The Global Math Circle

The Sum of Two Squares as a Math Circle Activity

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Special Session: Math Circle Activities as a Gateway to Research
Joint Mathematics Meetings San Francisco
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Bob and Ellen's vision

Engage all kids in math circles in order to share the creative, collaborative nature of mathematics, our lost native language.

Which numbers are the sum of two squares?

$$1^2 + 1^2 = 2$$

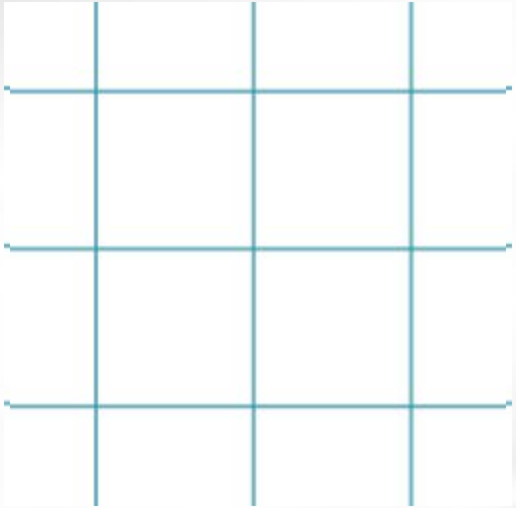
$$1^2 + 2^2 = 5$$

$$2^2 + 2^2 = 8$$

$$2^2 + 3^2 = 13$$

A different approach to the same question:

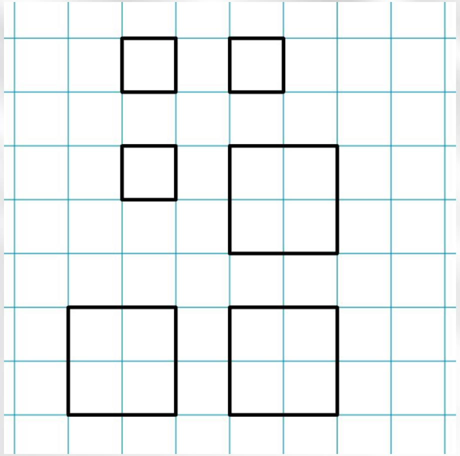
Draw a square on a grid that has an area of exactly two square units.

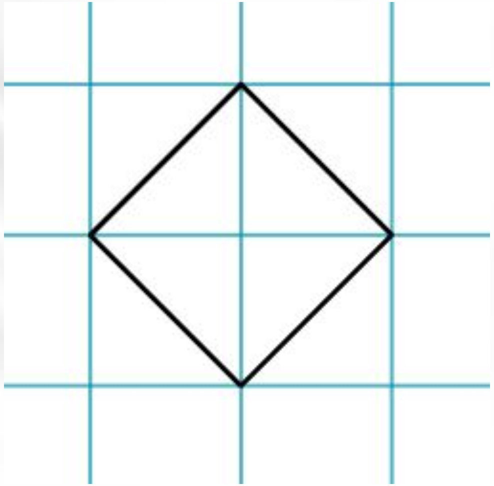


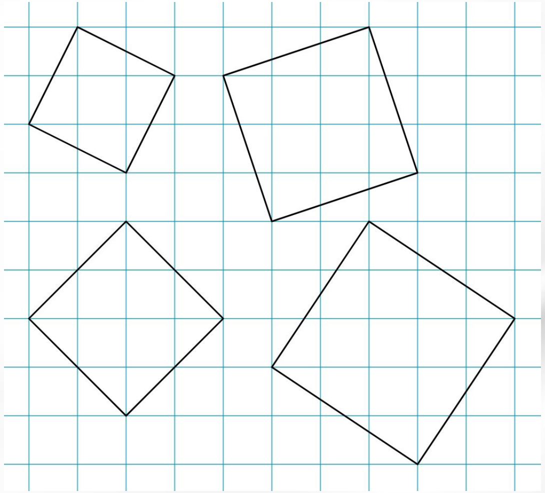
How do these two questions relate?

Which numbers are the sum of two squares?

Draw a square on a grid that has an area of exactly two square units.







Connection to tilted squares

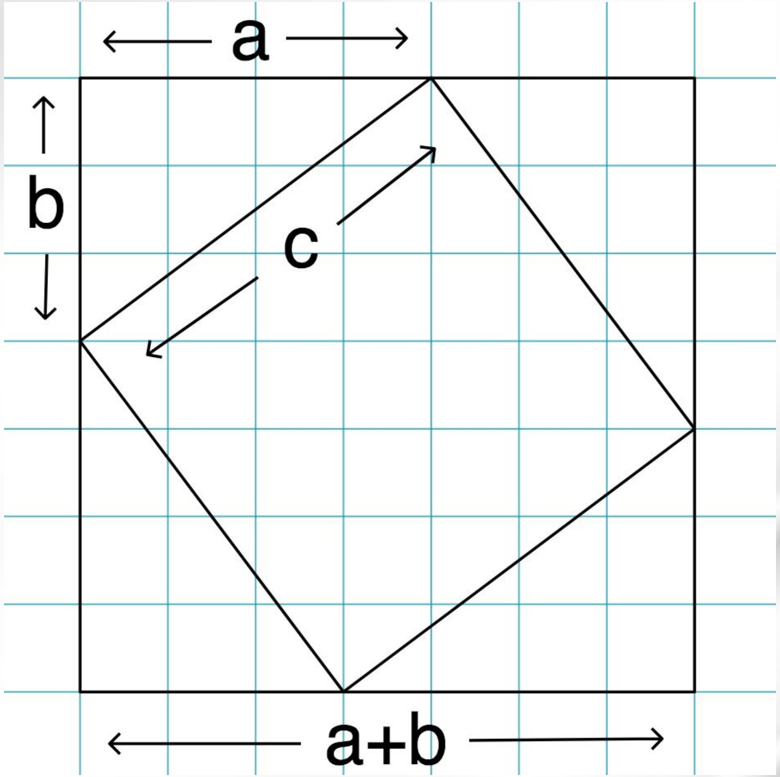
- Triangles: $\underline{4 \times (\frac{1}{2}ab)} = \boxed{2ab}$
- Outside square:

