The Game of Tones

Jeff Suzuki

Department of Mathematics
Brooklyn College

Gamification

Mathematics is a game:

We agree to a set of rules.

We decide on a "winning condition."

Then play the game abiding by those rules.

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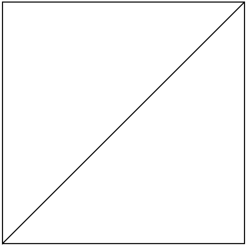

Truths We Tell Our Students That They Don't Care About

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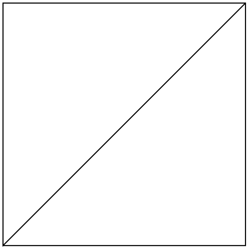

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Anyone who plays piano or guitar knows this isn't true.

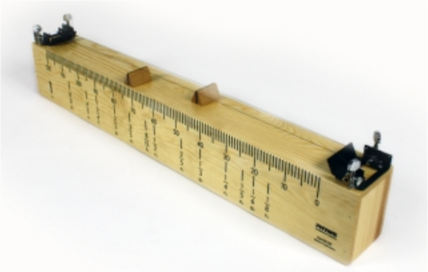
Frequencies

The most likely story about Pythagoras is that he invented the monochord.

This would allow for density, thickness, and tension to be fixed.

Leaving length as the only variable.

We'll use an online frequency generator that allows you to play multiple simultaneous frequencies.



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The Composer's goal is to find the highest frequency that CAN'T be distinguished from 240 by the Listener.

Interlude

Two tones will be played.

Can you tell the difference?

We Got the Beat

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So what happens if the audibly identical 240 and 241 are played simultaneously?

If the difference in frequencies is small, the two tones played together produce an unpleasant “wa-wa.”

Don't Stand So Close To Me

Now consider a set of frequencies ("notes") with a constant difference:

240   280   320   360   400

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Is there a pattern?

Consonances and Dissonances

Students pick pairs of frequencies (notes) and play them simultaneously.

(Implementation note: The higher frequency needs to be played at a lower volume, otherwise it will overwhelm it.)

Then they decided whether the resulting sound combination is pleasant (a consonance) or unpleasant (a dissonance).

Interlude

A pair of frequencies is played, first separately then simultaneously.

Decide whether the simultaneous notes form a consonance or a dissonance.

The Clash

Tastes vary, but:

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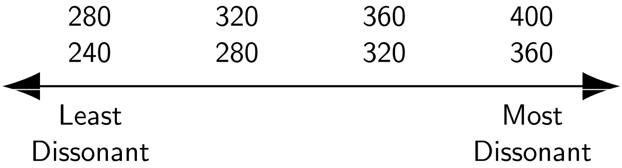
And in general, the dissonances were perceived as worse as the frequencies increased:



A Rationale for Ratios

We have two ways of comparing two quantities:

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So let's look at the ratios.

Consonance . . .

Almost all students regarded the $360, 240$ pair as consonant, and

$$\frac{360}{240} = \frac{3}{2}$$

Most also regarded the $320, 240$ pair as consonant,

$$\frac{320}{240} = \frac{4}{3}$$

. . . and Dissonance

Meanwhile dissonance seemed associated with ratios of larger numbers, like

$$\frac{400}{360} = \frac{10}{9}$$

and the "wa-wa"

$$\frac{241}{240} = \frac{241}{240}$$

This gives Pythagoras's observation:

Consonance corresponds to ratios of small numbers.

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But try to sing it!

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(We'll get to why this is important)

The Game of Tones

In the Game of Tones, we want to find a set of frequencies that produce as many pairwise consonances as possible:

- Choose a starting frequency: this will be the lowest note of our scale.
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The goal is to create a scale with the fewest notes but the most consonances.

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(Remember the game nature: it doesn't matter how a pair "sounds," what matters is the frequency ratio)

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Similarly, 288 gives us a $4:3$ and a $3:2$ consonance, so the four four notes

$$216 \qquad 288 \qquad 324 \qquad 432$$

give us four consonances.

Tone Deaf

If our scale consists of

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then the ratio between consecutive notes is

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Today this is called a tone.

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So it's impossible to solve!

Also Impossible

Similarly, if we try to interpolate frequencies between 216 and 324, we have

$$\left(\frac{9}{8}\right)^n = \frac{3}{2}$$

Consequently

$$2 \cdot 9^n = 3 \cdot 8^n$$

But this reduces to

$$3 \cdot 9^{n-1} = 4 \cdot 8^{n-1}$$

and again we have a product of odds equal to a product of events.

The First (?) Impossibility Theorem

Western music uses octave equivalence.

This leads to closing the circle of fifths:

If we ascend by 3 : 2 consonances, will we eventually reach our "starting" note (when notes in a 2 : 1 consonance are regarded as the same).

Mathematically:

$$\left(\frac{3}{2}\right)^n = 2^m$$

But this means

$$3^n = 2^{n+m}$$

which is impossible.

Historical Postlude

The early history of irrational numbers is uncertain.

The geometric origin is plausible, but it presupposes geometrical knowledge that can't be traced earlier than the 5th century BC (100 years after Pythagoras).

A minority view (Borzacchini [2007]) suggests a musical origin.

At the very least:

- The impossibility inherent in the tuning problem was known to the early Pythagoreans,
- The proof of the impossibility comes from purely arithmetic considerations.