Taxman Game's Optimal Second Move

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The taxman game was developed by Diane Resek around 1970 and published as a computer game by the Minnesota Educational Consortium. It is a single-player game that starts with all the integers from 1 to some integer N. The player can take a number m from the list only if there exists proper divisors of m on the list. When the player takes a number m, the taxman collects all the remaining divisors of m, the "tax." Thus, the player is not allowed to take any number that results in no tax. When no legal moves remain, the taxman collects the rest of numbers on the list. Whoever has the largest sum of numbers wins.

As an example of how to play the game, suppose $N = 6$. A greedy strategy takes the largest number, 6, from the list 1, 2, 3, 4, 5, 6. The taxman then gets 1, 2, and 3 as proper divisors. The taxman then gets 4 and 5 since these have no proper divisors left on the list, i.e., no more legal moves are possible. This means that the taxman wins 15 to 6.

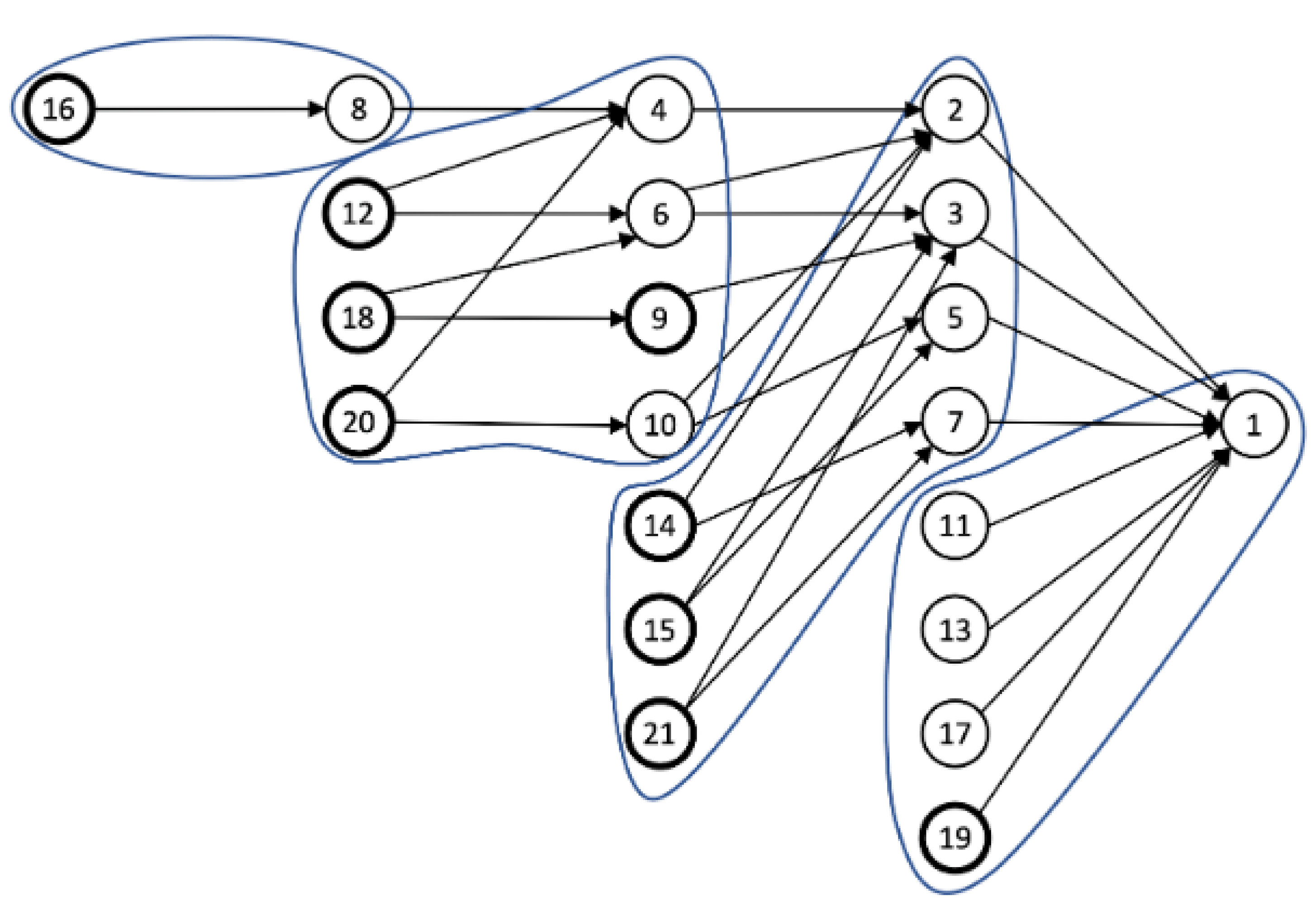
An Optimal Strategy

An *optimal strategy* for the taxman game is a strategy that always yields the largest possible score. The player's optimal first move is to take the largest prime on the list because it limits the taxman's take to one since prime numbers are only divisible by one and themselves. In [1], Moniot shows the largest square of a prime is the optimal second pick for all $N \leq 49$ other than $N = 8$ and $N = 20$.