$$\underline{(a+b)^2} = a^2 + \boxed{2ab} + b^2$$

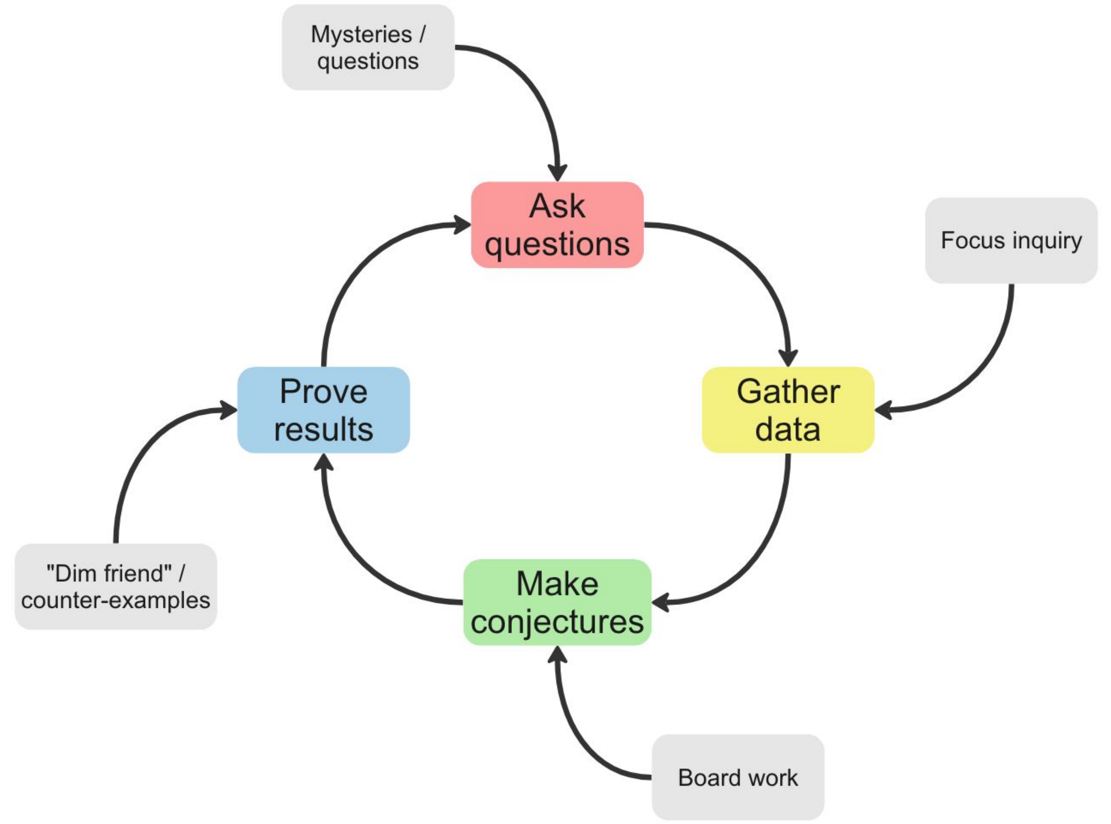
- Inside square:

