Ulam Sequences

Chaos and Order and Connections Between the Two

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Some History

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- Although the definition is fairly simple, this sequence is chaotic and difficult to predict. Many basic questions are completely open.
- Ever since then, people have been looking at various generalizations of this sequence, making conjectures, and occasionally proving results.
- Many of the known partial results are due to undergraduate students. Let me show you some.

Notation

- We'll write $\mathrm{U}(a, b)$ to mean the sequence of integers starting with a, b (where $0 < a < b$), such that every subsequent term is the next smallest integer that can be written as the sum of two distinct prior terms in exactly one way.

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- So, $\mathrm{U}(1, 2)$ was Ulam's original sequence.

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- Some specific sequences are known to grow linearly.

Periodicity

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Is $\mathrm{U}(a,b)$ *eventually structured? For example, is the sequence of consecutive differences eventually periodic? (That is,* $\mathrm{U}(a,b)$ *is regular.)*

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- Unknown for most a, b. (Conjectured to be true/false for roughly half.)
- If there are finitely many even numbers in $\mathrm{U}(a, b)$, then it is regular (Finch 1992).
- This is known to occur when
  1. $a = 2,\ b \geq 5$, (Schmerl and Spiegel, 1994)
  2. $a = 4,\ b = 1 \mod 4$ (Cassaigne and Finch 1995)
  3. (a, b) in the following table (Joshua Hinman 2019)

$(4, 11)$	$(4, 19)$	$(6, 7)$	$(6, 11)$	$(7, 8)$	$(7, 10)$	$(7, 12)$
$(7, 16)$	$(7, 18)$	$(7, 20)$	$(8, 9)$	$(8, 11)$	$(9, 10)$	$(9, 14)$
$(9, 16)$	$(9, 20)$	$(10, 11)$	$(10, 13)$	$(10, 17)$	$(11, 12)$	$(11, 14)$
$(11, 16)$	$(11, 18)$	$(11, 20)$	$(12, 13)$	$(12, 17)$	$(13, 14)$	

"Congruence" Restrictions

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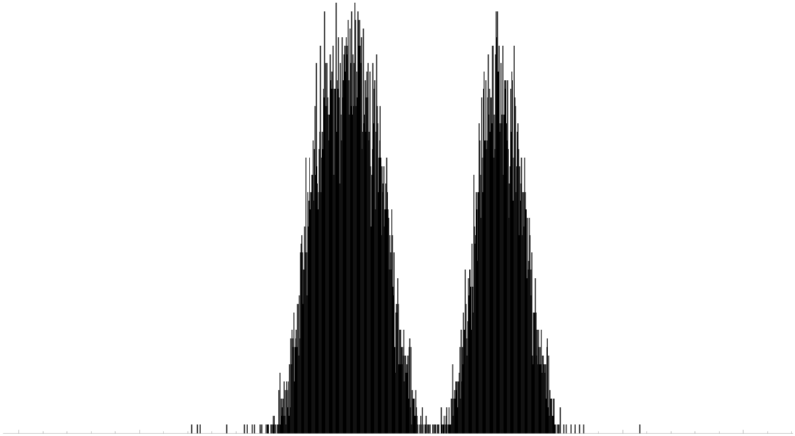
- Conjecturally, if $\mathrm{U}(a, b)$ is not regular, then it equidistributes in all congruence classes.
- *However*, in 2015, Stefan Steinerberger discovered the existence of a "magic number" for $\mathrm{U}(1, 2)$: $\lambda_{1,2} \approx 2.44344$. If you take the first N elements of $\mathrm{U}(1, 2)$ modulo $\lambda_{1,2}$ and take a histogram, something odd happens.