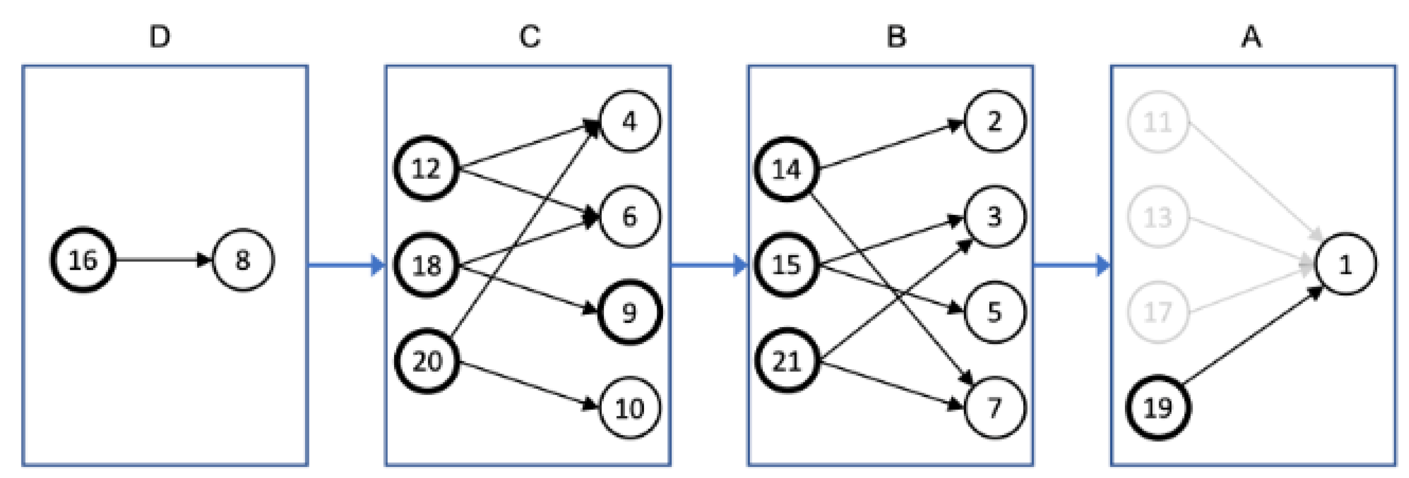
In [3], Chess presents an algorithm and uses it to find the optimal sequence of moves for $N = 1$ to $N = 1000$, which shows that the largest square of a prime is the optimal second move for all $N \leq 1000$ other than 8, 20, and 120. We illustrate Chess's algorithm for $N = 21$.

Step 1: Create a graph where each number connects to its maximal factors. A number f is a maximal factor of n if n/f is prime. If n itself is prime, then 1 is its maximal factor.

Step 2: Associate numbers with no composites ("source nodes") with their maximal factors. These are called *frames*.



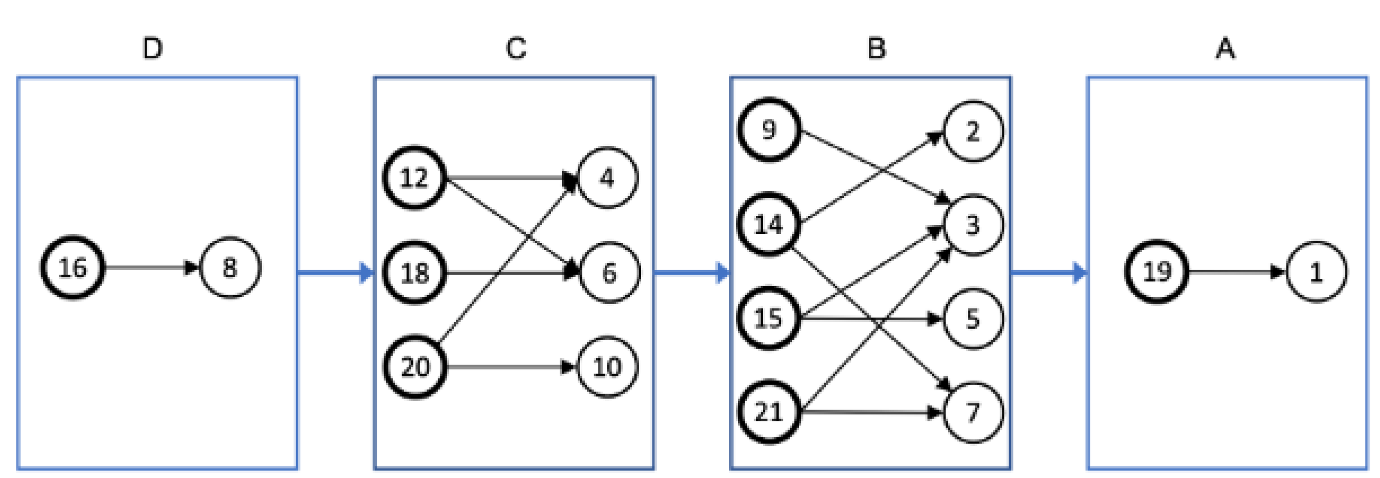
Step 3: Each frame from A to D is a mini-game. Eliminate source nodes that should not be selected within each frame. In Frame A we take the largest prime, 19.



