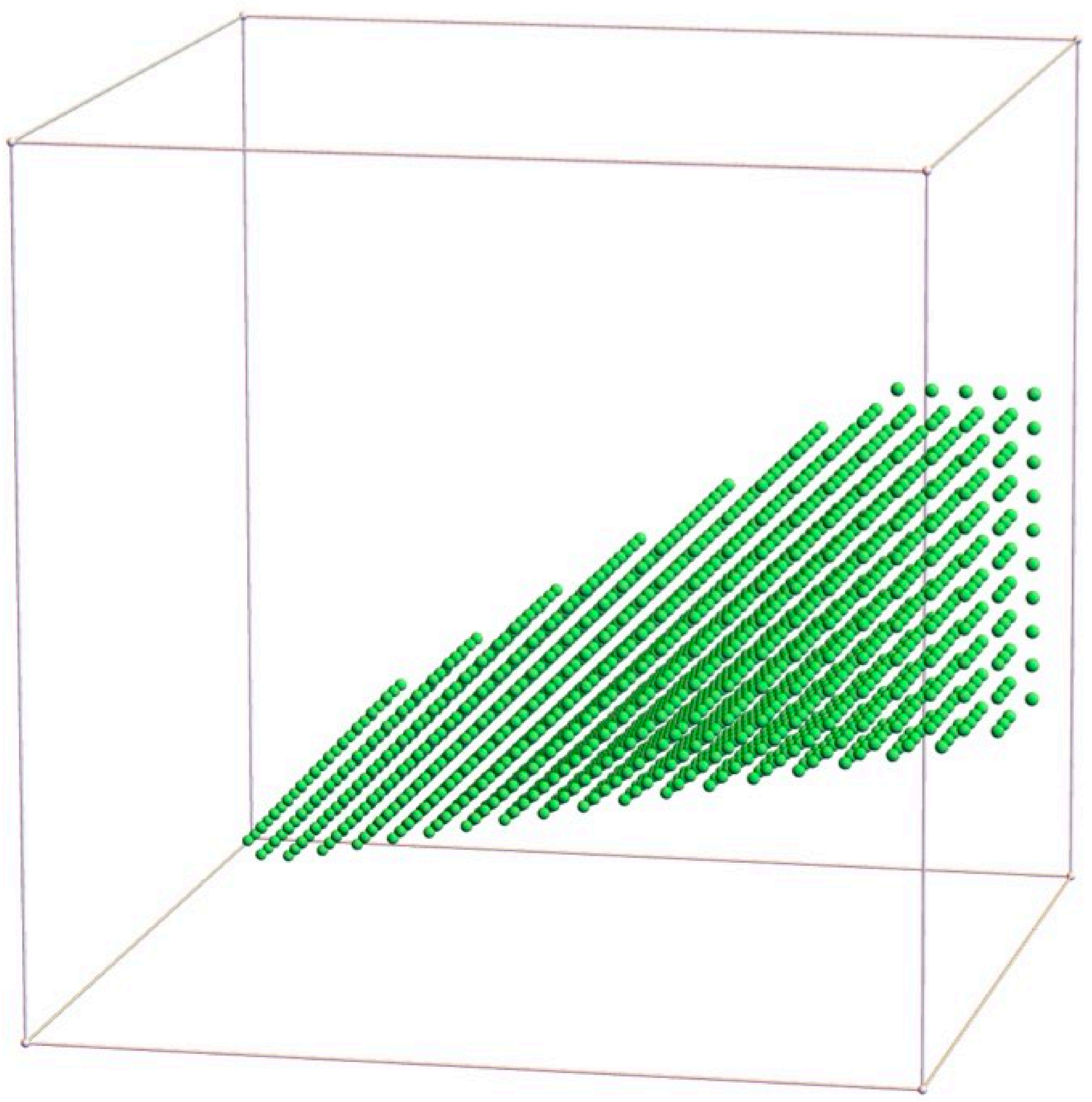
$ab - ac - bc \le a + b + c$ *(using rule of 5/6)*

$2ac - ab - bc \le a + b + c$ *(using rule of 9/10)*

$ab + ac - 4bc \le -3a + 7b + 7c + 12$ *(r.15/16)*

$-2ab + ac + 3bc \le 3a + 5b - 7c + 8$ *(r.24/25)*

$-2ab + 2ac + bc \le 4a + 3b - 5c + 6$ *(r.18/25)*



The survivors, after "linear shaving"

Inequalities Expose the Super-Sector

Each picture eliminates another 'bad' part of the cube, leaving the eligible supersector empty.

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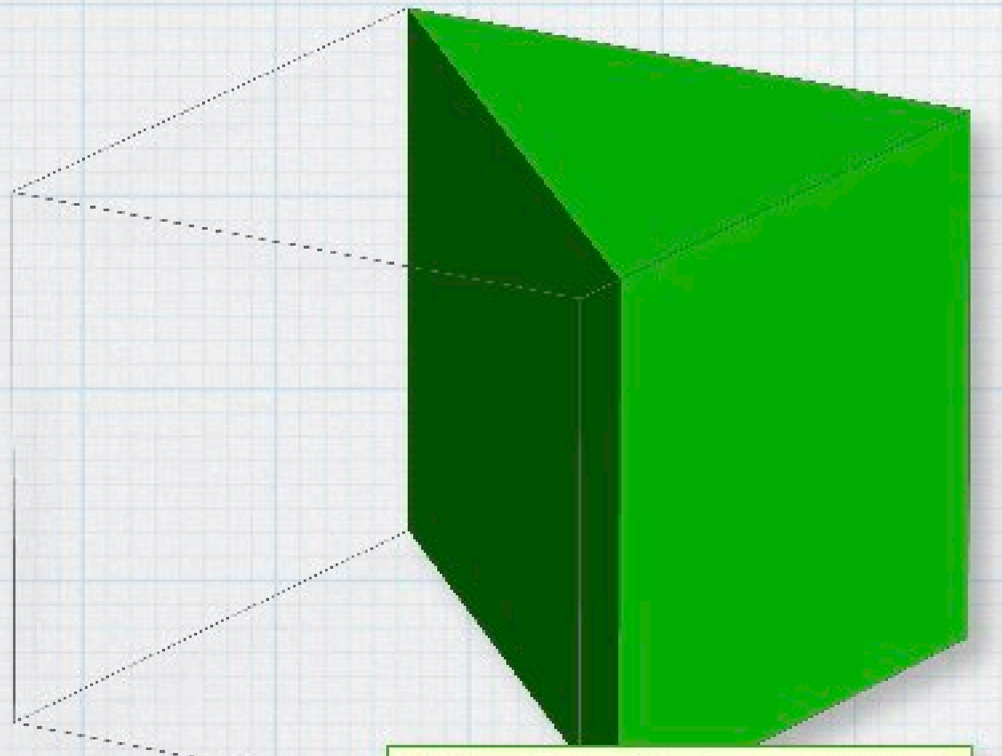
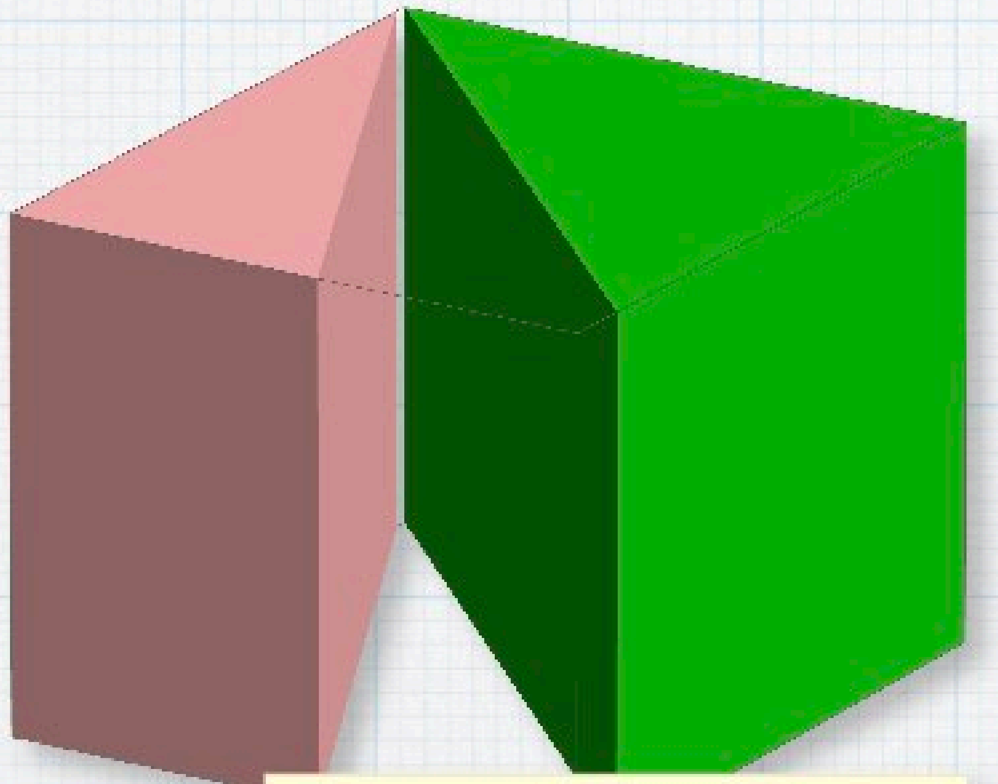
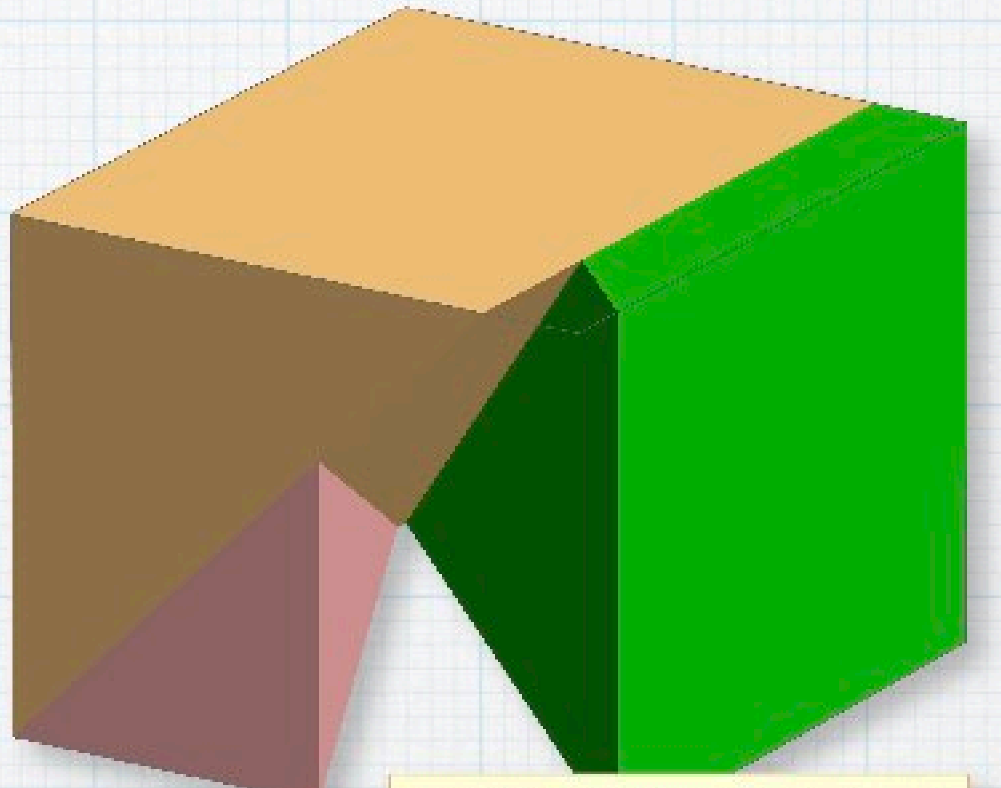