$$\underline{(a+b)^2} - \underline{4 \times (\frac{1}{2}ab)} = a^2 + b^2$$

- Inside square: $a^2 + b^2 = c^2$



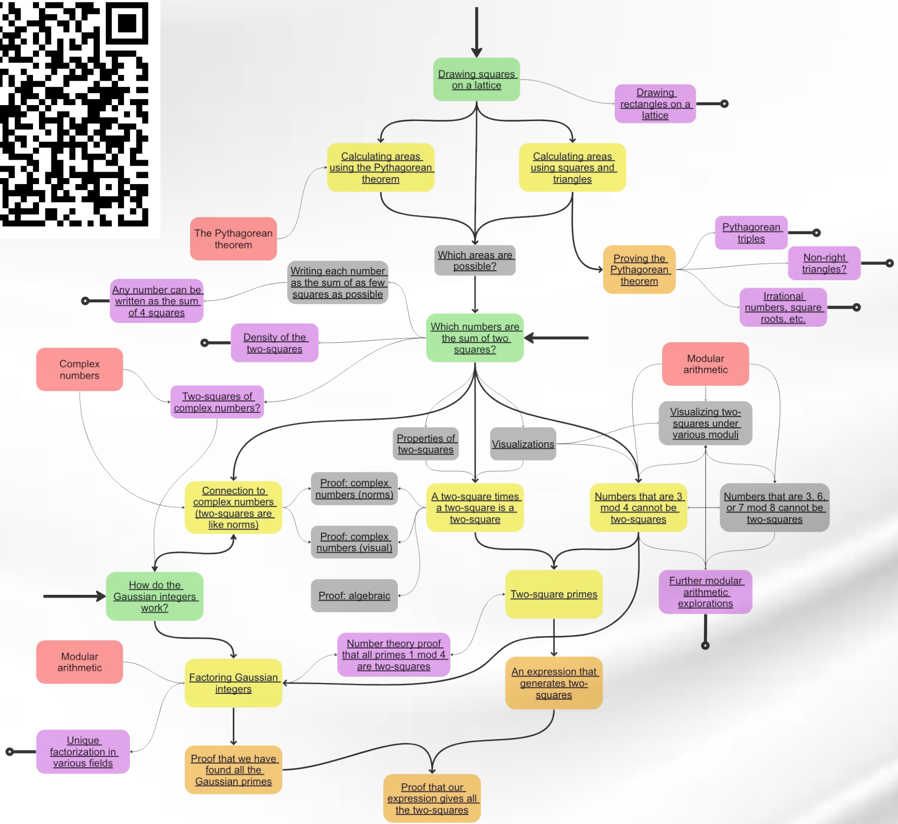
Mathing



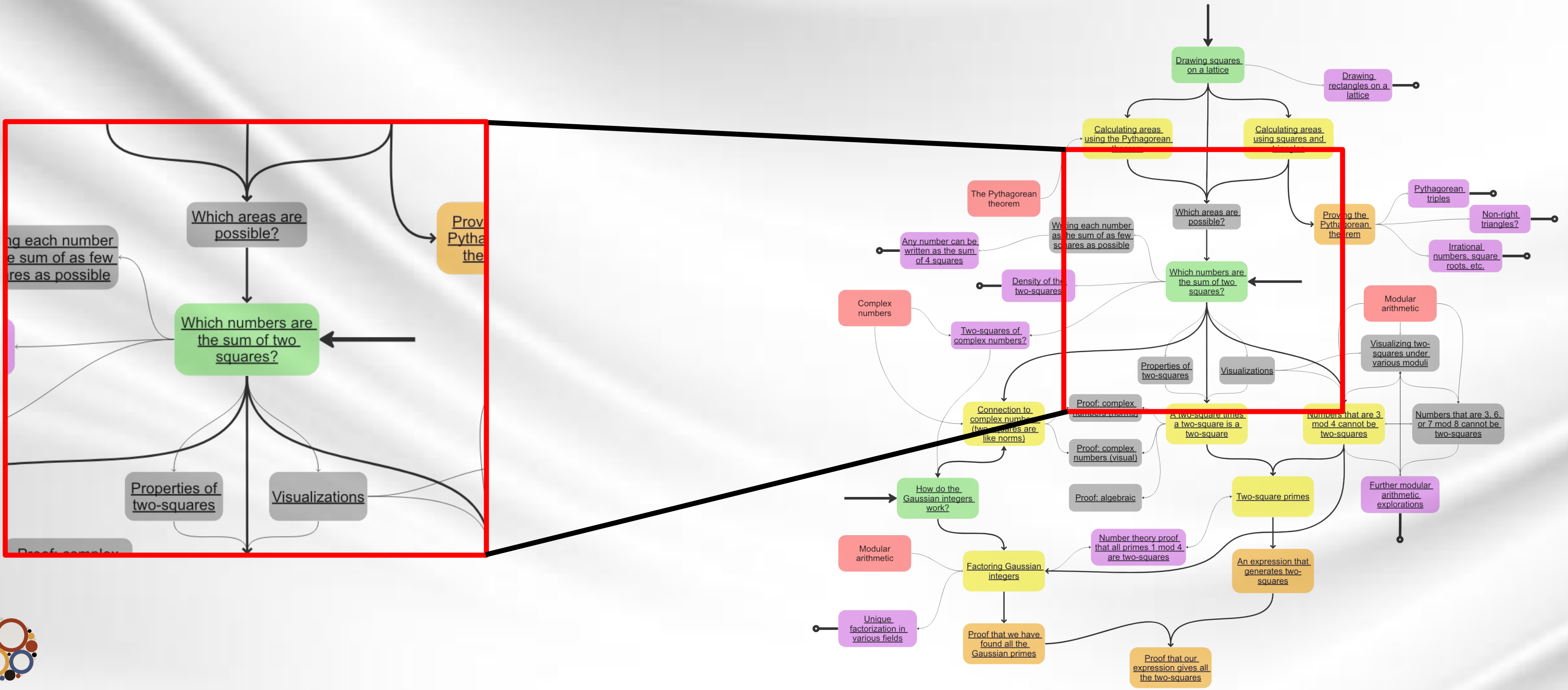
Mysteries / questions
Ask questions
Focus inquiry
Gather data
Prove results
Make conjectures
"Dim friend" / counter-examples
Board work

The Nexus

- Shows many possible paths of mathematical inquiry
- Each cell links to a description
- Multiple entry points, multiple meaningful results
- Prerequisites and further explorations
- Ever-growing; leaders contribute



Which numbers are the sum of two squares?



Here's a list of some sums of two squares. Let's math!

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

What do you notice?

