Crossing the Desert with Mathematics

MAA MathFest

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The Problem

What is “The Jeep Problem”

Problem

A jeep has to cross a desert but does not have enough fuel to complete the distance in one trip. To accomplish the trip, fuel caches must be set up at particular points along the way to allow for refueling during the trip.

Inspiration

The American Mathematical Monthly ›
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Original Articles

A Jeep Crossing a Desert of Unknown Width

Richard E. Korf

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Abstract

The classic jeep problem concerns crossing a desert wider than the range of the jeep, with the aid of preplaced fuel caches. There has been a lot of work on this problem and its variations, and the optimal strategy is well known, but all previous work assumes that we know the width of the desert. We consider the case where we don't know the distance in advance. We evaluate a strategy by its competitive ratio, which is the worst-case ratio of the cost of the strategy, divided by the cost of an optimal solution had we known the distance in advance. We show that no strategy with a fixed sequence of caches can achieve a finite competitive ratio. The optimal strategy is an iterative one that uses the optimal known-distance strategy to reach a sequence of target distances, emptying all caches between iterations. An optimal iterative strategy doubles the cost of each successive iteration, and achieves a competitive ratio of four.

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   - Problem level is right, but math is obtuse.
3. More searching led to Michiel de Bondt's paper.
   - Simpler approach reveals the kernel of the problem.

If $x \leq 1$, the problem is trivial. If x exceeds 1, however, gas dumps will have to be established at various points along the way. It will be convenient to take the path of the jeep along the positive x-axis, starting at x and ending at the origin. The gas dumps will then form a subdivision σ of the interval $(0, x)$:

$$\sigma: \quad 0 < x_1 < x_2 < \cdots < x_r < x,$$

in which the x_i denote the positions of the dumps (assumed to be finite in number). If z is any non-negative number less than x, the subdivision σ induces a subdivision of $(0, z)$ by deletion of all the stations to the right of z. There will be no ambiguity if we refer to this induced subdivision by the same symbol, σ. Other subdivisions will be denoted by σ', σ^*, and so forth. If all the stations (points of division) of σ are contained among the stations of σ', we shall say that σ' is a *refinement* of σ, written $\sigma' < \sigma$.

We may now rephrase our problem. Once a subdivision is fixed, the amount of gas required is still a function of the method of establishing and employing its stations. We shall denote by $f(x, \sigma)$ the greatest lower bound of this amount for all possible methods, and by $f(x)$ the greatest lower bound of $f(x, \sigma)$ for all possible subdivisions σ. Our task is to discover the form of $f(x)$.

In §2 we introduce the *standard* method of establishing and using the stations of a given subdivision σ, and we prove that this method is at least as economical as any other. This enables us to determine $f(x, \sigma)$ in §3. A rather surprising application of the standard method leads to the result (§4) that if $\sigma' < \sigma$, then $f(x, \sigma') \leq f(x, \sigma)$. In §5 we determine criteria for non-improvement

5. Properties of σ^*. Our problem will be solved if we can find a subdivision σ^* for which

$$\text{(A)} \qquad f(x, \sigma^*) \leqq f(x, \sigma) \quad \textit{for every } \sigma.$$

We can bring to bear the results of §4 by proving that any σ^* which satisfies (A) also satisfies (B) that follows, and conversely.

$$\text{(B)} \qquad f(x, \sigma^*) = f(x, \sigma') \quad \textit{for every} \quad \sigma' < \sigma^*.$$

Clearly, (A) and (9) imply (B). Conversely, suppose that (B) is satisfied, and let σ be any subdivision whatsoever. We choose for σ' the common refinement of σ and σ^*. From (B), $f(x, \sigma^*) = f(x, \sigma')$; another application of (9) shows that $f(x, \sigma') \leqq f(x, \sigma)$. Combining these we obtain (A).

Using the criterion established at the end of §4, we find that (B) is equivalent to

(C) *For every* $t = 1, 2, \cdots, r+1$, *and for every* y *such that* $x_{t-1} < y \leqq x_t$,

$$k' \equiv \{f(y, \sigma^*)\} = k_t \equiv \{f(x_t, \sigma^*)\}.$$

We shall now show that (C) is equivalent to (D):

(D) *For every* $m = 1, 2, \cdots, [f(x, \sigma^*)]$, *there exists an integer* s *such that* $f(x_s, \sigma^*) = m$.

Let's give it a go

Digging into "The Jeep Problem"

Question

A jeep has to cross a desert but does not have enough fuel to complete the distance in one trip. To accomplish the trip, fuel caches must be set up at particular points along the way to allow for refueling during the trip. Unlimited fuel is available at the start of the desert.

What questions does the problem induce?

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Question

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What questions does the problem induce?

1. *Where along the route should the fuel caches be placed?*
2. *How much fuel in total will be required? (Including the fuel used in placing the fuel caches.)*
3. *What is the longest desert that can be crossed in this way?*
4. *What is the relationship between fuel used and distance traveled?*

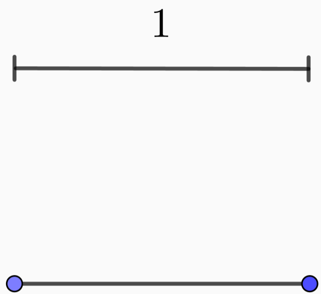
Early Question Prompts

1. Assume for simplicity the jeep can hold 1 unit of fuel which can travel 1 unit of desert. (Or 1 gallon and 1 mile if you wish.) There are two big questions we aim to answer:
   (a) How far can you travel on N gallons of fuel, assuming the jeep can only carry 1 at a time, but can cache fuel along the way?
   (b) How many gallons does it take to cross a desert of N miles (and how do you do it)?
2. Choose a problem and start thinking about it. What ideas do you have? What problem solving strategies might you use? If you need some prompting/guidance, let me know.