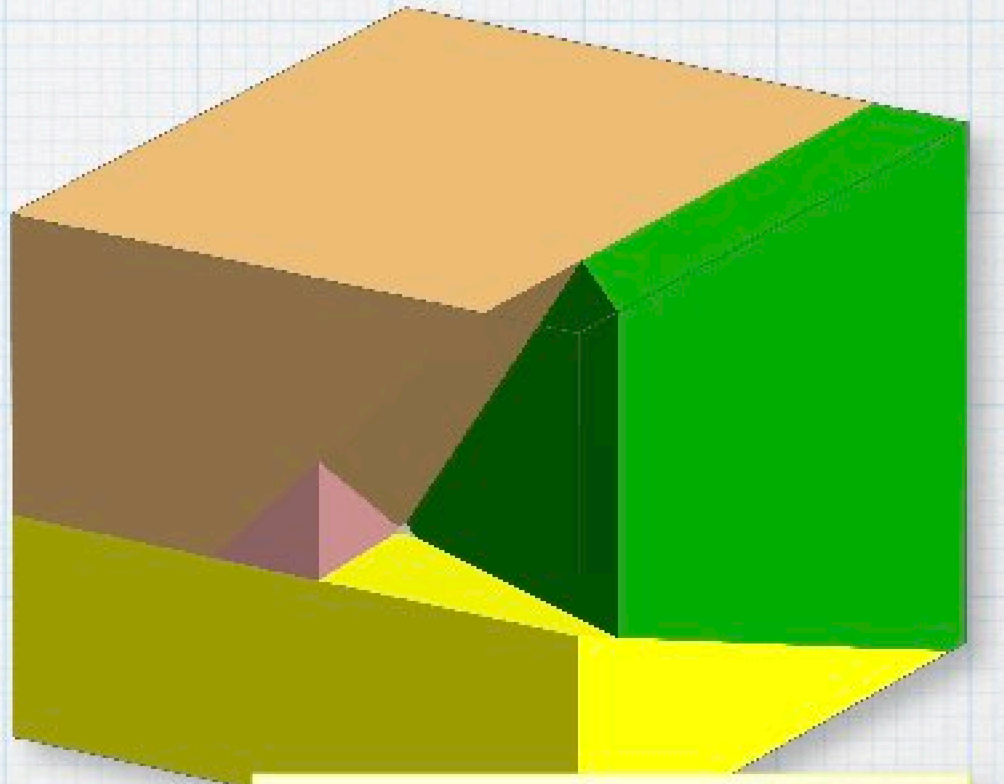
2/3 rule: eliminate $a < b$



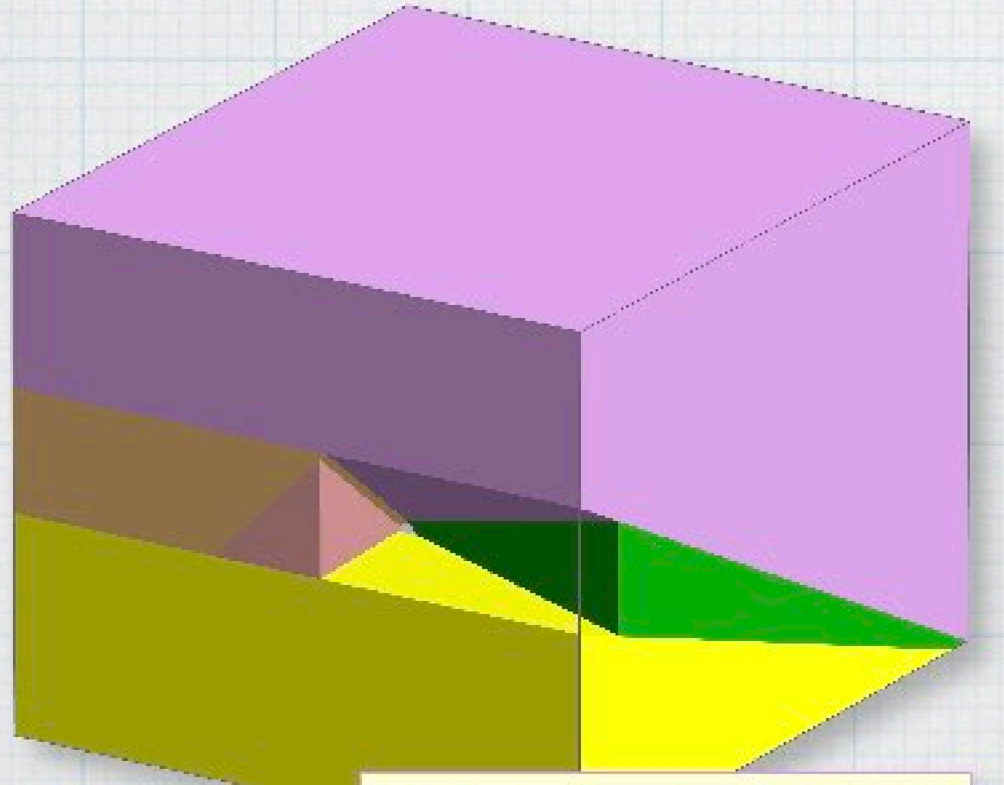
3/4 rule: eliminate $a > 2b + 2$



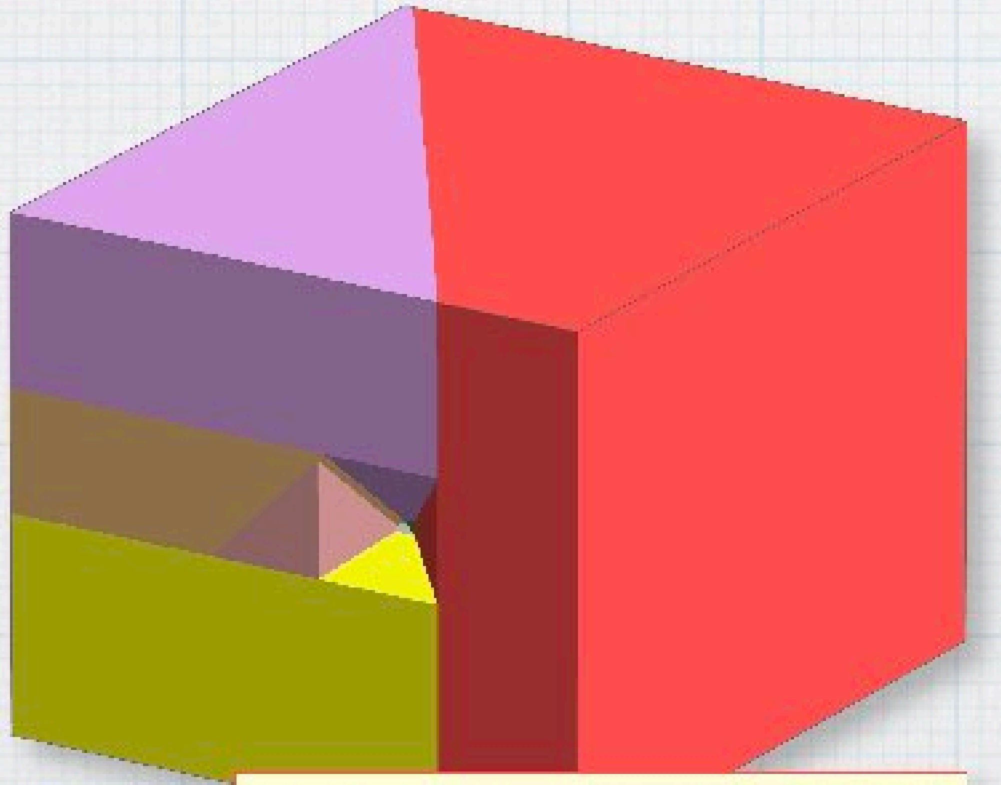
3/5 rule: eliminate $b < c$



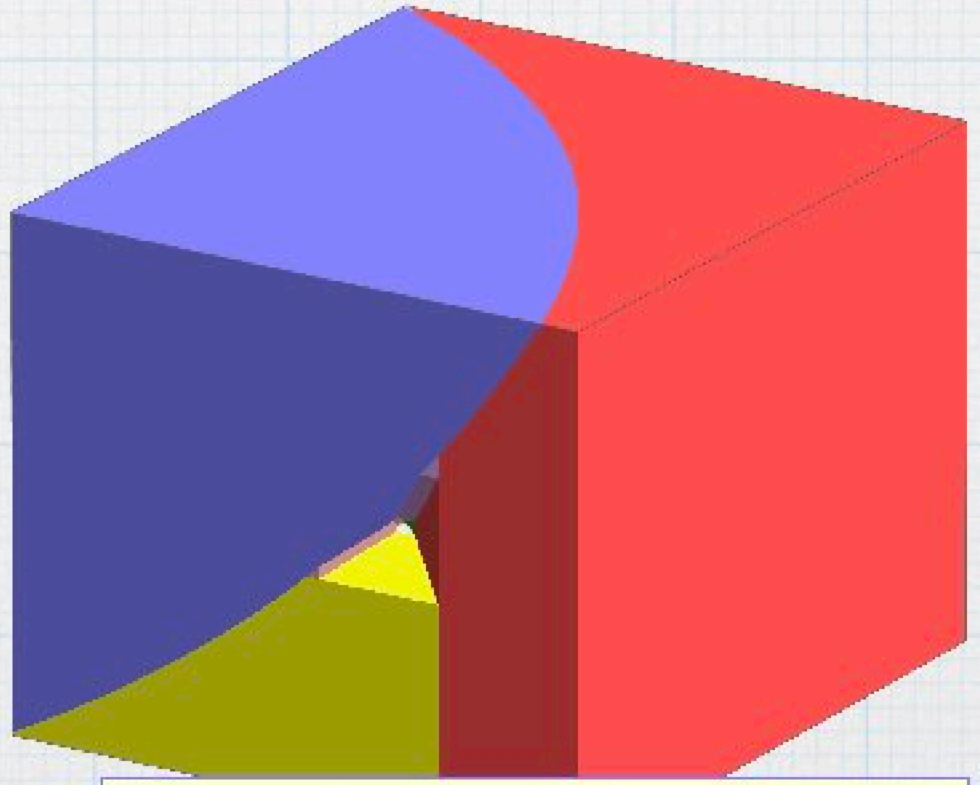
5/8 rule: eliminate $a > 3c + 4$



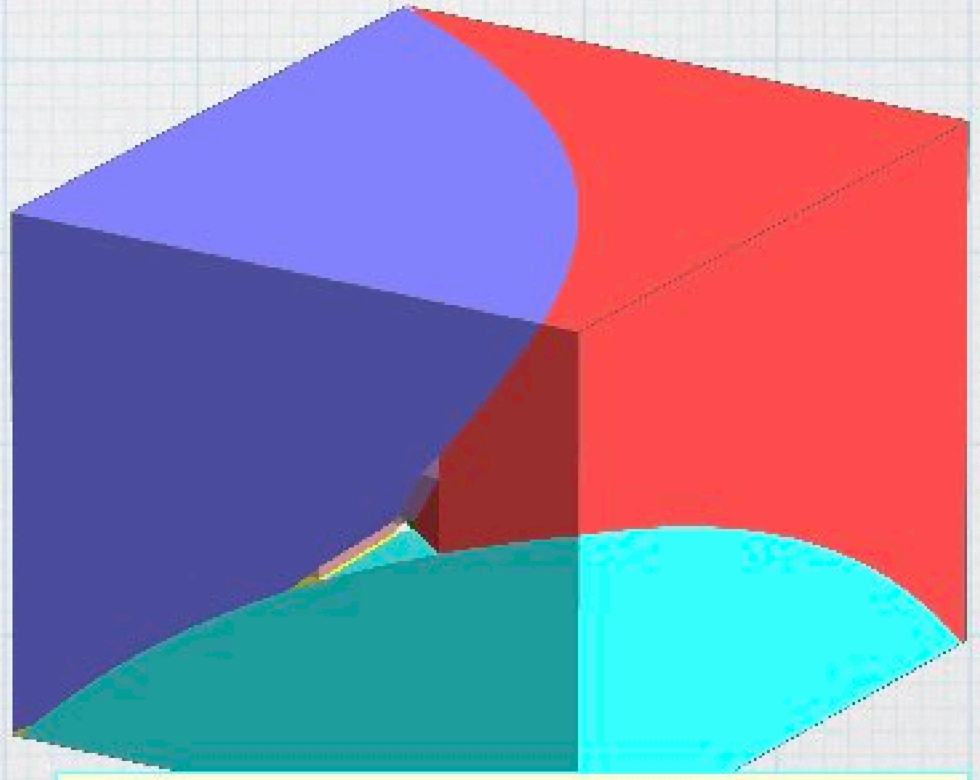
4/5 rule: eliminate $a < 2c$



8/9 rule: eliminate $2a < 3b - 4$



9/10 rule: elim $2ac - ab - bc > a + b + c$



5/6 rule: elim $ab - ac - bc > a + b + c$

A Bigger Box for the Super-Sector

Each new picture eliminates another 'bad' part of the cube, leaving the super-sector empty.