Patterns in the Data



$N = 10^4$

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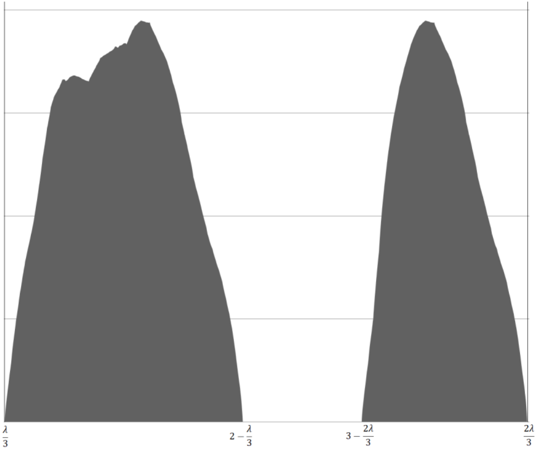
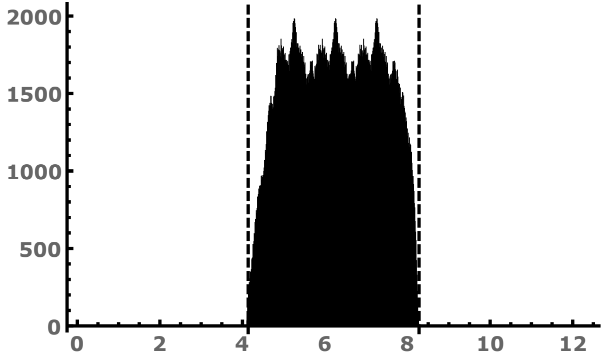


$N \approx 7.4 \cdot 10^{10}$

The Patterns Persist

- Similar things occur for other Ulam sequences, for other "magic numbers."



$U(1,4)$

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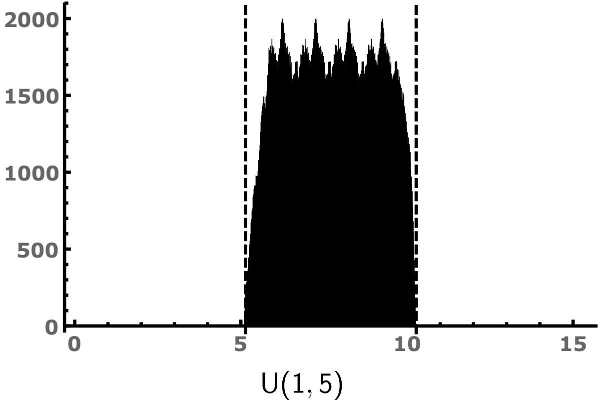

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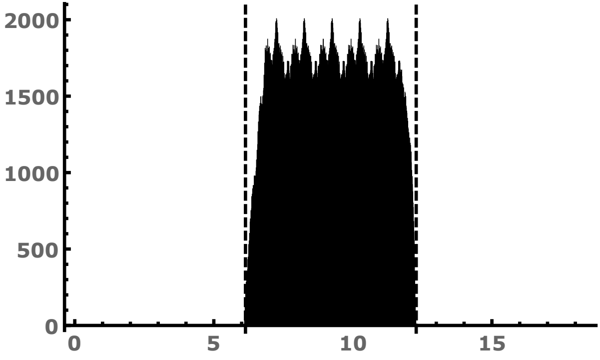

$U(1,6)$

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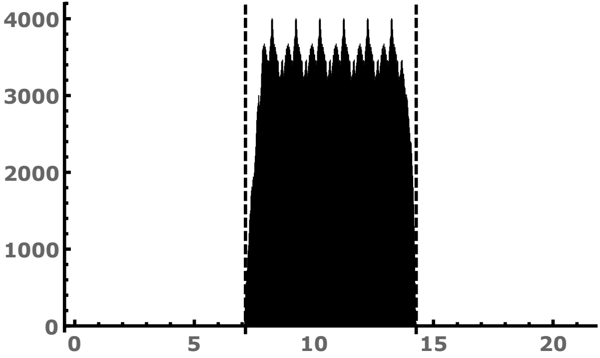

U(1, 7)