Other representations

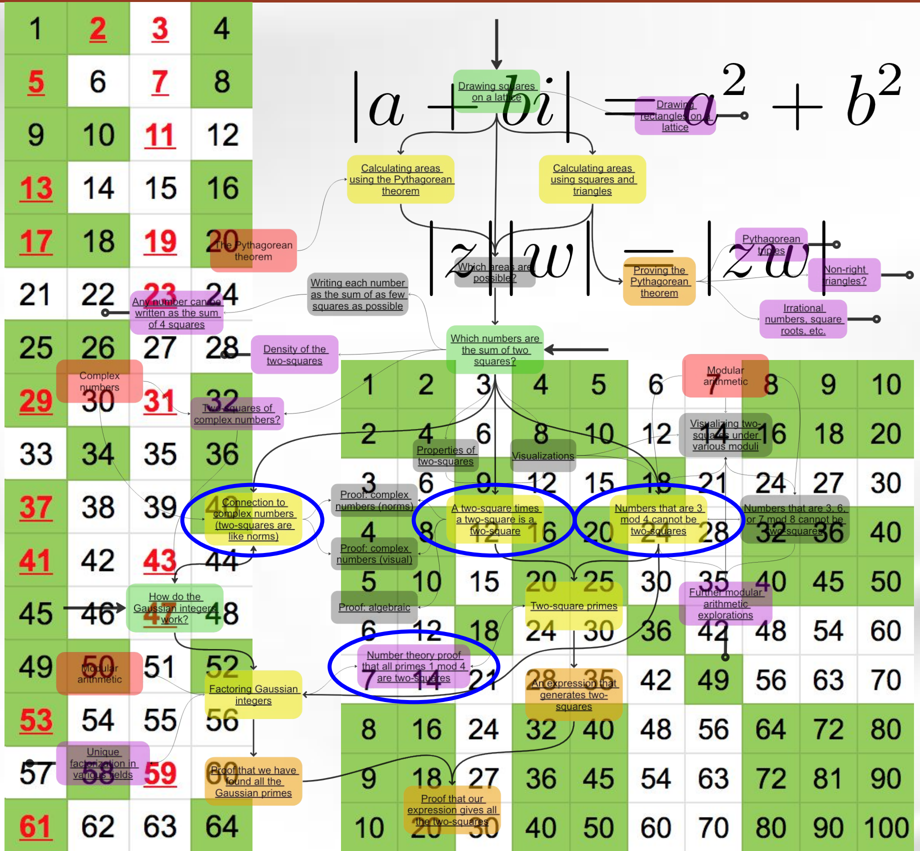
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64
65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80

What do you notice?

Intermediate results

- Numbers 3 mod 4 cannot be two-squares
- Primes 1 mod 4 are two-squares
- Two-squares are the norms of Gaussian integers
- A two-square times a two-square is a two-square

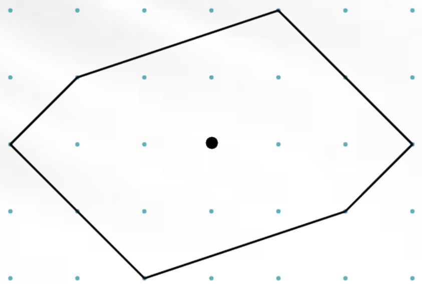


An expression that gives all 2-squares

$$2^a \prod p_i^{b_i} \prod q_j^{2c_j}$$

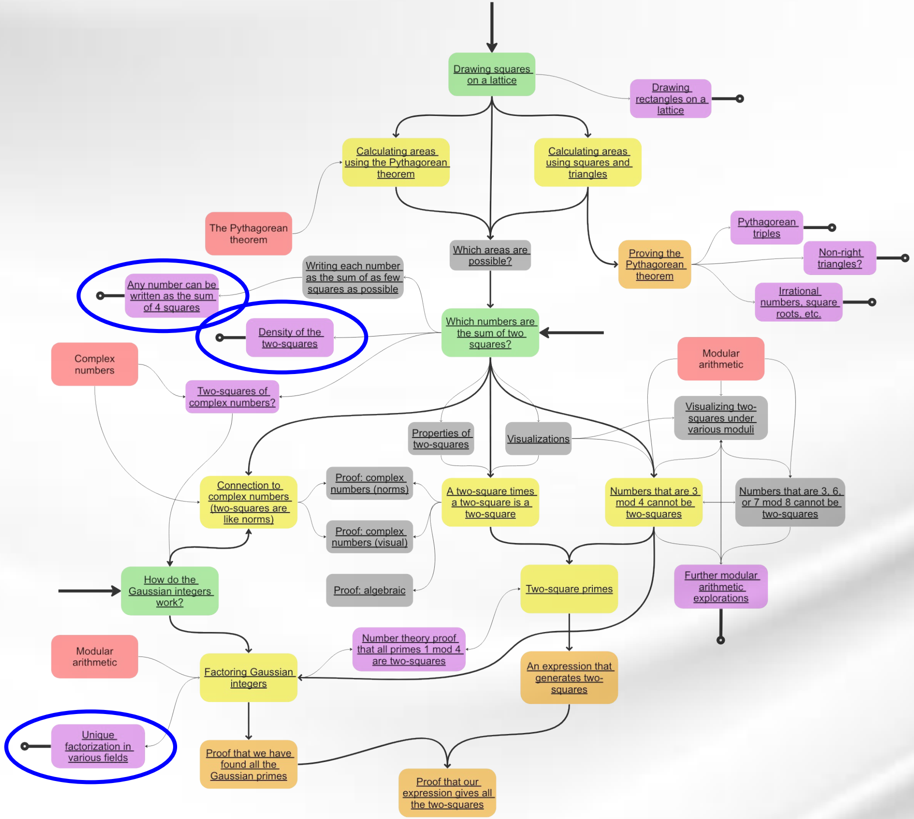
For a, b, c nonnegative integers, p primes 1 mod 4, and q primes 3 mod 4

Can be proved using Gaussian integers, Minkowski's theorem, etc.