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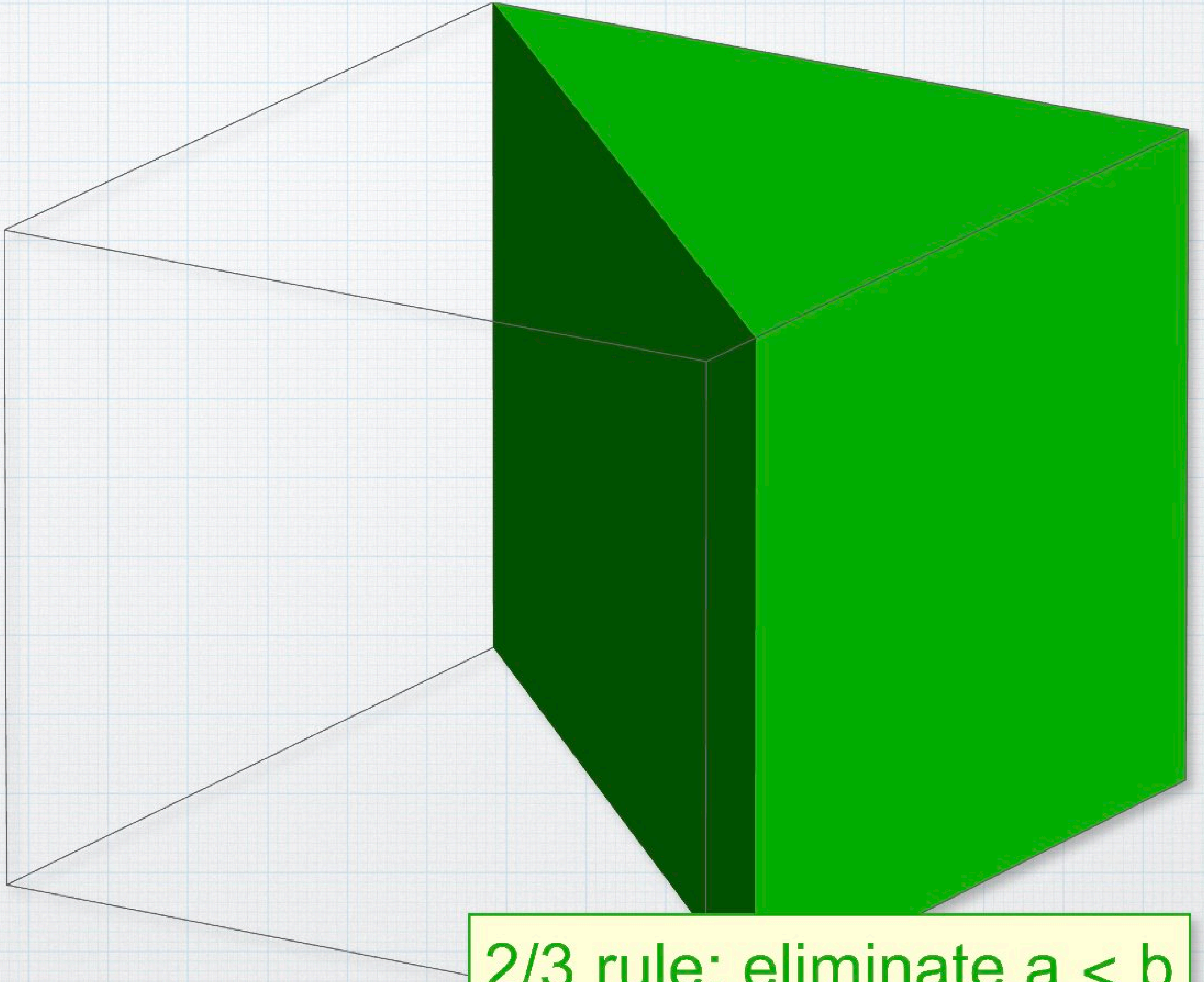

2/3 rule: eliminate a < b

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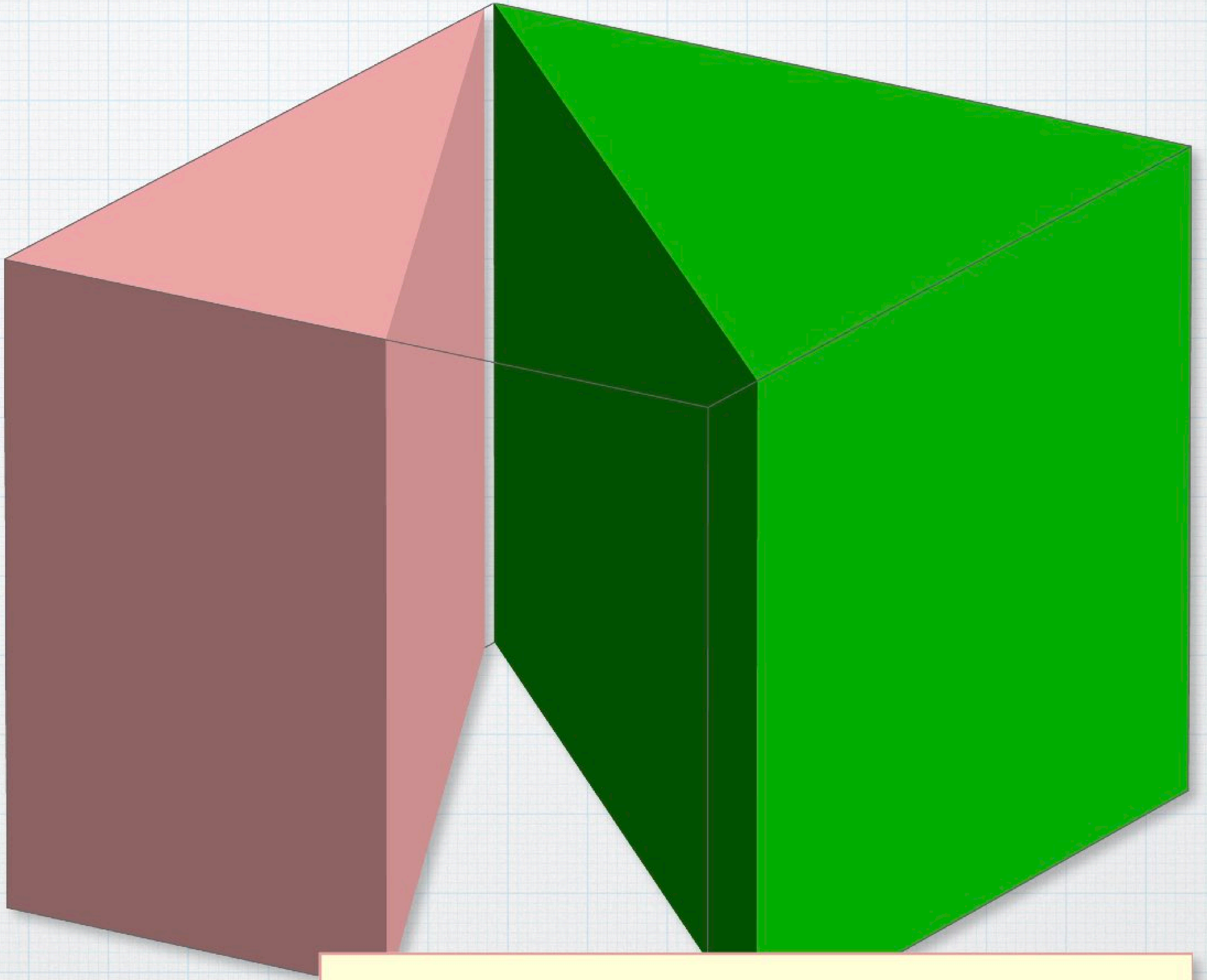

3/4 rule: eliminate a > 2b + 2

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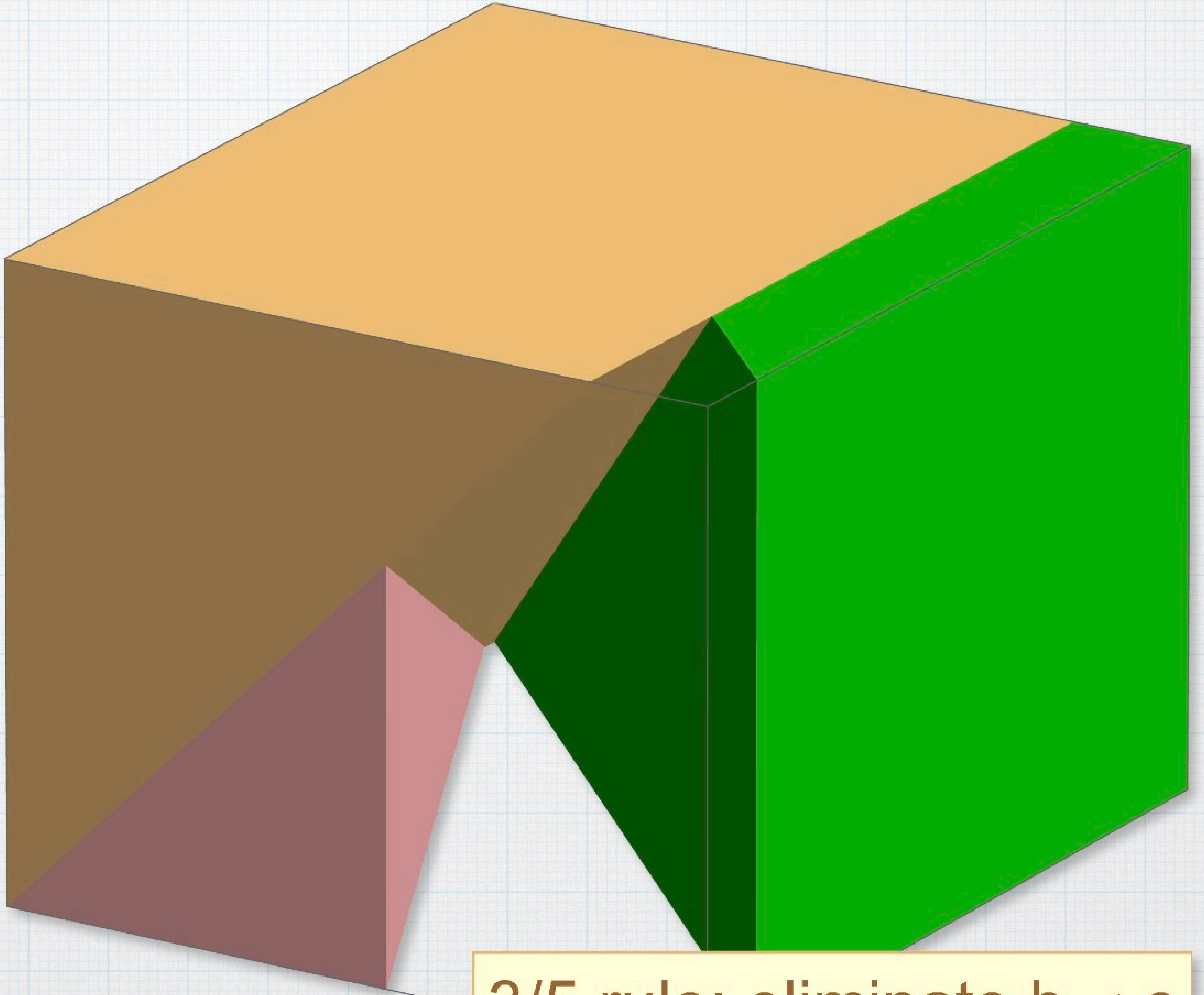

