Avoiding triples in the card game Spot it!

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Math Circle Activities as a Gateway into Research

Subtitle: How to use the card game ***Spot It!*** to learn about the Projective Plane and introduce students to fun combinatorial problems.

How do you play Spot it!?

1. Lay out 2 cards and look for the symbol in common.
2. The person who finds the symbol first takes the cards, and two new cards are dealt.
3. The player with the most cards at the end wins.

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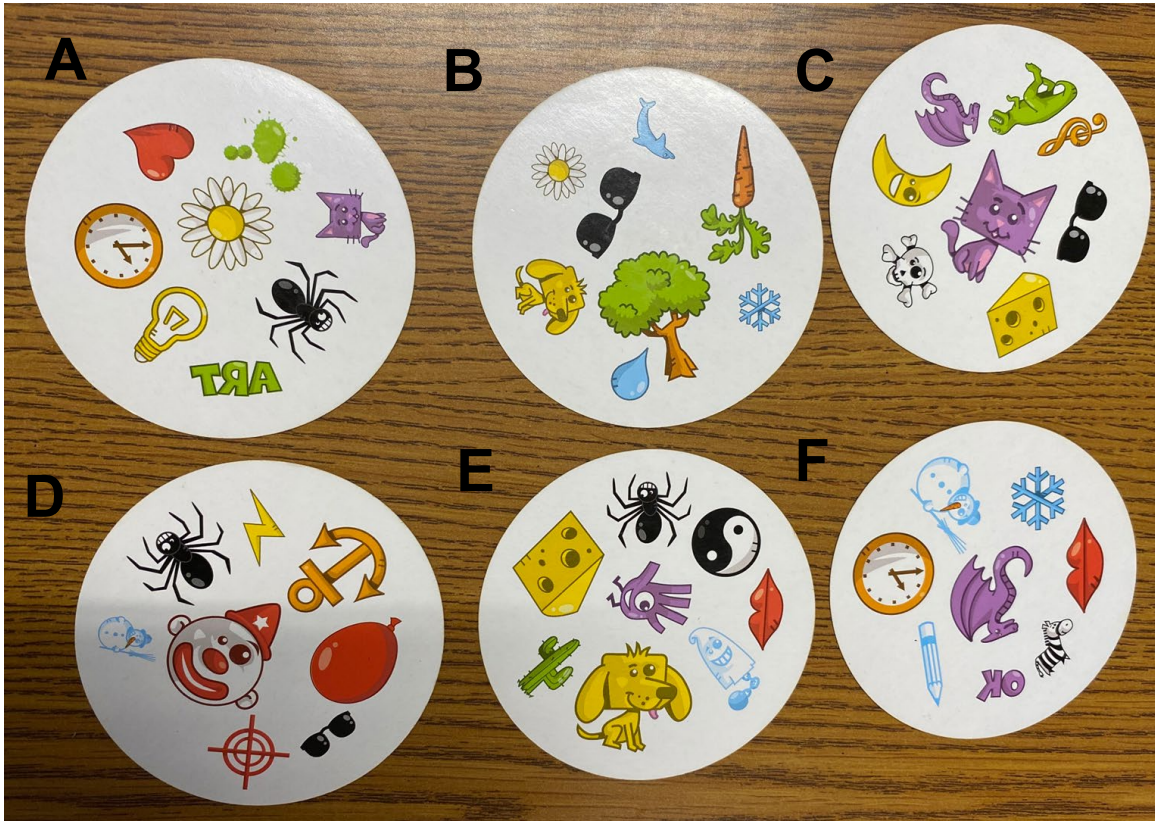
★ Pros: Any age can play, no reading required!

★ Cons: No obvious strategy or math.

Variation: Spot it Triplets

A **triplet** is three cards that share the same symbol.

Are there any triplets in these cards?