Research directions

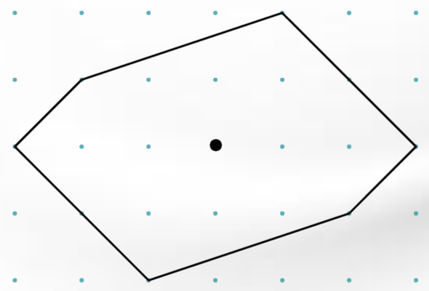
- The Four-Square Theorem
- Unique factorization in different fields
- Density of the two-squares
- Other questions



The Four-Square Theorem

$$1 = 1^2$$
$$2 = 1^2 + 1^2$$
$$3 = 1^2 + 1^2 + 1^2$$
$$4 = 2^2$$
$$5 = 2^2 + 1^2$$
$$6 = 2^2 + 1^2 + 1^2$$
$$7 = 2^2 + 1^2 + 1^2 + 1^2$$
$$8 = 2^2 + 2^2$$
$$9 = 3^2$$
$$10 = 3^2 + 1^2$$
$$11 = 3^2 + 1^2 + 1^2$$
$$12 = 2^2 + 2^2 + 2^2$$
$$13 = 3^2 + 2^2$$

- Multiple possible proofs:
  - Quaternions
  - Number theory
  - Minkowski's theorem



Unique factorization in different fields

- Is there unique factorization in the Gaussian integers?

$$5 = (1+2i)(1-2i) = (2-i)(2+i)$$

  - Unique factorization up to units (1, i, -1, -i)

- What about in other fields? How about numbers of the form $a + b\sqrt{-5}$?

$$6 = 2 \cdot 3 = (1+\sqrt{-5})(1-\sqrt{-5})$$

  - Only have unique factorization with -1, -2, -3, -7, -11, -19, -43, -67, or -163 under the square root. Why?

Density of the 2-squares

- How many two-squares are there under a certain value n?
- Should be roughly proportional to n, but no…

Threshold	Two-squares	Ratio
100	43	0.43
1000	330	0.33
10000	2749	0.2749
100000	24028	0.24028

- Does the ratio converge?

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400	441	484	529	576	625
1	1	2	5	10	17	26	37	50	65	82	101	122	145	170	197	226	257	290	325	362	401	442	485	530	577	626
2	4	5	8	13	20	29	40	53	68	85	104	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	365	404	445	488	533	580	629
3	9	10	13	18	25	34	45	58	73	90	109	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	370	409	450	493	538	585	634
4	16	17	20	25	32	41	52	65	80	97	116	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	377	416	457	500	545	592	641
5	25	26	29	34	41	50	61	74	89	106	125	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	386	425	466	509	554	601	650
6	36	37	40	45	52	61	72	85	100	117	136	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	397	436	477	520	565	612	661
7	49	50	53	58	65	74	85	98	113	130	149	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	410	449	490	533	578	625	674
8	64	65	68	73	80	89	100	113	128	145	164	185	208	233	260	289	320	353	388	425	464	505	548	593	640	689
9	81	82	85	90	97	106	117	130	145	162	181	202	225	250	277	306	337	370	405	442	481	522	565	610	657	706
10	100	101	104	109	116	125	136	149	164	181	200	221	244	269	296	325	356	389	424	461	500	541	584	629	676	725
11	121	122	125	130	137	146	157	170	185	202	221	242	265	290	317	346	377	410	445	482	521	562	605	650	697	746
12	144	145	148	153	160	169	180	193	208	225	244	265	288	313	340	369	400	433	468	505	544	585	628	673	720	769
13	169	170	173	178	185	194	205	218	233	250	269	290	313	338	365	394	425	458	493	530	569	610	653	698	745	794
14	196	197	200	205	212	221	232	245	260	277	296	317	340	365	392	421	452	485	520	557	596	637	680	725	772	821
15	225	226	229	234	241	250	261	274	289	306	325	346	369	394	421	450	481	514	549	586	625	666	709	754	801	850
16	256	257	260	265	272	281	292	305	320	337	356	377	400	425	452	481	512	545	580	617	656	697	740	785	832	881
17	289	290	293	298	305	314	325	338	353	370	389	410	433	458	485	514	545	578	613	650	689	730	773	818	865	914
18	324	325	328	333	340	349	360	373	388	405	424	445	468	493	520	549	580	613	648	685	724	765	808	853	900	949
19	361	362	365	370	377	386	397	410	425	442	461	482	505	530	557	586	617	650	685	722	761	802	845	890	937	986
20	400	401	404	409	416	425	436	449	464	481	500	521	544	569	596	625	656	689	724	761	800	841	884	929	976	1025
21	441	442	445	450	457	466	477	490	505	522	541	562	585	610	637	666	697	730	765	802	841	882	925	970	1017	1066
22	484	485	488	493	500	509	520	533	548	565	584	605	628	653	680	709	740	773	808	845	884	925	968	1013	1060	1109
23	529	530	533	538	545	554	565	578	593	610	629	650	673	698	725	754	785	818	853	890	929	970	1013	1058	1105	1154
24	576	577	580	585	592	601	612	625	640	657	676	697	720	745	772	801	832	865	900	937	976	1017	1060	1105	1152	1201
25	625	626	629	634	641	650	661	674	689	706	725	746	769	794	821	850	881	914	949	986	1025	1066	1109	1154	1201	1250

$$\frac{\pi}{8}\ \frac{\pi}{8} n^{-2}$$

What questions about 2-squares pique your interest?