3/5 rule: eliminate b < c

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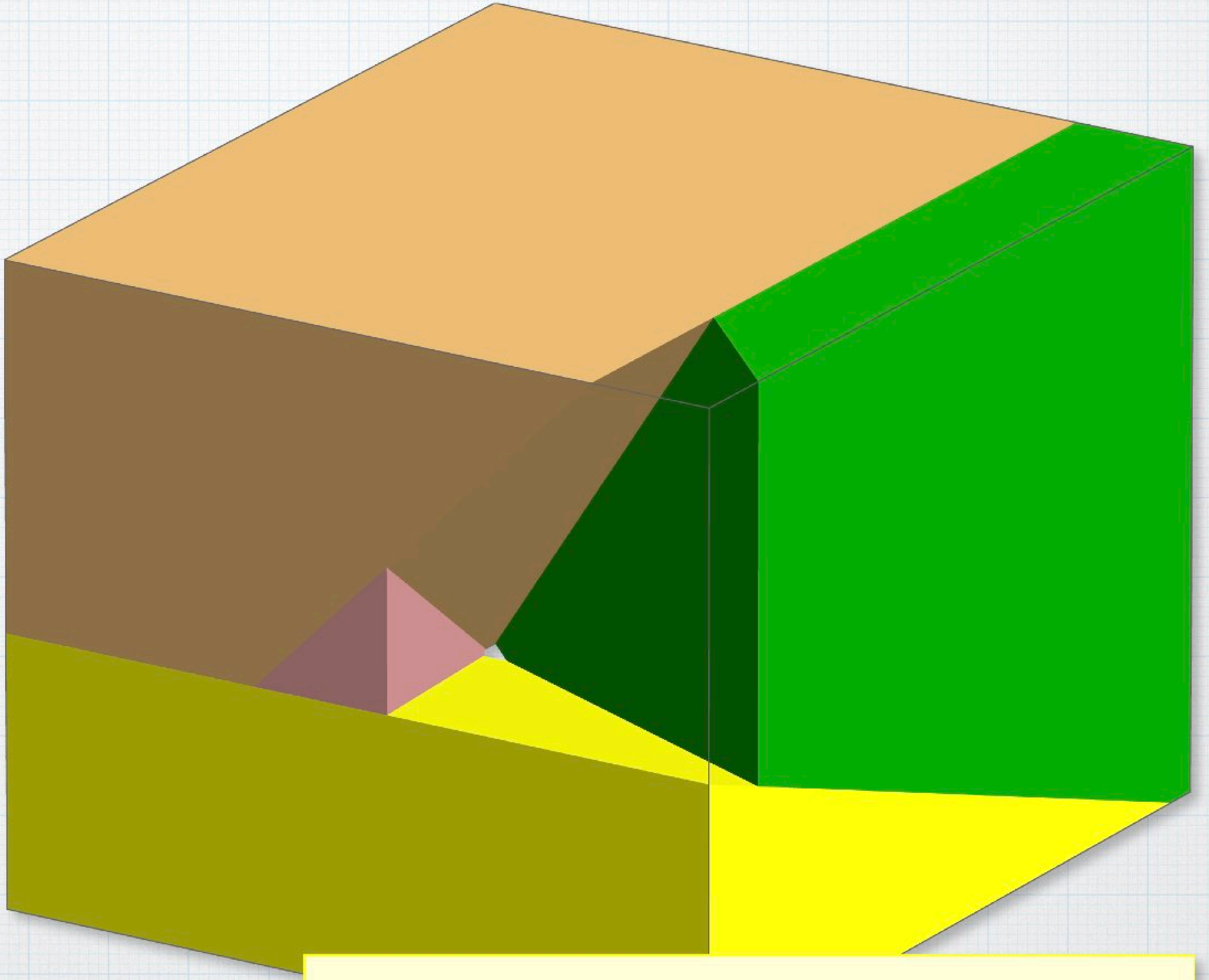

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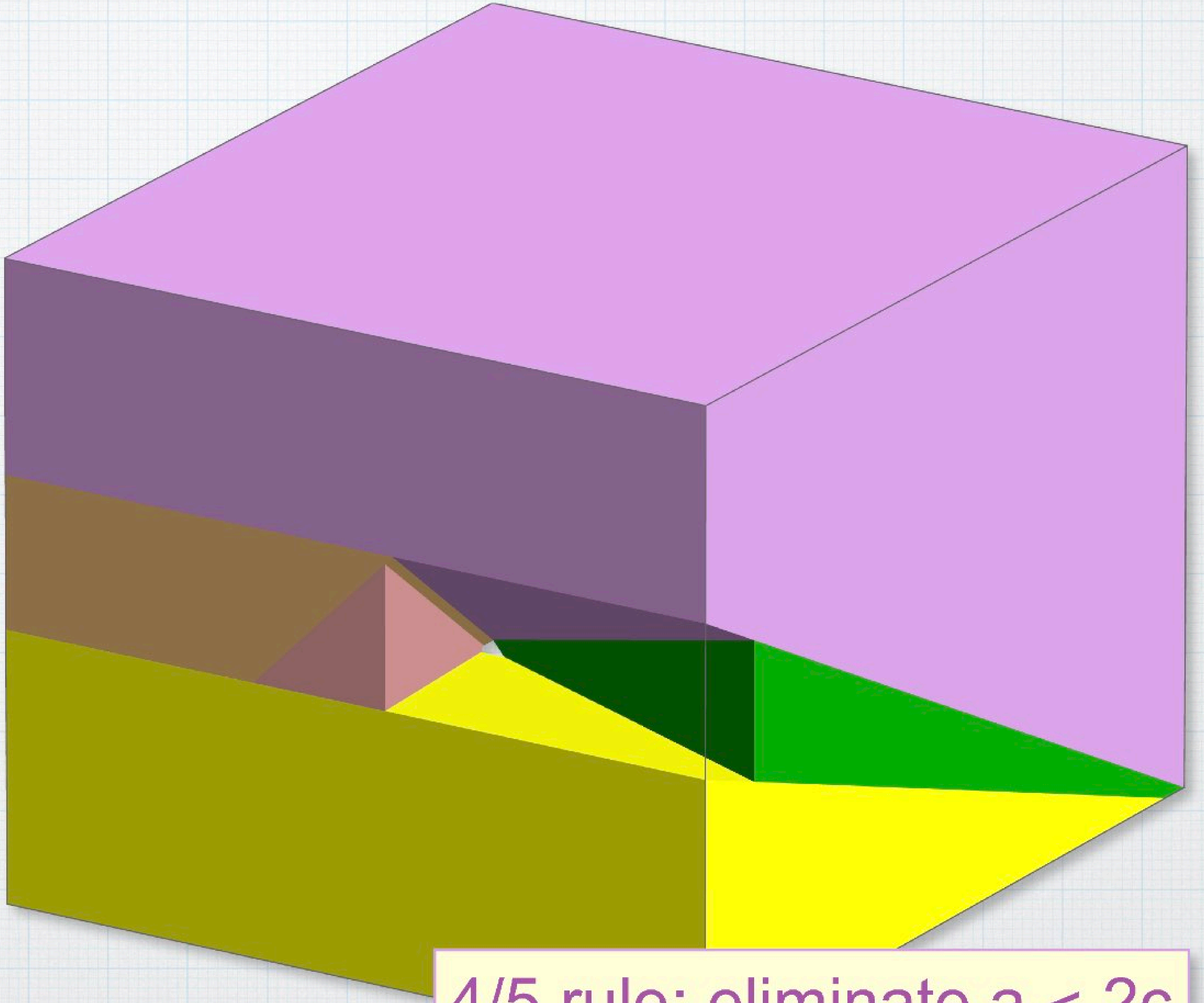

4/5 rule: eliminate a < 2c

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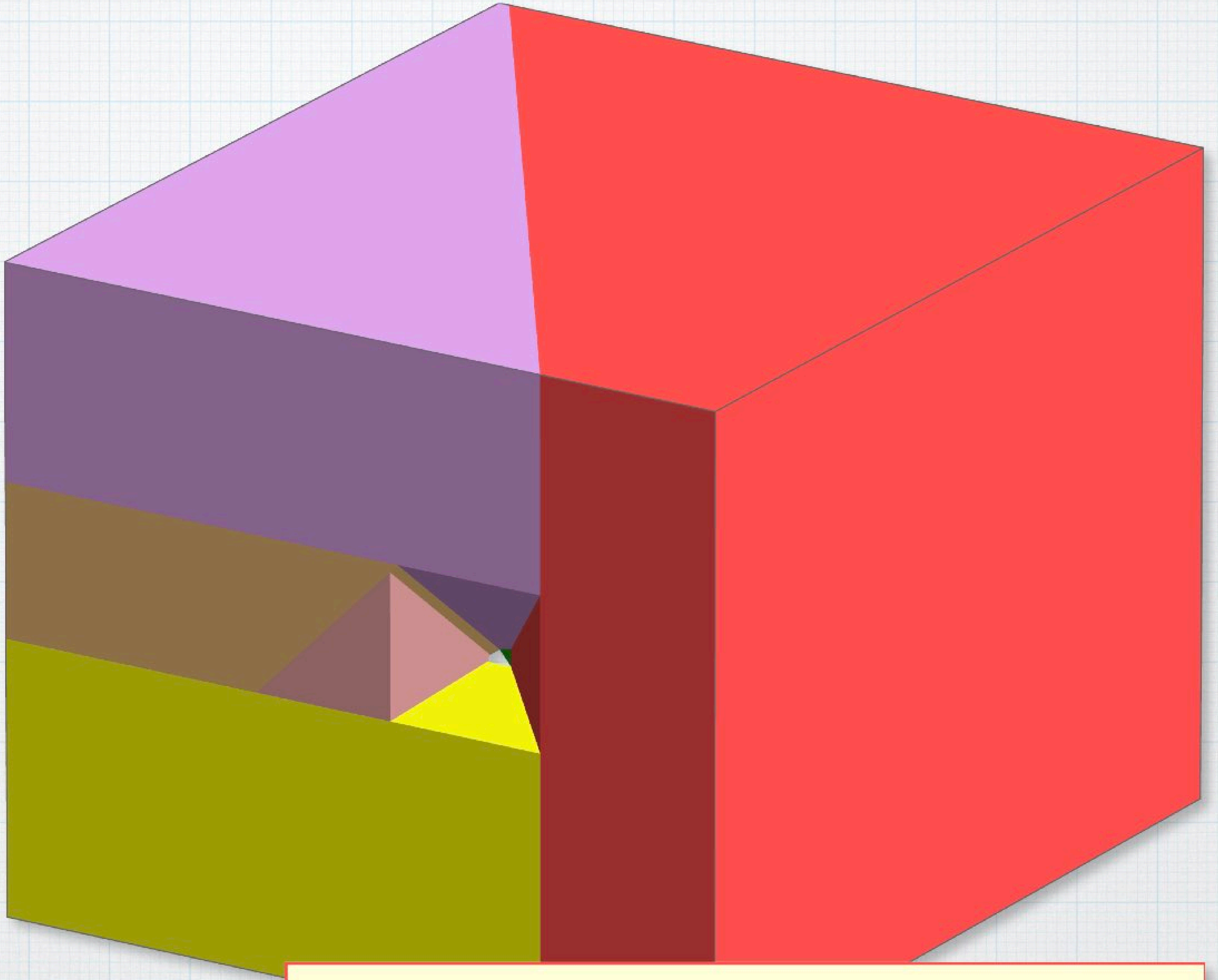