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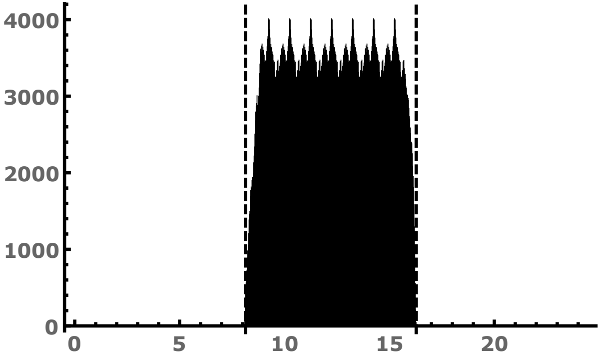

$U(1,8)$

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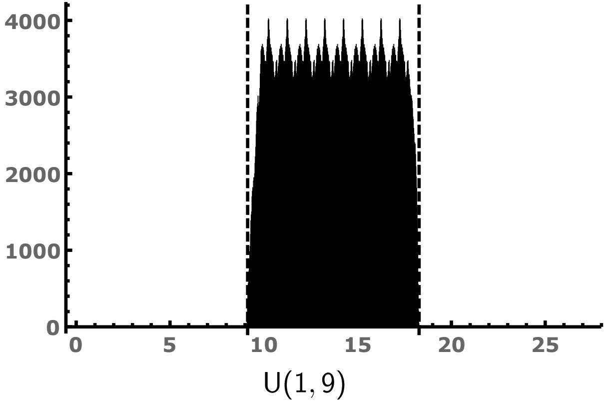

A Hidden Signal

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- We do have a *lot* of numerical evidence, due to work by Judson and Gibbs in 2017.

Conjecture (Judson, Gibbs 2017)

There exists a real number $\lambda_{1,2} \approx 2.44344$ *such that for every* $\epsilon > 0$, *the set*

$$\left\{ u \in \mathrm{U}(1,2) \middle| u \mod \lambda_{1,2} \notin \left(\frac{\lambda_{1,2}}{3} - \epsilon, \frac{2\lambda_{1,2}}{3} + \epsilon \right) \right\}$$

is finite.

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is finite.

- We are reasonably sure similar "magic numbers" $\lambda_{1,n}$ exist for all Ulam sequences $\mathrm{U}(1,n)$, and probably for other families as well.

Patterns within Families

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U(1, 2) :	1,	2,	3,	4,	6,	8,	11,	13,	16,	18,	26,	28,	36,	38,	47 ...
U(1, 3) :	1,	3,	4,	5,	6,	8,	10,	12,	17,	21,	23,	28,	32,	34,	39 ...
U(1, 4) :	1,	4,	5,	6,	7,	8,	10,	16,	18,	19,	21,	31	32,	33,	42 ...
U(1, 5) :	1,	5,	6,	7,	8,	9,	10,	12,	20,	22,	23,	24,	26,	38,	39 ...
U(1, 6) :	1,	6,	7,	8,	9,	10,	11,	12,	14,	24,	26,	27,	28,	29,	31 ...

Patterns within Families

U(1, 2) :	$\boxed{1}$,	$\boxed{2, 3, 4}$,	$\boxed{6}$,	8,	11,		13 . . .
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U(1, 4) :	$\boxed{1}$,	$\boxed{4, 5, 6, 7, 8}$,	$\boxed{10}$,	$\boxed{16}$,	$\boxed{18, 19}$,		$\boxed{21}$ . . .
U(1, 5) :	$\boxed{1}$,	$\boxed{5, 6, 7, 8, 9, 10}$,	$\boxed{12}$,	$\boxed{20}$,	$\boxed{22, 23, 24}$,		$\boxed{26}$ . . .
U(1, 6) :	$\boxed{1}$,	$\boxed{6, 7, 8, 9, 10, 11, 12}$,	$\boxed{14}$,	$\boxed{24}$,	$\boxed{26, 27, 28, 29}$,		$\boxed{31}$ . . .
U(1, 7) :	$\boxed{1}$,	$\boxed{7, 8, 9, 10, 11, 12, 13, 14}$,	$\boxed{16}$,	$\boxed{28}$,	$\boxed{30, 31, 32, 33, 34}$,		$\boxed{36}$ . . .
U(1, 8) :	$\boxed{1}$,	$\boxed{8, 9, 10, 11, 12, 13, 14, 15, 16}$,	$\boxed{18}$,	$\boxed{32}$,	$\boxed{34, 35, 36, 37, 38, 39}$,		$\boxed{41}$ . . .