$1 = 1^2$
$2 = 1^2 + 1^2$
$3 = 1^2 + 1^2 + 1^2$
$4 = 2^2$
$5 = 2^2 + 1^2$
$6 = 2^2 + 1^2 + 1^2$
$7 = 2^2 + 1^2 + 1^2 + 1^2$
$8 = 2^2 + 2^2$
$9 = 3^2$
$10 = 3^2 + 1^2$
$11 = 3^2 + 1^2 + 1^2$
$12 = 2^2 + 2^2 + 2^2$
$13 = 3^2 + 2^2$

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25
0	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400	441	484	529	576	625
1	1	2	5	10	17	26	37	50	65	82	101	122	145	170	197	226	257	290	325	362	401	442	485	530	577	626
2	4	5	8	13	20	29	40	53	68	85	104	125	148	173	200	229	260	293	328	365	404	445	488	533	580	629
3	9	10	13	18	25	34	45	58	73	90	109	130	153	178	205	234	265	298	333	370	409	450	493	538	585	634
4	16	17	20	25	32	41	52	65	80	97	116	137	160	185	212	241	272	305	340	377	416	457	500	545	592	641
5	25	26	29	34	41	50	61	74	89	106	125	146	169	194	221	250	281	314	349	386	425	466	509	554	601	650
6	36	37	40	45	52	61	72	85	100	117	136	157	180	205	232	261	292	325	360	397	436	477	520	565	612	661
7	49	50	53	58	65	74	85	98	113	130	149	170	193	218	245	274	305	338	373	410	449	490	533	578	625	674
8	64	65	68	73	80	89	100	113	128	145	164	185	208	233	260	289	320	353	388	425	464	505	548	593	640	689
9	81	82	85	90	97	106	117	130	145	162	181	202	225	250	277	306	337	370	405	442	481	522	565	610	657	706
10	100	101	104	109	116	125	136	149	164	181	200	221	244	269	296	325	356	389	424	461	500	541	584	629	676	725
11	121	122	125	130	137	146	157	170	185	202	221	242	265	290	317	346	377	410	445	482	521	562	605	650	697	746
12	144	145	148	153	160	169	180	193	208	225	244	265	288	313	340	369	400	433	468	505	544	585	628	673	720	769
13	169	170	173	178	185	194	205	218	233	250	269	290	313	338	365	394	425	458	493	530	569	610	653	698	745	794
14	196	197	200	205	212	221	232	245	260	277	296	317	340	365	392	421	452	485	520	557	596	637	680	725	772	821
15	225	226	229	234	241	250	261	274	289	306	325	346	369	394	421	450	481	514	549	586	625	666	709	754	801	850
16	256	257	260	265	272	281	292	305	320	337	356	377	400	425	452	481	512	545	580	617	656	697	740	785	832	881
			293	298	305	314	325	338	353	370	389	410	433	458	485	514	545	578	613	650						
			328	333	340	349	360	373	388	405	424	445	468	493	520	549	580	613	648	685						
			365	370	377	386	397	410	425	442	461	482	505	530	557	586	617	650	685	722						
			404	409	416	425	436	449	464	481	500	521	544	569	596	625	656	689	724	761						
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			488	493	500	509	520	533	548	565	584	605	628	653	680	709	740	773	808	845						
			533	538	545	554	565	578	593	610	629	650	673	698	725	754	785	818	853	890						
			580	585	592	601	612	625	640	657	676	697	720	745	772	801	832	865	900	937						
			629	634	641	650	661	674	689	706	725	746	769	794	821	850	881	914	949	986						

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
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5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

1	2	3	4	5	6	7	8
9	10	11	12	13	14	15	16
17	18	19	20	21	22	23	24
25	26	27	28	29	30	31	32
33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48
49	50	51	52	53	54	55	56
57	58	59	60	61	62	63	64
65	66	67	68	69	70	71	72
73	74	75	76	77	78	79	80