8/9 rule: eliminate 2a < 3b – 4

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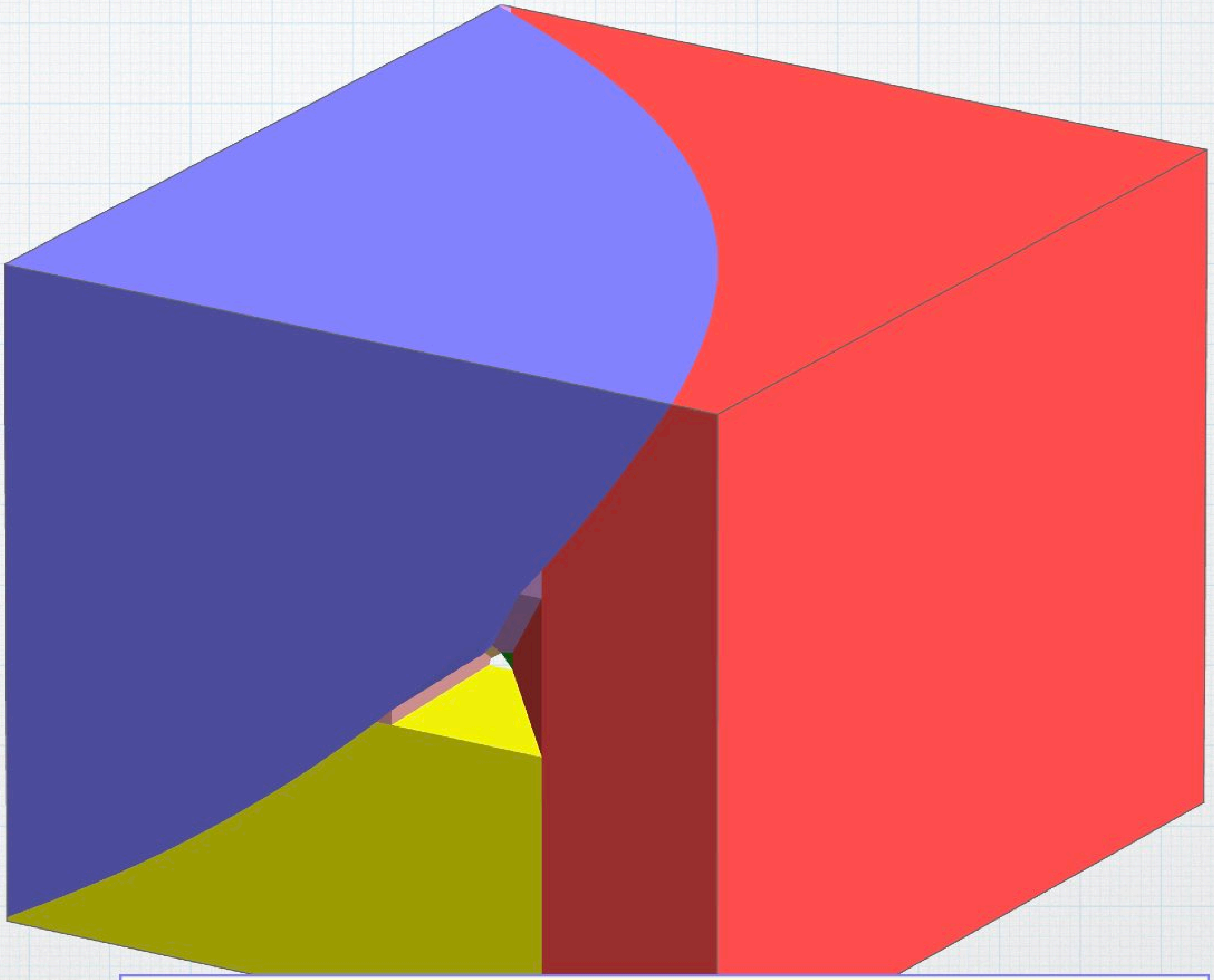

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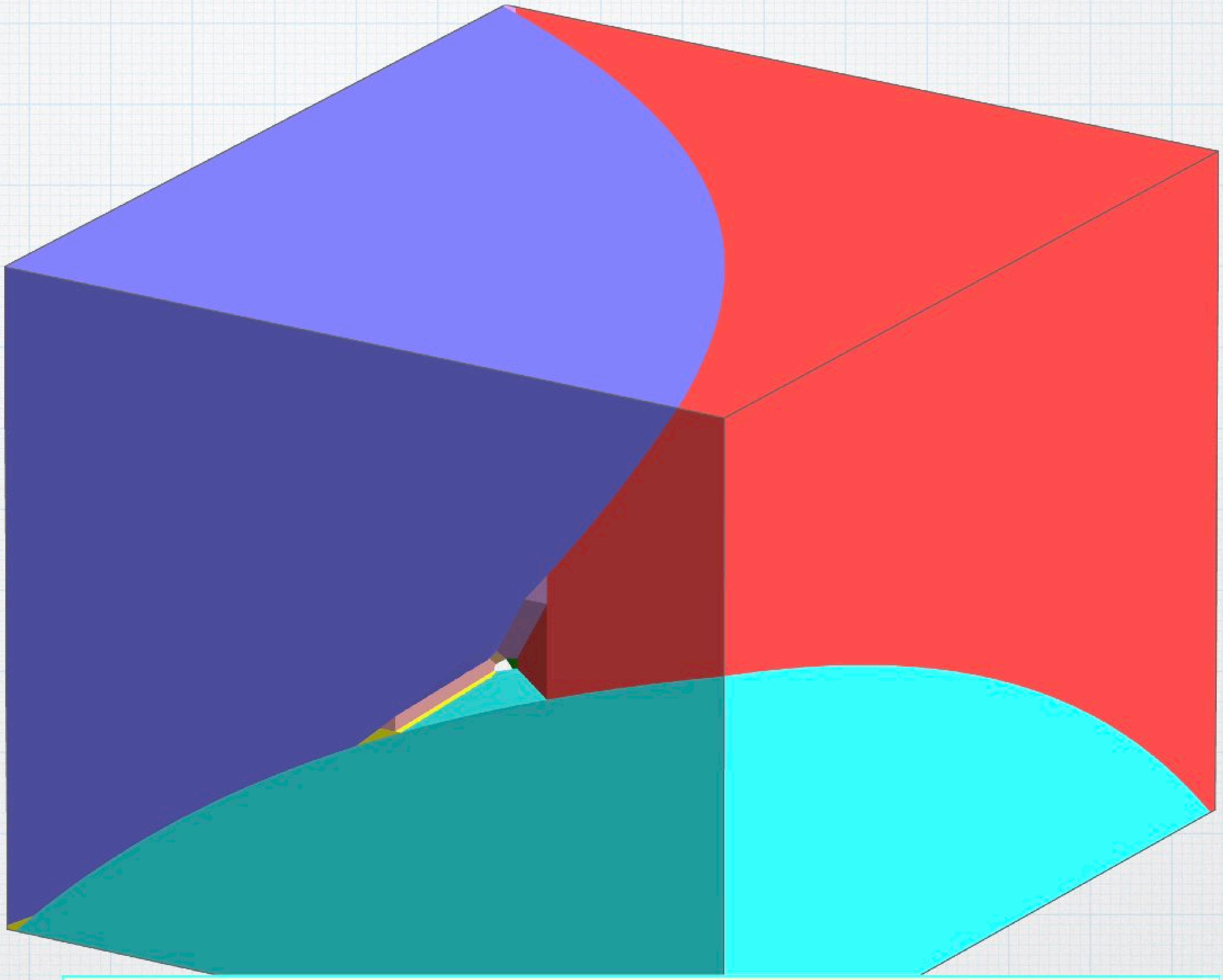

5/6 rule: elim ab – ac – bc > a + b + c

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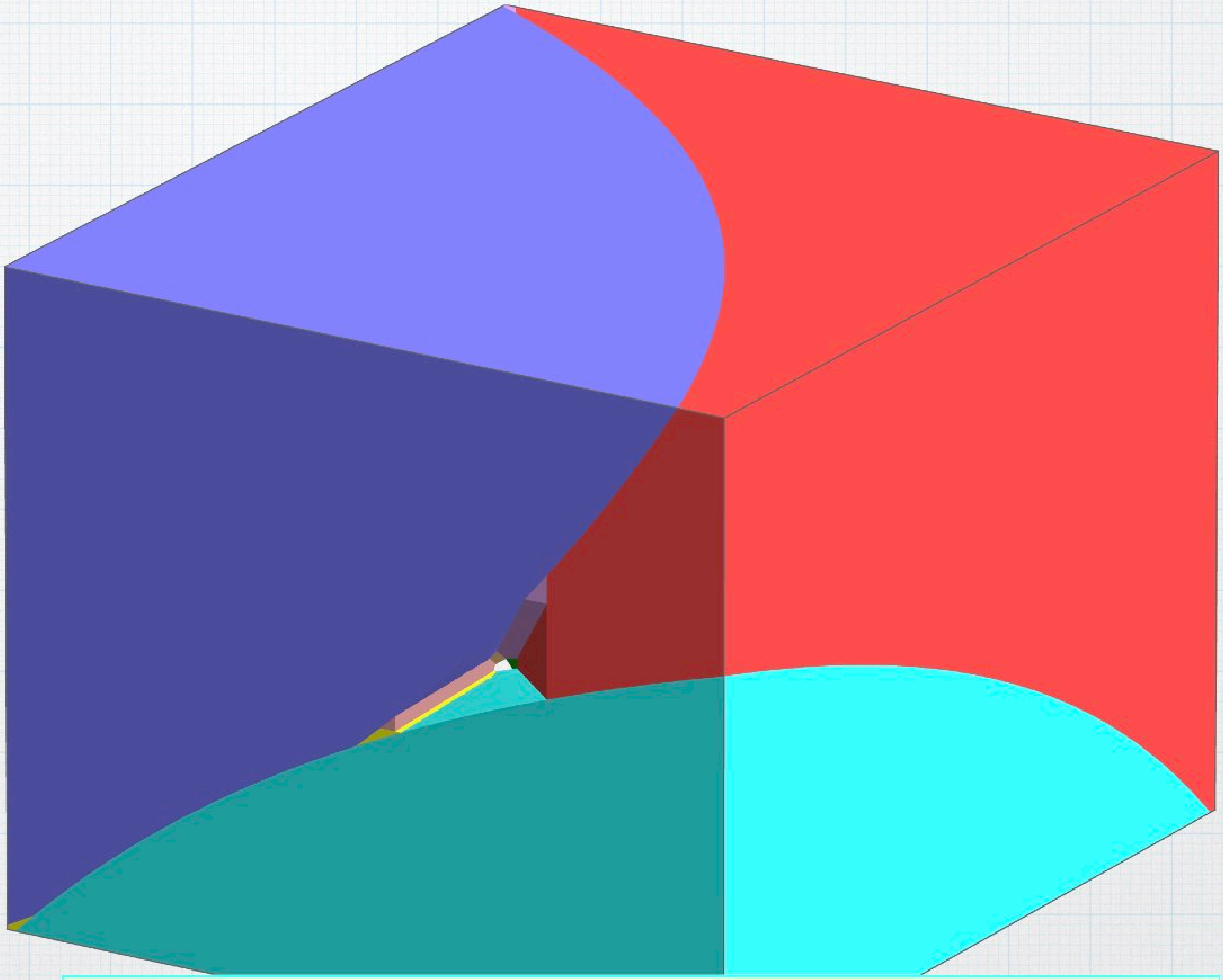
The cube represents all exponents of $2^a\ 3^b\ 5^c$ with $0 \le a \le 100$, $0 \le b \le 90$, $0 \le c \le 75$.