Patterns within Families

$$
\begin{array}{r|llllll}
U(1,4): & \boxed{1}, & \boxed{4,\ldots 8}, & \boxed{10}, & \boxed{16}, & \boxed{18,19}, & \boxed{21}\ldots \\
U(1,5): & \boxed{1}, & \boxed{5,\ldots 10}, & \boxed{12}, & \boxed{20}, & \boxed{22,\ldots 24}, & \boxed{26}\ldots \\
U(1,6): & \boxed{1}, & \boxed{6,\ldots 12}, & \boxed{14}, & \boxed{24}, & \boxed{26,\ldots 29}, & \boxed{31}\ldots \\
U(1,7): & \boxed{1}, & \boxed{7,\ldots 14}, & \boxed{16}, & \boxed{28}, & \boxed{30,\ldots 34}, & \boxed{36}\ldots \\
U(1,8): & \boxed{1}, & \boxed{8,\ldots 16}, & \boxed{18}, & \boxed{32}, & \boxed{34,\ldots 39}, & \boxed{41}\ldots \\
U(1,n): & \boxed{1}, & \boxed{n,\ldots 2n}, & \boxed{2n+2}, & \boxed{4n}, & \boxed{4n+2,\ldots 5n-1}, & \boxed{5n+1}\ldots
\end{array}
$$

The Rigidity Conjecture

Conjecture (HKSS 2018)

There exists an N (probably 4) and integer coefficients $\{a_i\}_{i=0}^{\infty}, \{b_i\}_{i=0}^{\infty}, \{c_i\}_{i=0}^{\infty}, \{d_i\}_{i=0}^{\infty}$ such that for all $n \geq N$,

$$\mathrm{U}(1, n) = \bigcup_{i=0}^{\infty} \left[a_i n + b_i, c_i n + d_i\right],$$

where $c_i n + d_i + 1 < a_{i+1} n + b_{i+1}$ for all i.

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- This is currently open, but we do have some interesting partial results.

The Rigidity Theorem

Theorem (HKSS 2019)

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$$\mathrm{U}(1, n) \cap [1, c_k n + d_k] = \bigcup_{i=0}^{k} [a_i n + b_i, c_i n + d_i],$$

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- If we could prove that N_k does not depend on k, we would prove the Rigidity Conjecture.
- Our original proof used model theory; there is now a constructive proof using a generalization of Ulam sequences.

Generalizations with Vector Spaces

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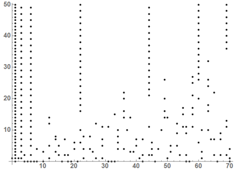
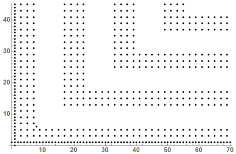
- Sometimes, these are apparently chaotic. Sometimes, they are eventually periodic. (Alexander Schlesinger 2019)



$$U\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}4\\0\end{pmatrix}\right)$$



$$U\left(\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}8\\6\end{pmatrix}\right)$$

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- For example, you could consider the free group on two generators—call them "0" and "1"—and take word length to be the measure of size.
- You get an Ulam set

$$\{0, 1, 00, 01, 10, 11, 0000, 0001, \ldots\},$$

and you can prove various results about when words of special type appear in this set. (Bade, Cui, Labelle, Li 2020) (Mandelshtam 2022)

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Definition (S. 2021)

Given $0 < a < b$ in $\langle 1, X \rangle$, an *Ulam sequence* starting with a, b is a set $\mathcal{U} \subset \langle 1, X \rangle$ such that

1. $\mathcal{U} \cap (-\infty, b] = \{a, b\}$,
2. for all $p < q \in \langle 1, X \rangle$, $\mathcal{U} \cap [p, q]$ has both a minimum and a maximum, and
3. for every $p \in (b, \infty)$, $p \in \mathcal{U}$ if and only if it is the smallest element in the set

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- For any a, b, there always exists such a set, but it is not always unique.