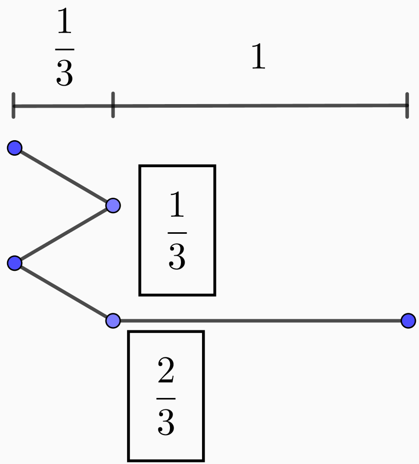
Key Leading Question Prompts

1. How far can you get on 1 gallon? On 2 gallons? On 3 gallons?
2. How far can you travel if you need to use only 1 gallon of fuel to deliver N gallons to the next fuel cache?

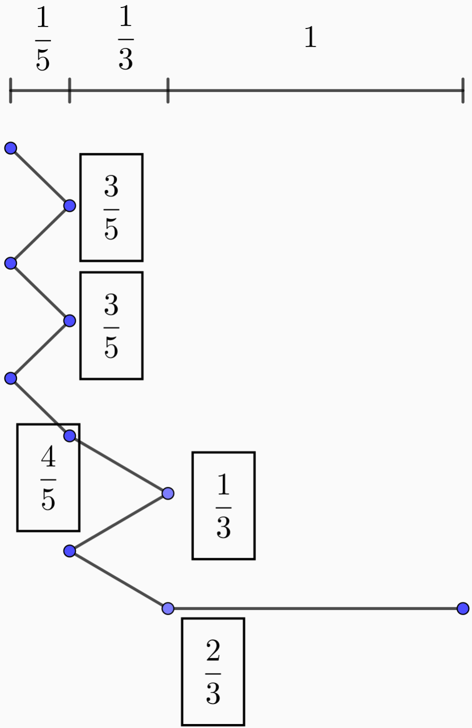
Solution Idea: 1 Gallon



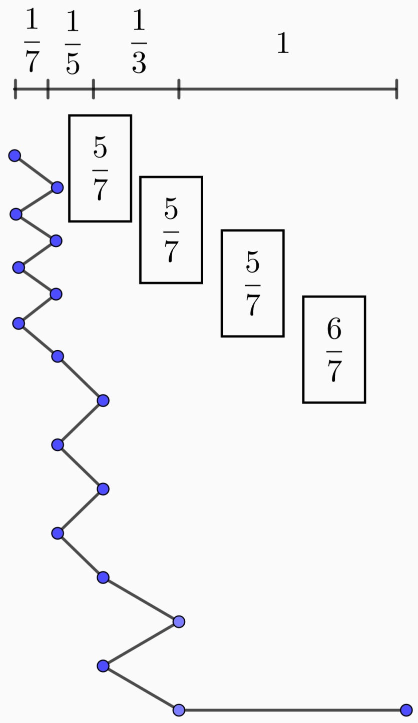
Solution Idea: 2 Gallons



Solution Idea: 3 Gallons



Solution Idea: 4 Gallons



Solution Summary

Solution

- *For any odd integer* n*, making* n *legs of length* $\frac{1}{n}$ *will result in* $\frac{n-1}{2}$ *units of fuel to be deposited a distance of* $\frac{1}{n}$ *ahead.*
- *Or another way, given* f *units of fuel (an integer), the distance one can travel is* $1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{2f-1}$.

Solution Summary

Solution

- *For any odd integer n, making n legs of length $\frac{1}{n}$ will result in $\frac{n-1}{2}$ units of fuel to be deposited a distance of $\frac{1}{n}$ ahead.*
- *Or another way, given f units of fuel (an integer), the distance one can travel is $1 + \frac{1}{3} + \frac{1}{5} + ... + \frac{1}{2f-1}$.*
- *Choosing $n = 1, 3, 5, ...$ will result in traveling $1 + \frac{1}{3} + \frac{1}{5} + ...$ units away.*

Question

What do you notice about $1 + \frac{1}{3} + \frac{1}{5} + ...$?

Further Investigations...

Question

1. *What if the fuel* f *is not an integer? How far can you travel on* $\frac{3}{2}$ *units of fuel?*
2. *What is the function that gives distance* (y) *as a function of fuel* (x)*?*

Question

1. *What if the fuel f is not an integer? How far can you travel on $\frac{3}{2}$ units of fuel?*
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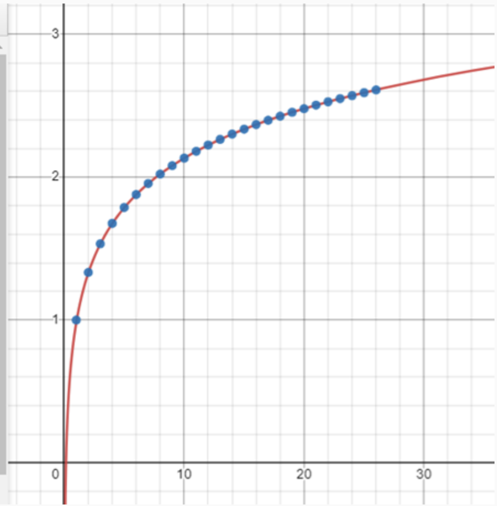
$$f(x) = \left(\frac{1}{2}\right)(\ln x + .577215665 + 2\ln 2)$$

x_1	y_1
1	1
2	1.333333333
3	1.533333333
4	1.676190476
5	1.787301587
6	1.878210678
7	1.955133755
8	2.021800422
9	2.080623951
10	2.13325553



Thanks For Your Attention!

Thanks

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Any Questions?

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