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Even More Superiority

Supercomposite numbers are at *local* maxima on the d(n) graph, but the graph can be ***distorted*** by dividing by n to a power.

If an s.c. number is a ***global*** max of $f(n) = \mathbf{d(n)/(n^k)}$ for some k, then it's called a ***superior*** supercomposite.

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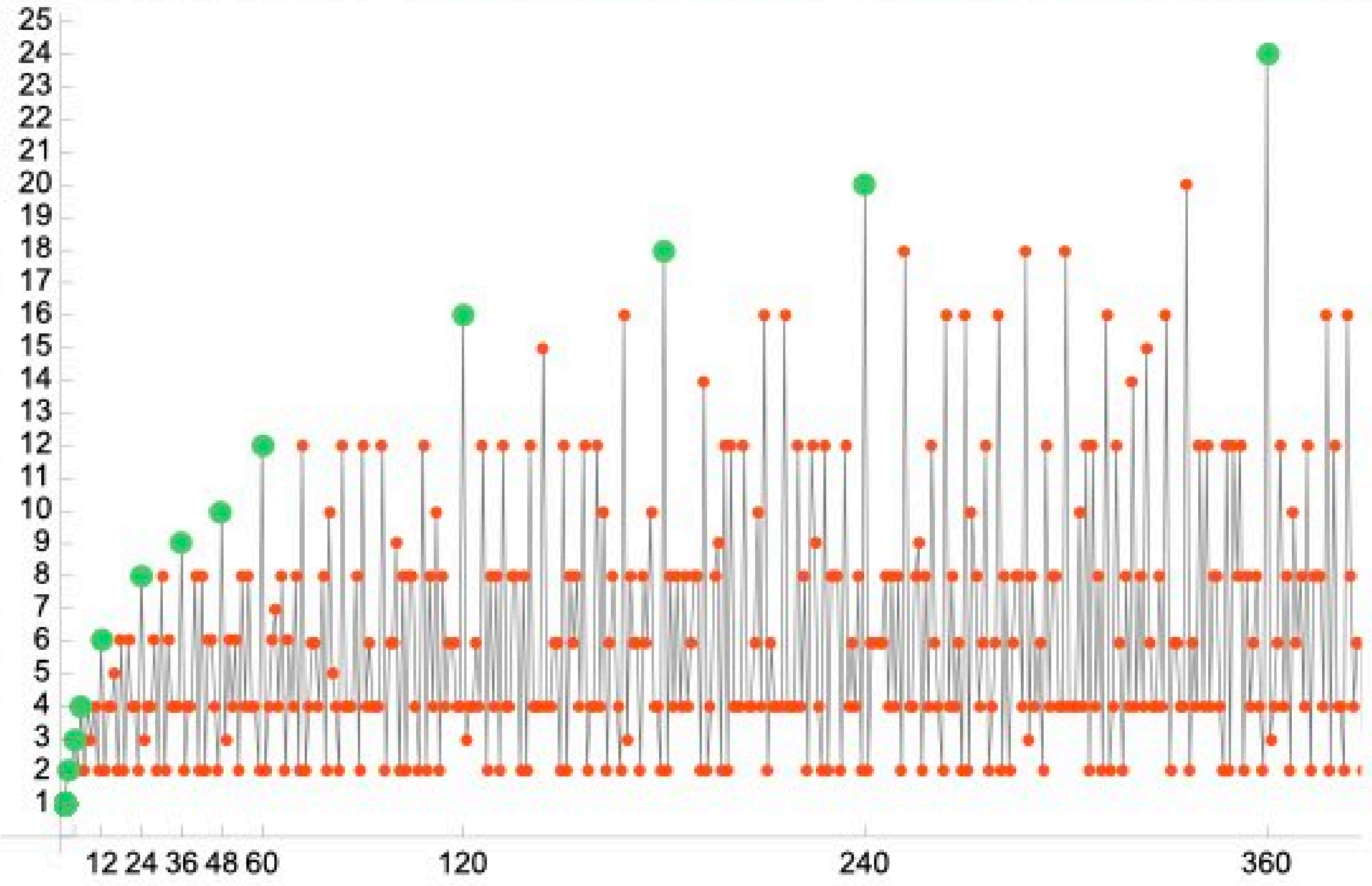

k = 0 **no max**

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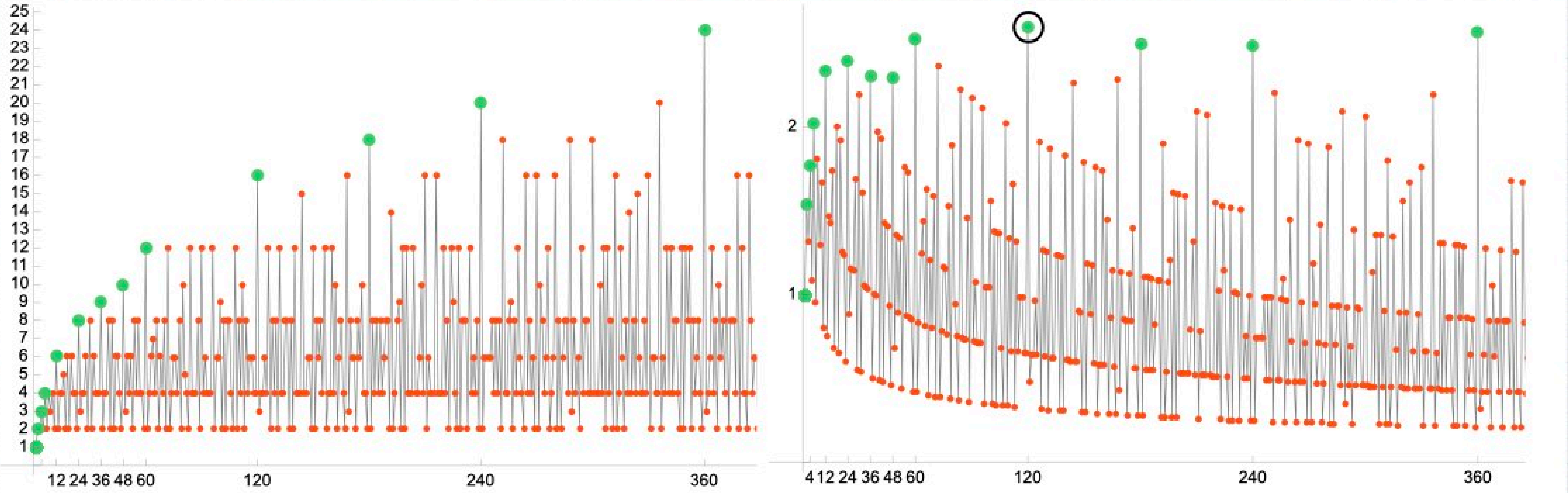

k = 0 no max

k = 0.38 max at 120

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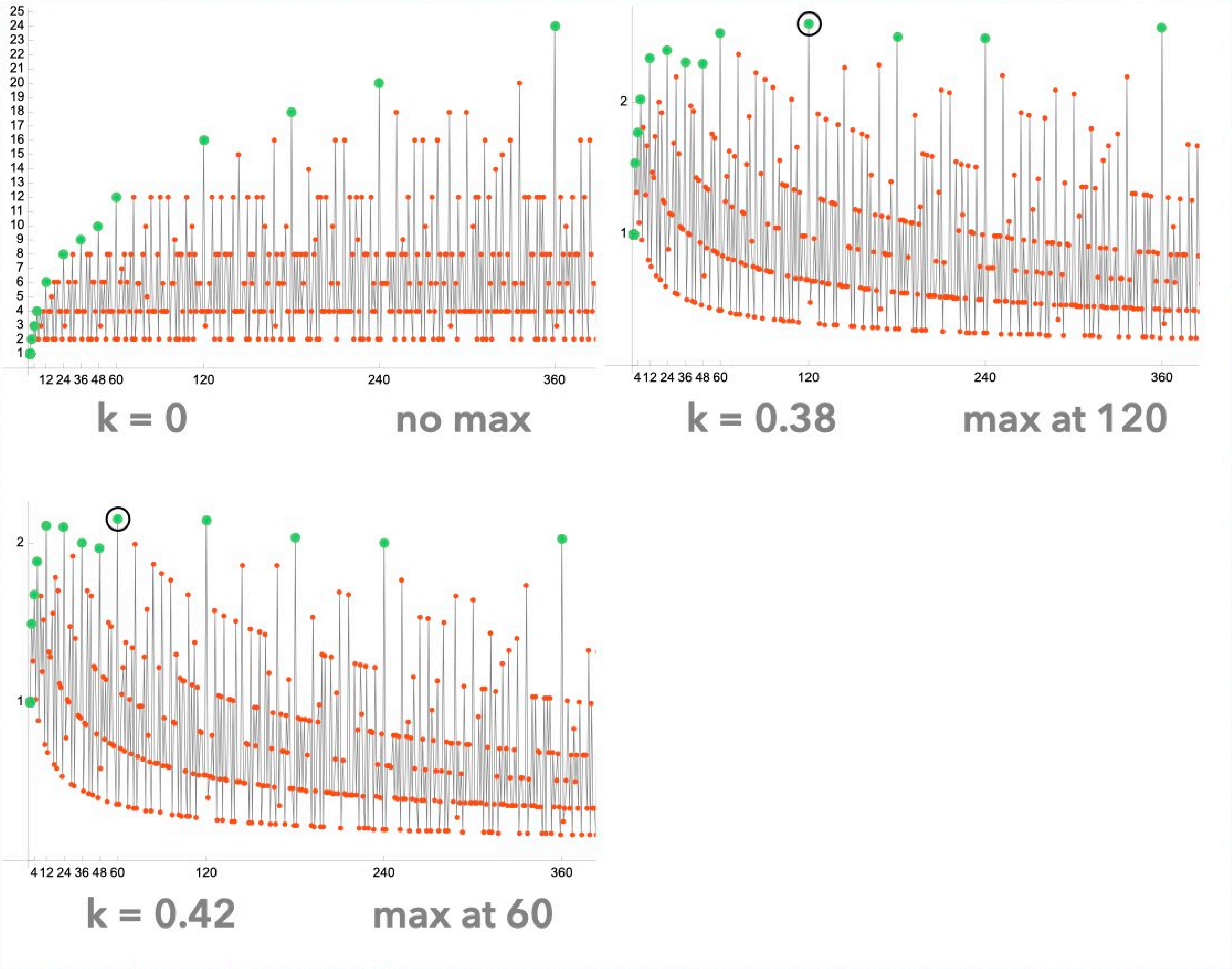

