The Sum of Two Squares as a Math Circle Activity

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Special Session: Math Circle Activities as a Gateway to Research
Joint Mathematics Meetings San Francisco
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Bob and Ellen's vision

Engage all kids in math circles in order to share the creative, collaborative nature of mathematics, our lost native language.
Which numbers are the sum of two squares?

$1^2 + 1^2 = 2$

$1^2 + 2^2 = 5$

$2^2 + 2^2 = 8$

$2^2 + 3^2 = 13$
A different approach to the same question:

Draw a square on a grid that has an area of exactly two square units.
How do these two questions relate?

Which numbers are the sum of two squares?

Draw a square on a grid that has an area of exactly two square units.
Connection to tilted squares

- **Triangles:** \( 4 \times \left( \frac{1}{2}ab \right) = 2ab \)

- **Outside square:**
  \[
  (a + b)^2 = a^2 + 2ab + b^2
  \]

- **Inside square:**
  \[
  (a + b)^2 - 4 \times \left( \frac{1}{2}ab \right) = a^2 + b^2
  \]

- **Inside square:**
  \[
  a^2 + b^2 = c^2
  \]
Mathing

1. Ask questions
2. Gather data
3. Make conjectures
4. Prove results
5. "Dim friend" / counter-examples
6. Focus inquiry

Mysteries / questions
Board work
The Nexus

- Shows many possible paths of mathematical inquiry
- Each cell links to a description
- Multiple entry points, multiple meaningful results
- Prerequisites and further explorations
- Ever-growing; leaders contribute
Which numbers are the sum of two squares?
Here's a list of some sums of two squares. Let's math!

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What do you notice?
Other representations

What do you notice?
Intermediate results

- Numbers 3 mod 4 cannot be two-squares
- Primes 1 mod 4 are two-squares
- Two-squares are the norms of Gaussian integers
- A two-square times a two-square is a two-square
An expression that gives all 2-squares

$$2^a \prod p_i^{b_i} \prod q_j^{2c_j}$$

For $a, b, c$ nonnegative integers, $p$ primes 1 mod 4, and $q$ primes 3 mod 4

Can be proved using Gaussian integers, Minkowski’s theorem, etc.
Research directions

- The Four-Square Theorem
- Unique factorization in different fields
- Density of the two-squares
- Other questions
The Four-Square Theorem

1 = 1^2
2 = 1^2 + 1^2
3 = 1^2 + 1^2 + 1^2
4 = 2^2
5 = 2^2 + 1^2
6 = 2^2 + 1^2 + 1^2
7 = 2^2 + 1^2 + 1^2 + 1^2
8 = 2^2 + 2^2
9 = 3^2
10 = 3^2 + 1^2
11 = 3^2 + 1^2 + 1^2
12 = 2^2 + 2^2 + 2^2
13 = 3^2 + 2^2

● Multiple possible proofs:
  ○ Quaternions
  ○ Number theory
  ○ Minkowski’s theorem
Unique factorization in different fields

- Is there unique factorization in the Gaussian integers?
  \[ 5 = (1 + 2i)(1 - 2i) = (2 - i)(2 + i) \]
  - Unique factorization up to units (1, i, -1, -i)

- What about in other fields? How about numbers of the form \( a + b\sqrt{-5} \)?
  \[ 6 = 2 \cdot 3 = (1 + \sqrt{-5})(1 - \sqrt{-5}) \]
  - Only have unique factorization with -1, -2, -3, -7, -11, -19, -43, -67, or -163 under the square root. Why?
Density of the 2-squares

- How many two-squares are there under a certain value n?
- Should be roughly proportional to n, but no…
- Does the ratio converge?

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What questions about 2-squares pique your interest?

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The fruits of mathing

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The table above shows a grid of numbers, possibly representing a multiplication table or some other mathematical relationship. Each cell represents the product of the row index and the column index.
The fruits of mathing

Cells = row^2 + column^2

Highlighted if text contains row mod column or column mod row
The fruits of mathing

Cells = row² + column²

Highlighted if text contains col mod row
Future mathing with the Nexus

- Prepare instructors to follow participants’ interests
- Discover effective methods
- An ever-growing map of math
Thank you!

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Taylor Yeracaris
taylor@theglobalmathcircle.org

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Nic Ford
Erika Go
Bob & Ellen Kaplan
Avital Oliver

Nexus repository QR code

Nexus flow chart QR code