The fruits of mathing

11	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
0	0	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400
1	1	2	5	10	17	26	37	50	65	82	101	122	145	170	197	226	257	290	325	362	401
2	4	5	8	13	20	29	40	53	68	85	104	125	148	173	200	229	260	293	328	365	404
3	9	10	13	18	25	34	45	58	73	90	109	130	153	178	205	234	265	298	333	370	409
4	16	17	20	25	32	41	52	65	80	97	116	137	160	185	212	241	272	305	340	377	416
5	25	26	29	34	41	50	61	74	89	106	125	146	169	194	221	250	281	314	349	386	425
6	36	37	40	45	52	61	72	85	100	117	136	157	180	205	232	261	292	325	360	397	436
7	49	50	53	58	65	74	85	98	113	130	149	170	193	218	245	274	305	338	373	410	449
8	64	65	68	73	80	89	100	113	128	145	164	185	208	233	260	289	320	353	388	425	464
9	81	82	85	90	97	106	117	130	145	162	181	202	225	250	277	306	337	370	405	442	481
10	100	101	104	109	116	125	136	149	164	181	200	221	244	269	296	325	356	389	424	461	500
11	121	122	125	130	137	146	157	170	185	202	221	242	265	290	317	346	377	410	445	482	521
12	144	145	148	153	160	169	180	193	208	225	244	265	288	313	340	369	400	433	468	505	544
13	169	170	173	178	185	194	205	218	233	250	269	290	313	338	365	394	425	458	493	530	569
14	196	197	200	205	212	221	232	245	260	277	296	317	340	365	392	421	452	485	520	557	596
15	225	226	229	234	241	250	261	274	289	306	325	346	369	394	421	450	481	514	549	586	625
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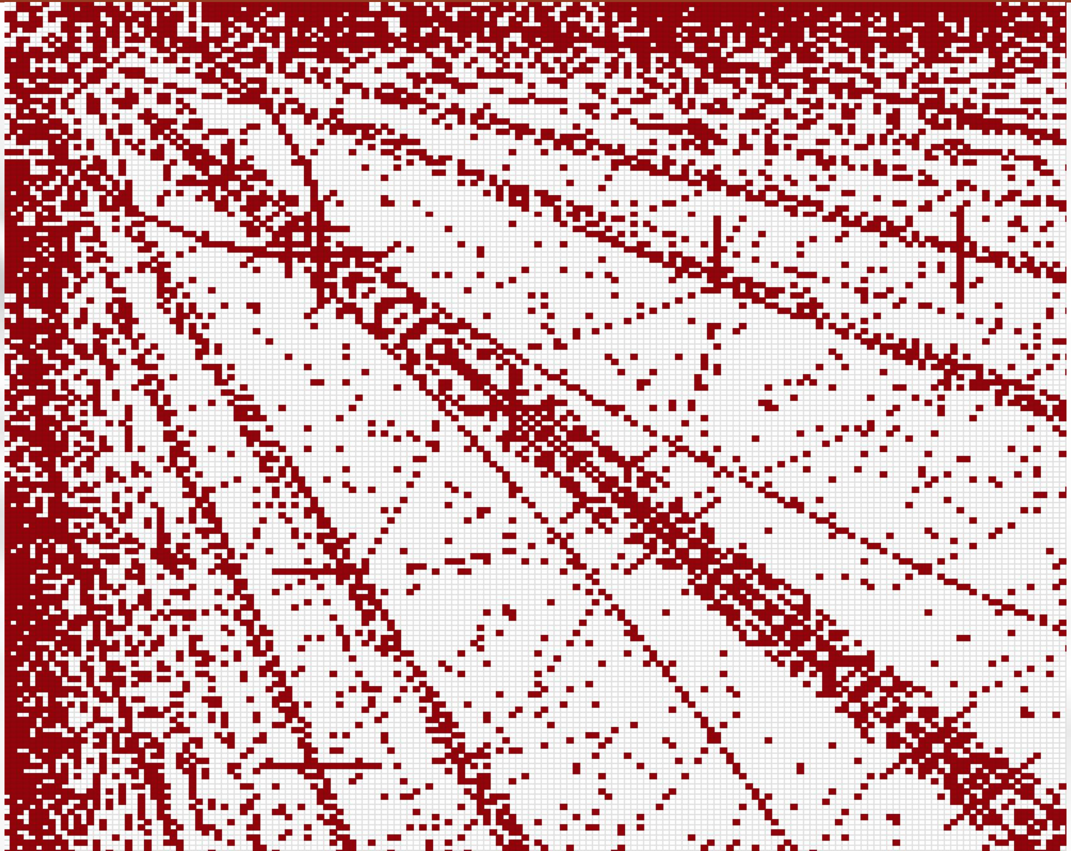
17	289	290	293	298	305	314	325	338	353	370	389	410	433	458	485	514	545	578	613	650	689
18	324	325	328	333	340	349	360	373	388	405	424	445	468	493	520	549	580	613	648	685	724
19	361	362	365	370	377	386	397	410	425	442	461	482	505	530	557	586	617	650	685	722	761
20	400	401	404	409	416	425	436	449	464	481	500	521	544	569	596	625	656	689	724	761	800
21	441	442	445	450	457	466	477	490	505	522	541	562	585	610	637	666	697	730	765	802	841
22	484	485	488	493	500	509	520	533	548	565	584	605	628	653	680	709	740	773	808	845	884
23	529	530	533	538	545	554	565	578	593	610	629	650	673	698	725	754	785	818	853	890	929
24	576	577	580	585	592	601	612	625	640	657	676	697	720	745	772	801	832	865	900	937	976
25	625	626	629	634	641	650	661	674	689	706	725	746	769	794	821	850	881	914	949	986	1025
26	676	677	680	685	692	701	712	725	740	757	776	797	820	845	872	901	932	965	1000	1037	1076
27	729	730	733	738	745	754	765	778	793	810	829	850	873	898	925	954	985	1018	1053	1090	1129
28	784	785	788	793	800	809	820	833	848	865	884	905	928	953	980	1009	1040	1073	1108	1145	1184