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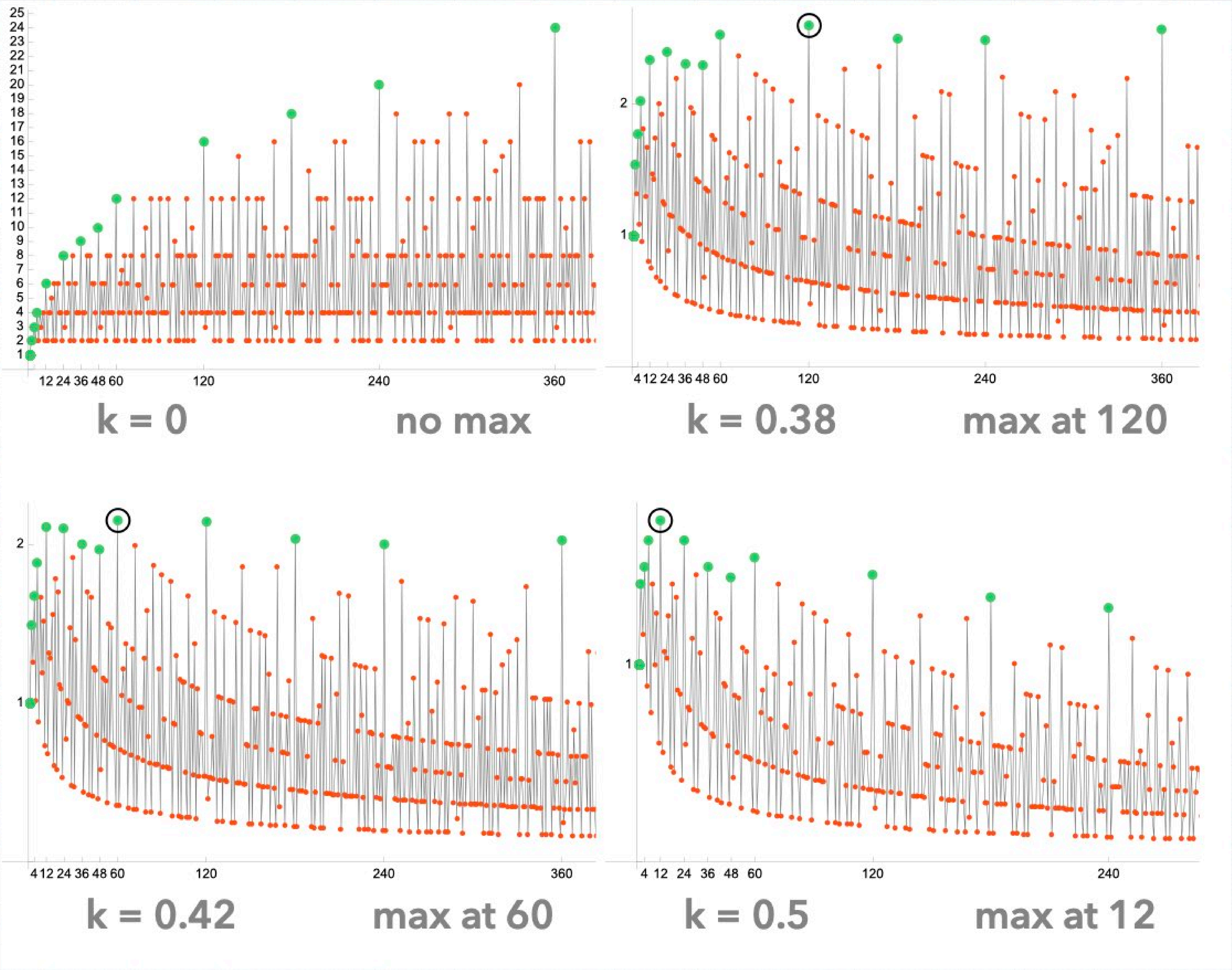
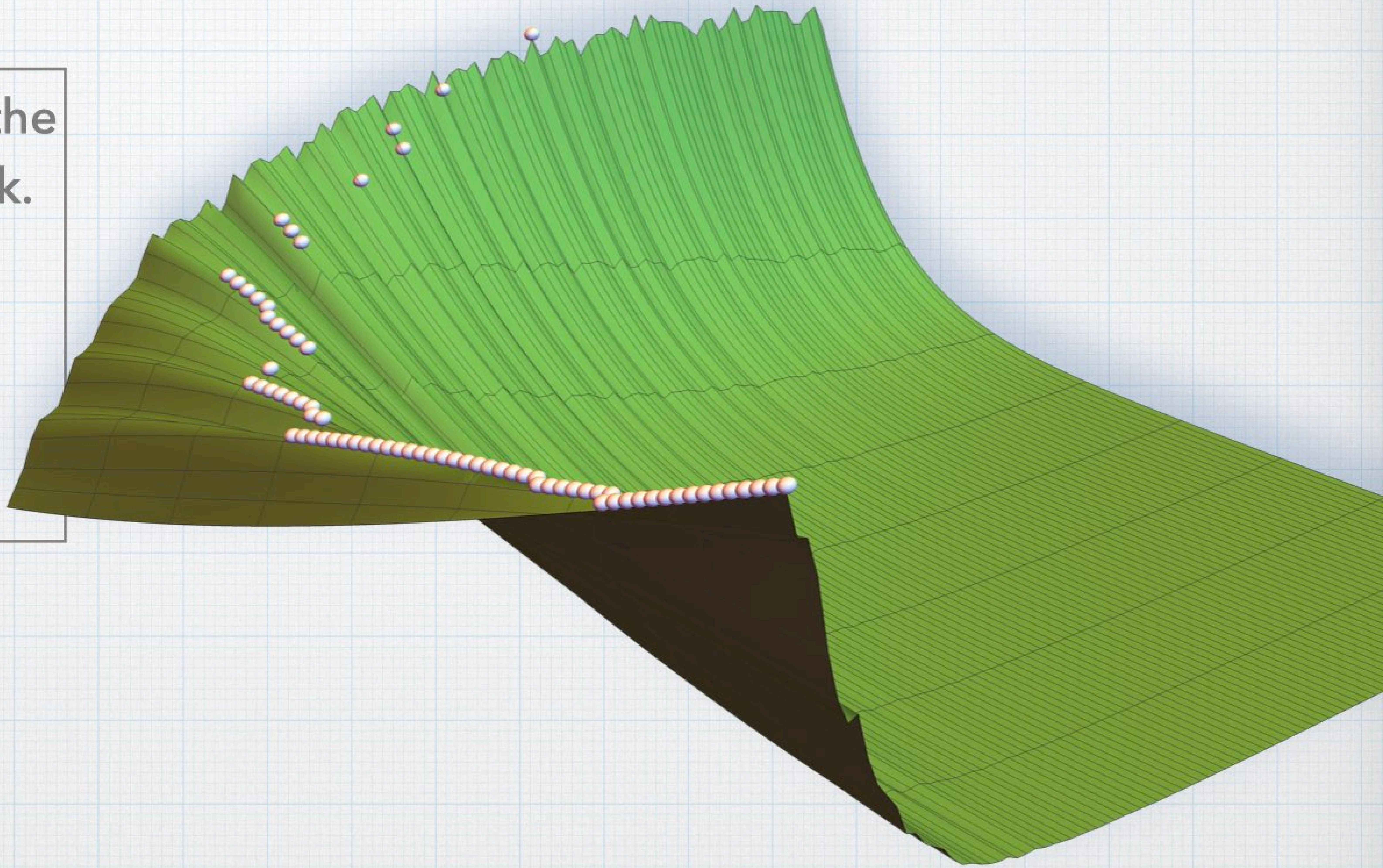


Superior Super-Eggs

Looking at $d(n)/n^k$ we spot the global max superior at each k.

Lay an egg there, and then move on to the next k.

Once in a while, the global max changes.



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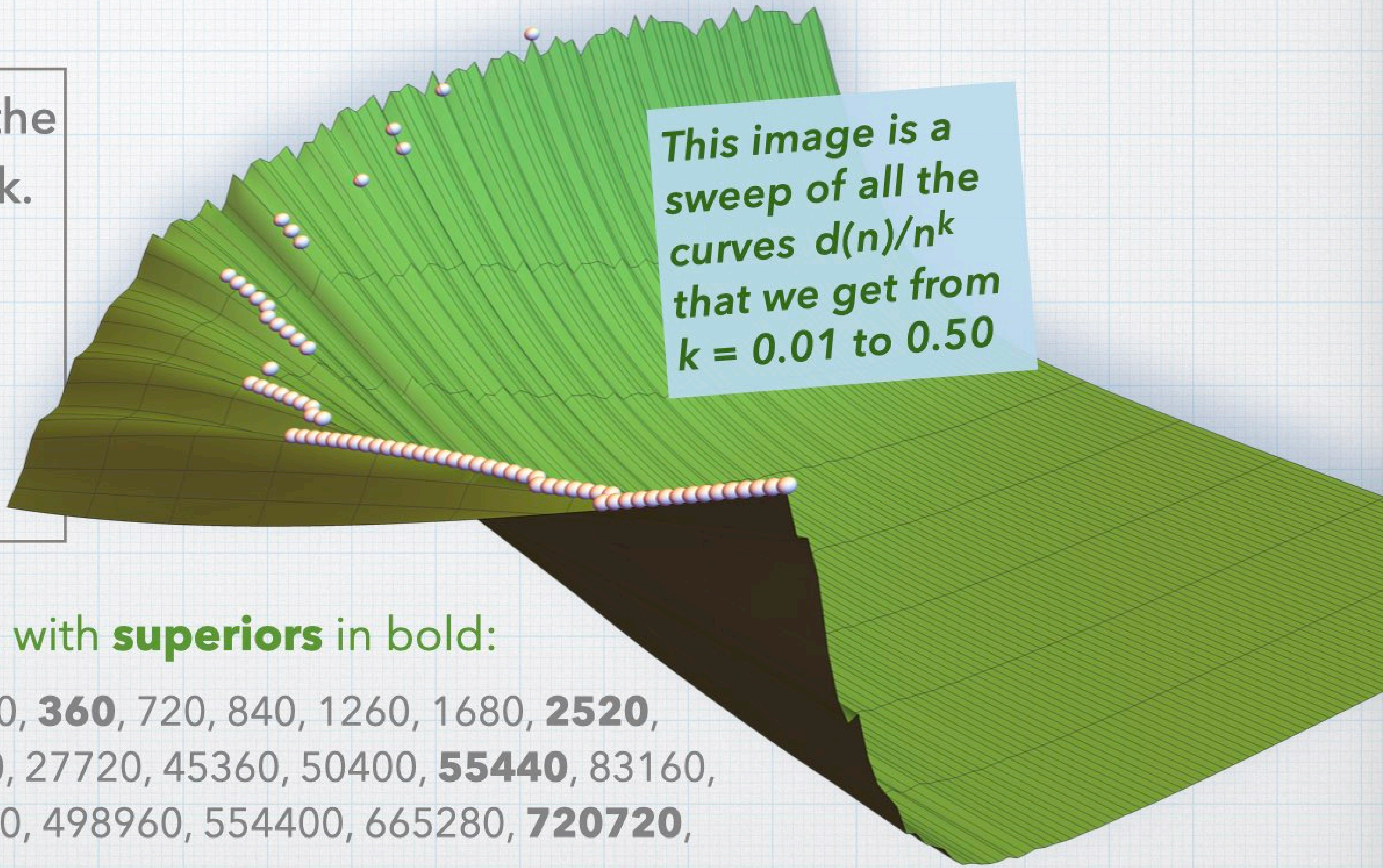

Supercomposites under a million, with **superiors** in bold:

1, **2**, 4, **6**, **12**, 24, 36, 48, **60**, **120**, 180, 240, **360**, 720, 840, 1260, 1680, **2520**, **5040**, 7560, 10080, 15120, 20160, 25200, 27720, 45360, 50400, **55440**, 83160, 110880, 166320, 221760, 277200, 332640, 498960, 554400, 665280, **720720**,

That's **10 superiors** out of the first 38 supers.

What about complex numbers?

Yes, Gaussian Integers do factor uniquely into their own little primes.