A Unique Ulam Sequence

X,	$X+1$,	$2X+1$,	$3X+1$,	$3X+2$,	$4X+1$,	$4X+3$,
$5X+1$,	$5X+4$	$6X+1$,	$6X+3$,	$6X+5$,	$7X+1$,	$7X+6$,
$8X+1$,	$8X+3$	$8X+5$,	$8X+7$,	$9X+1$,	$9X+8$,	$10X+1$,
$10X+3$,	$10X+5$,	$10X+7$,	$10X+9$,	$11X+1$,	$11X+10$,	$12X+1$,
$12X+3$,	$12X+5$,	$12X+7$,	$12X+9$	$12X+11$,	$13X+1$,	$13X+12$
$14X+1$,	$14X+3$,	$14X+5$,	$14X+7$,	$14X+9$	$14X+11$,	$14X+13$,
$15X+1$,	$15X+14$,	$16X+1$,	$16X+3$,	$16X+5$,	$16X+7$	$16X+9$,
$16X+11$,	$16X+13$,	$16X+15$,	$17X+1$,	$17X+16$,	$18X+1$,	$18X+3$,
$18X+5$,	$18X+7$,	$18X+9$,	$18X+11$,	$18X+13$,	$18X+15$,	$18X+17$,
$19X+1$,	$19X+18$,	$20X+1$,	$20X+3$,	$20X+5$,	$20X+7$,	$20X+9\ldots$

The Ulam Sequence Starting with $1, X$

$$
\begin{array}{lllllll}
\{1\} & \cup & [X, 2X] & \cup & \{2X+2\} & \cup & \{4X\} \\
 & \cup & [4X+2, 5X-1] & \cup & \{5X+1\} & \cup & [7X+3, 8X+1] \\
 & \cup & \{10X+2\} & \cup & \{11X+2\} & \cup & [13X+4, 14X+1] \\
 & \cup & \{16X+2\} & \cup & \{17X+2\} & \cup & \{19X+3\} \\
 & \cup & \{20X+2\} & \cup & \{22X+3\} & \cup & \{23X+4\} \\
 & \cup & [25X+4, 25X+5] & \cup & \{26X+3\} & \cup & \{28X+4\} \\
 & \cup & [31X+5, 32X+3] & \cup & \{34X+5\} & \cup & \{38X+6\} \\
 & \cup & \{40X+5\} & \cup & [40X+8, 41X+4] & \cup & [43X+7, 44X+4]\ldots
\end{array}
$$

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\begin{array}{llllll}
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\end{array}
$$

- By an easy induction argument, this is *also* uniquely determined and there exist integer coefficients $\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}$ such that

$$\mathrm{U}(1, X) = \bigcup_{i=0}^{\infty} [a_i X + b_i, c_i X + d_i].$$

Relating the Polynomial Set to the Integer Set

- We can find a polynomial-time algorithm $\mathcal{A}$ that can compute the first k coefficients $\{a_i\}, \{b_i\}, \{c_i\}, \{d_i\}$ such that

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- Furthermore, this algorithm can output an integer N_k such that for all $n \geq N_k$,

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- How? Every time the algorithm makes a comparison $aX + b < cX + d$, it computes the smallest n such that $an + b < cn + d$; N_k is the maximum of all these n's. If $n \geq N_k$, then all the comparisons are still valid even if we replace X by n everywhere.

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Finale

Thank you for the invitation!