0
1
2
3
4
5
6
7
8
9
10
11
12

The fruits of mathing

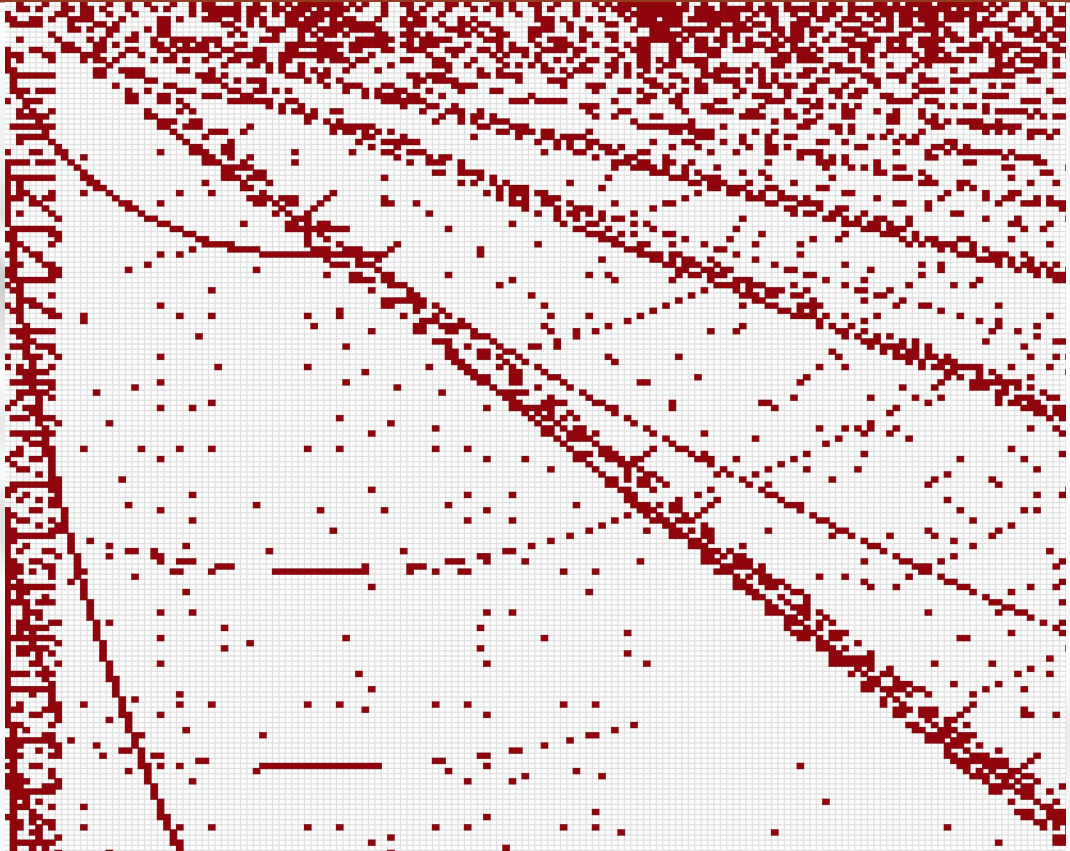
Cells = $row^2 + column^2$



Highlighted if text contains row mod column or column mod row

The fruits of mathing

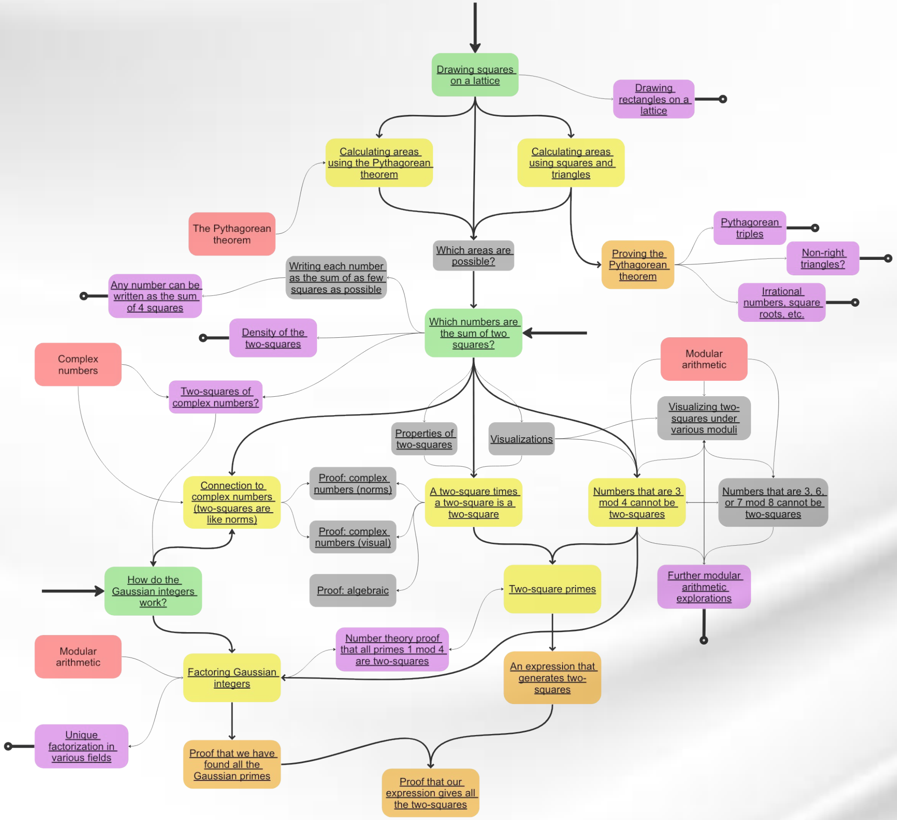
Cells = row^2 + column^2



Highlighted if text contains col mod row

Future mathing with the Nexus

- Prepare instructors to follow participants' interests
- Discover effective methods
- An ever-growing map of math



Thank you!

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Nexus repository QR code

Nexus flow chart QR code