$5 = (2 + i)(2 - i)$

$14 - 5i = (3 - 2i)(4 + i)$

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Each Gaussian integer has its list of divisors:

Divisors of 10 :

$\{1,\ 1 + i,\ 1 + 2i,\ 1 + 3i,\ 2,\ 2 + i,$
$2 + 4i,\ 3 + i,\ 4 + 2i,\ 5,\ 5 + 5i,\ 10\}$

$d(10) = 4$

$dg(10) = 12$

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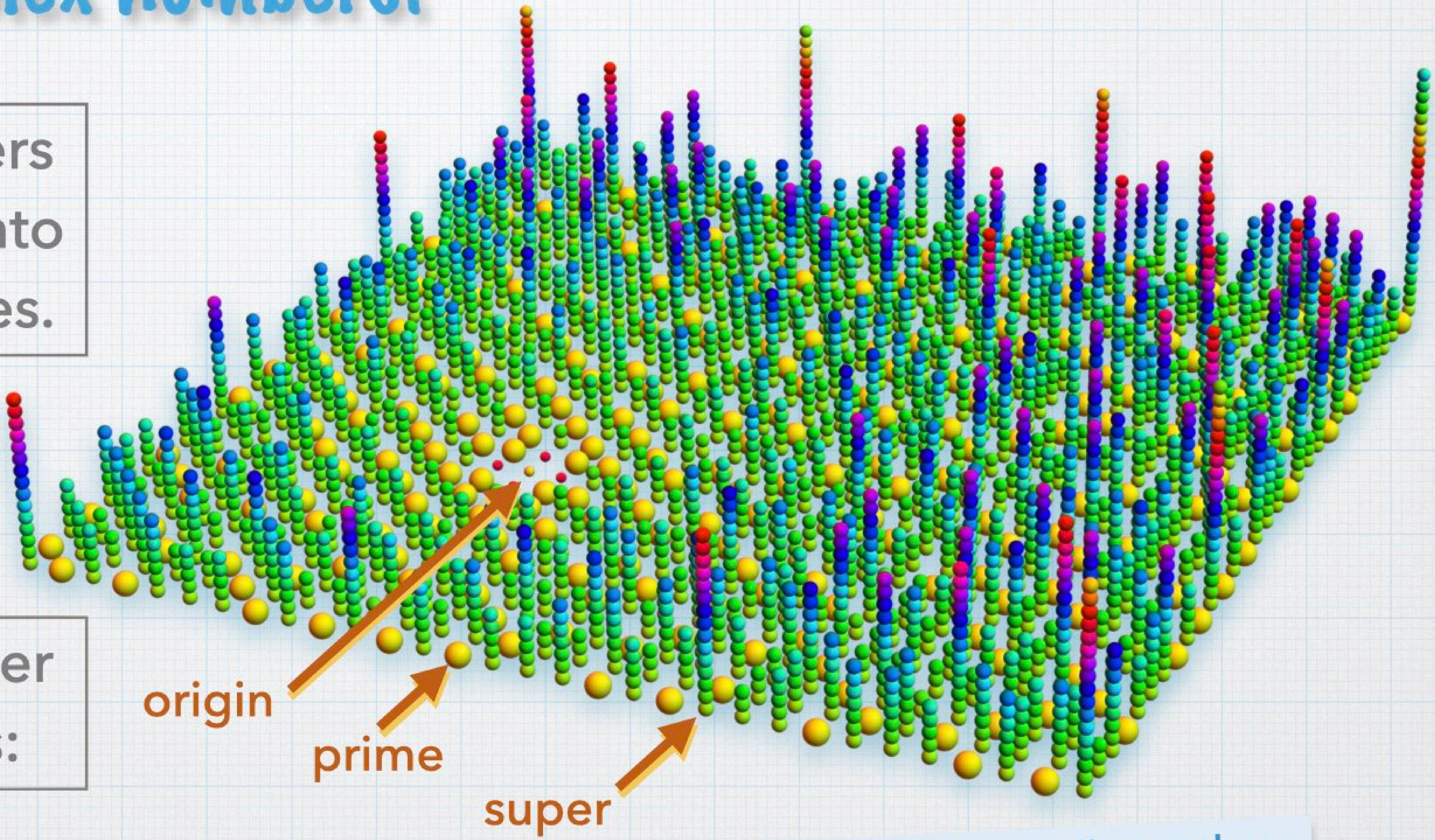

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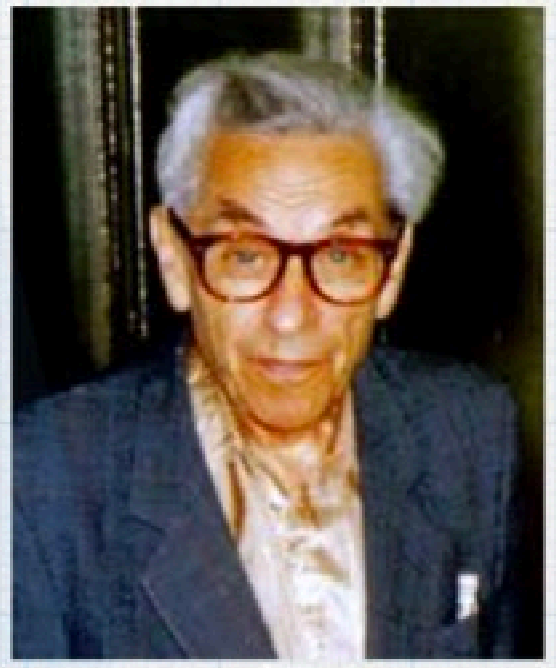
$dg(10) = 12$

The yellow balls are the Gaussian primes; the tallest stacks are supercomposite!

A Super Quartet...



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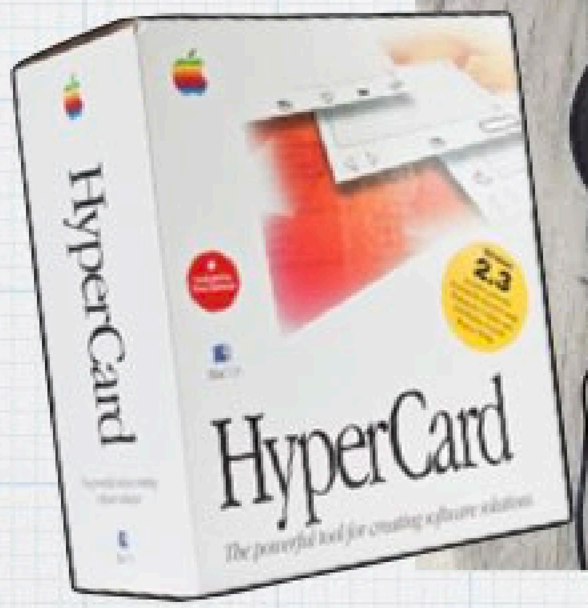
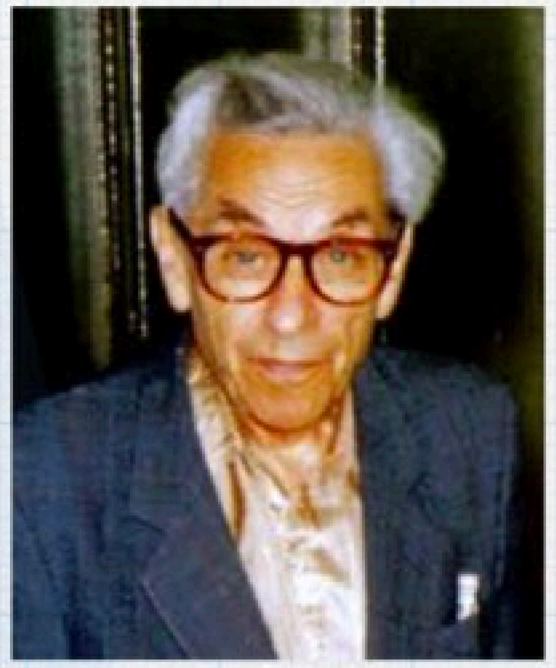


Somewhere I have a HyperCard stack of applying various 'r/s' rules to check for supercompositivity

James Grime and Brady Haran went over the basics of "Anti-Primes" *(2016)*

Srini Ramanujan also invented the super-composite numbers, and over 70 years before I did



The great Paul Erdös proved that

$$\lim_{n \to \infty} \frac{s_{n+1}}{s_n} = 1$$

On Highly Composite Numbers *(1944)*

MathArt at MathFest 2025 Art Catalog!

This catalog available for sale online at:

lulu.com - *bookstore* - *search "dan bach"*

MAA - MathArt at MathFest - 2025

Thank you !

MathArt at MathFest
(from Aug 7 2025)

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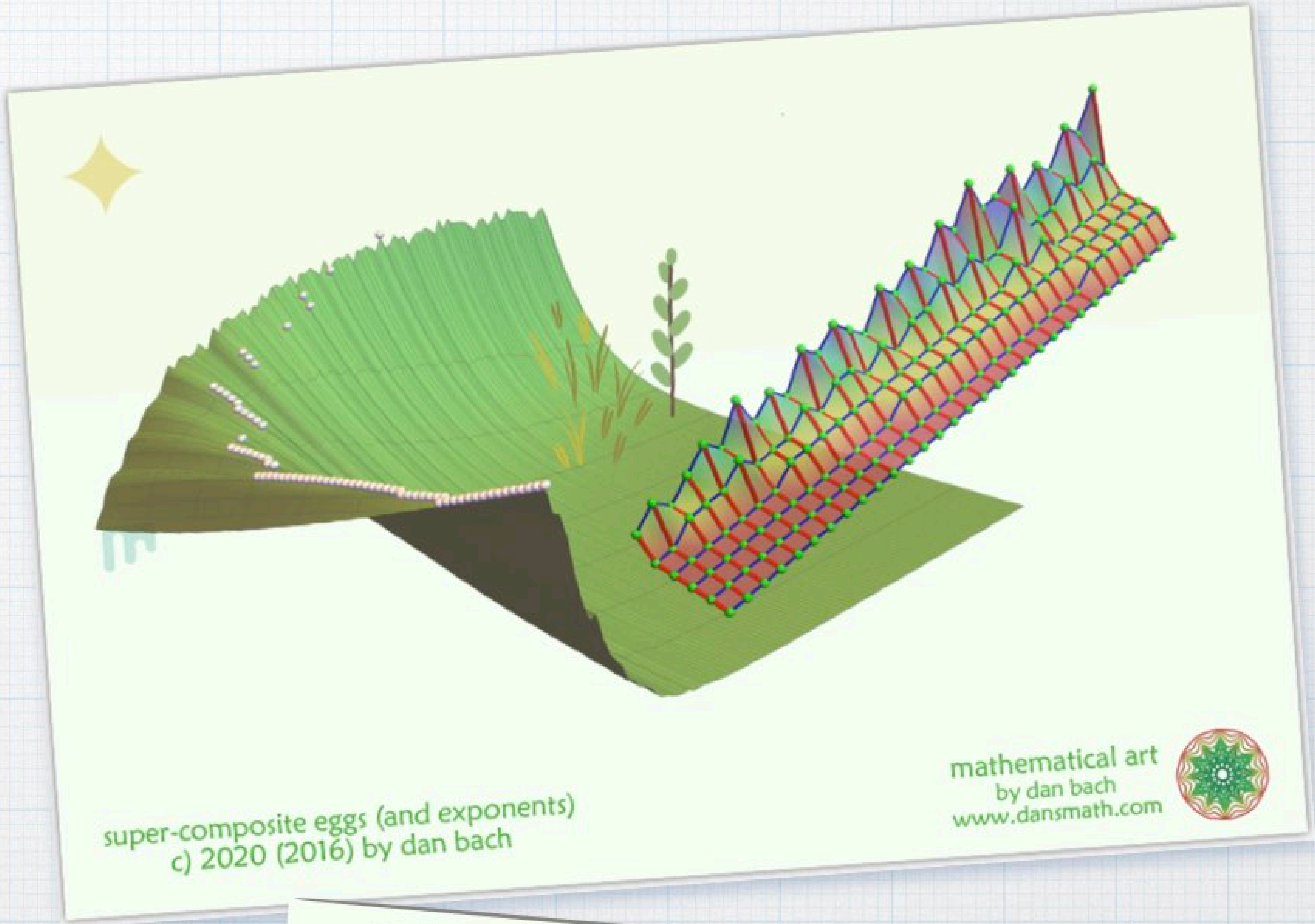

Did everybody get a
dansmathart postcard?