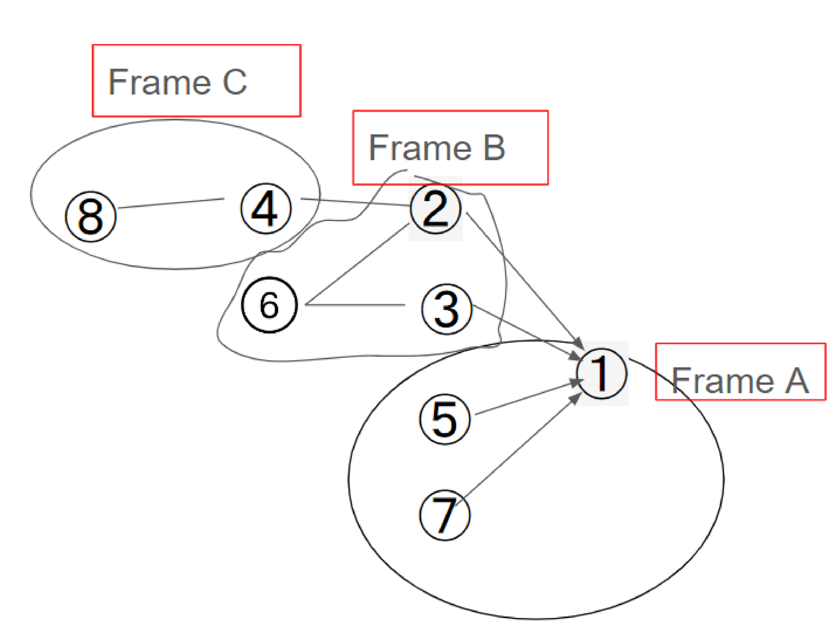
Step 4: Ignore Frames A and D because they have exactly as many factors as they need for all the source nodes to be selected.

Step 5: A "donating" frame must be able to use all its remaining source nodes without needing the "promoted" source. Meanwhile, the "receiving" frame must be able to use all its existing sources and also include the newly promoted source. In this example, Frame C can give up any of its factors, but 9 is optimal because B can not accept 10 and still select all its source nodes. After promoting 9 from Frame C to Frame B, all the frames close.

Step 6: We create a solution by solving each frame from front (A) to back (D). In this example we get the optimal sequence of pick of 19, 9, 21, 15, 14, 18, 12, 20, and 16, which yields the optimal score of 144 to the taxman's 87.



If $N = 8$, Chess's algorithm shows that the largest square of the prime, 4, is a receiver in Frame C that is not promoted. Instead 6 is the only source node that can be selected in Frame B and thus is the optimal second move despite not being the largest square of a prime.



If $N = 20$ the same process applies where 9, the largest square of a prime, is a receiver while 10 is the optimal second move since it is a source node that gets promoted to Frame B.

Likewise, if $N = 120$, 49 is a receiver while 25 is the optimal second move since it is a source node that gets promoted to Frame B.

The Optimal Second Move

We now show that the largest square of a prime is the optimal second move for all $N \geq 121$ by proving the following theorem.

Theorem If $N \geq 121$, then the largest square of a prime $\leq N$ is in the upper half of the list and thus cannot be a receiver.

Proof. If $N \geq 121$ and p^2 is the largest square of a prime $\leq N$, then $p \geq 11$. If $11 \leq p \leq 25$ and q is the next largest prime, then we can verify directly that $p^2 < q^2 < 2p^2$, so if $p^2 \leq N/2$, then $p^2 < q^2 < 2p^2 \leq N$, meaning that the largest square of a prime cannot be in the lower half of the list.

By [2], if $p \geq 25$, there is a prime q such that $p < q < p\sqrt{2}$, so if $p^2 \leq N/2$, then $p^2 < q^2 < 2p^2 \leq N$, meaning that the largest square of a prime cannot be in the lower half of the list.

Thus, the largest square of a prime $\leq N$ must be in the upper half of the list and thus cannot be a receiver. $\square$

Future Research

We will analyze Chess's algorithm to determine an optimal strategy. In Frame X, the receivers are products n of $X - 1$ primes such that $2n \leq N$ and the sources are products m of X primes such that $2m > N$. Each receiver n has as many sources as there are primes p such that $N < 2pn \leq 2N$ and contributes as many receivers as there are primes p such that $2pn \leq N$ to the next frame. For a receiver to be promoted, both the donating and receiving frames must have have more receivers than sources; determining which receiver is optimal to promote is our next step.

Acknowledgments

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References

[1] Robert K. Moniot, The taxman game. *Math Horizons*, 14(3):18–20, 2007.

[2] Jitsuro Nagura, On the interval containing at least one prime number. *Proceedings of the Japan Academy*, 28(4), 177-181, 1952.

[3] Brian Chess, `https://github.com/bvchess/taxman/wiki`