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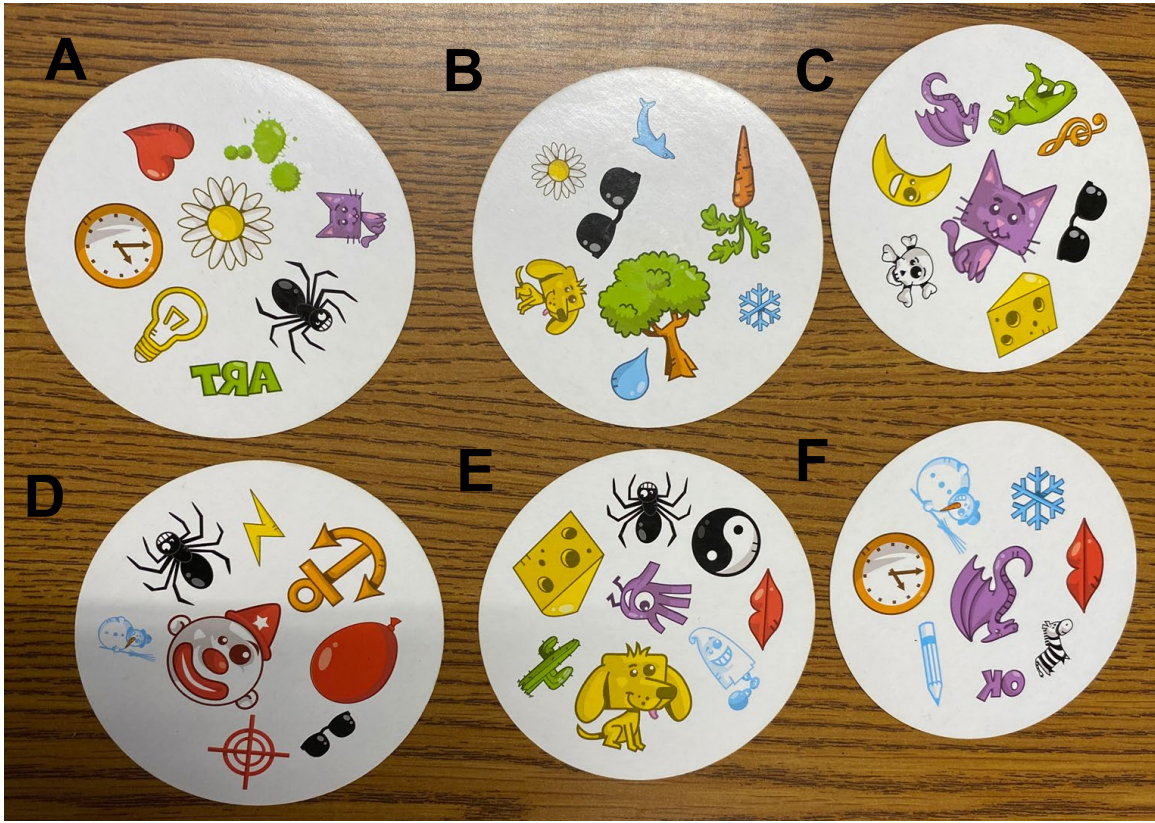

Answer:
BCD: glasses
ADE: spiders

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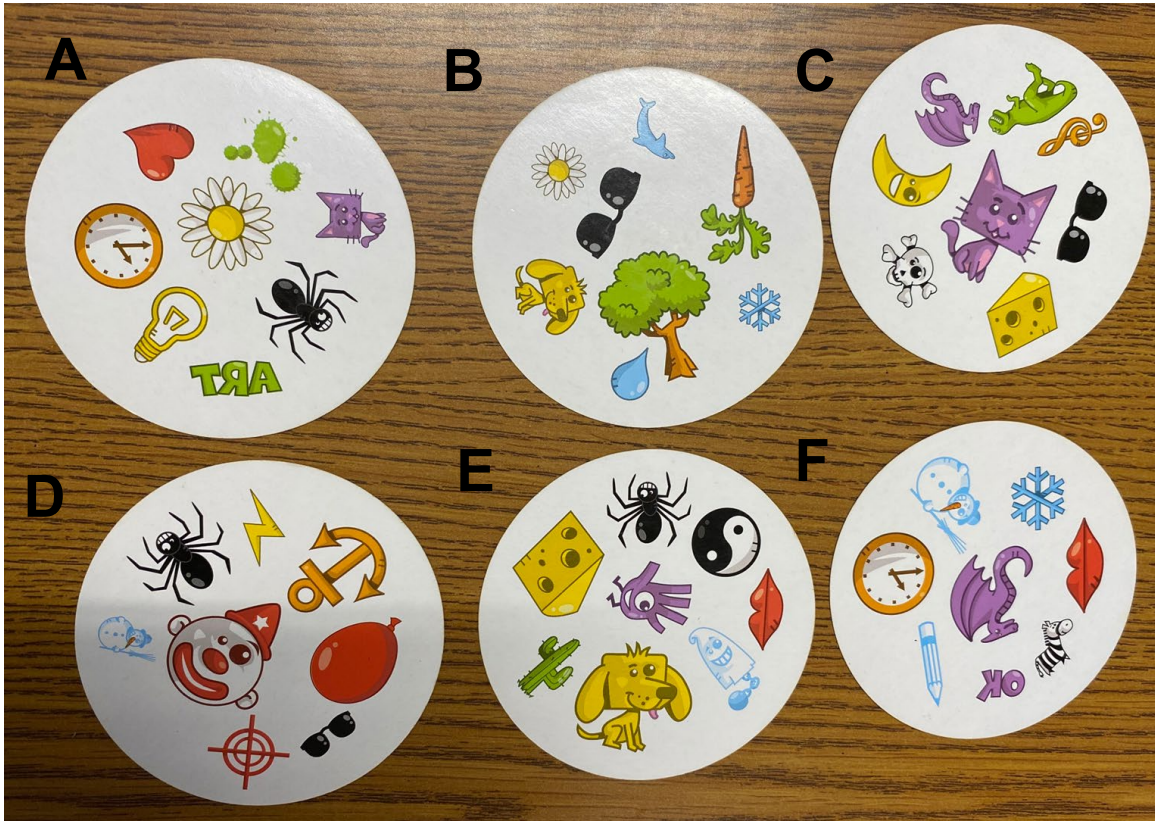
Mathematical Questions?



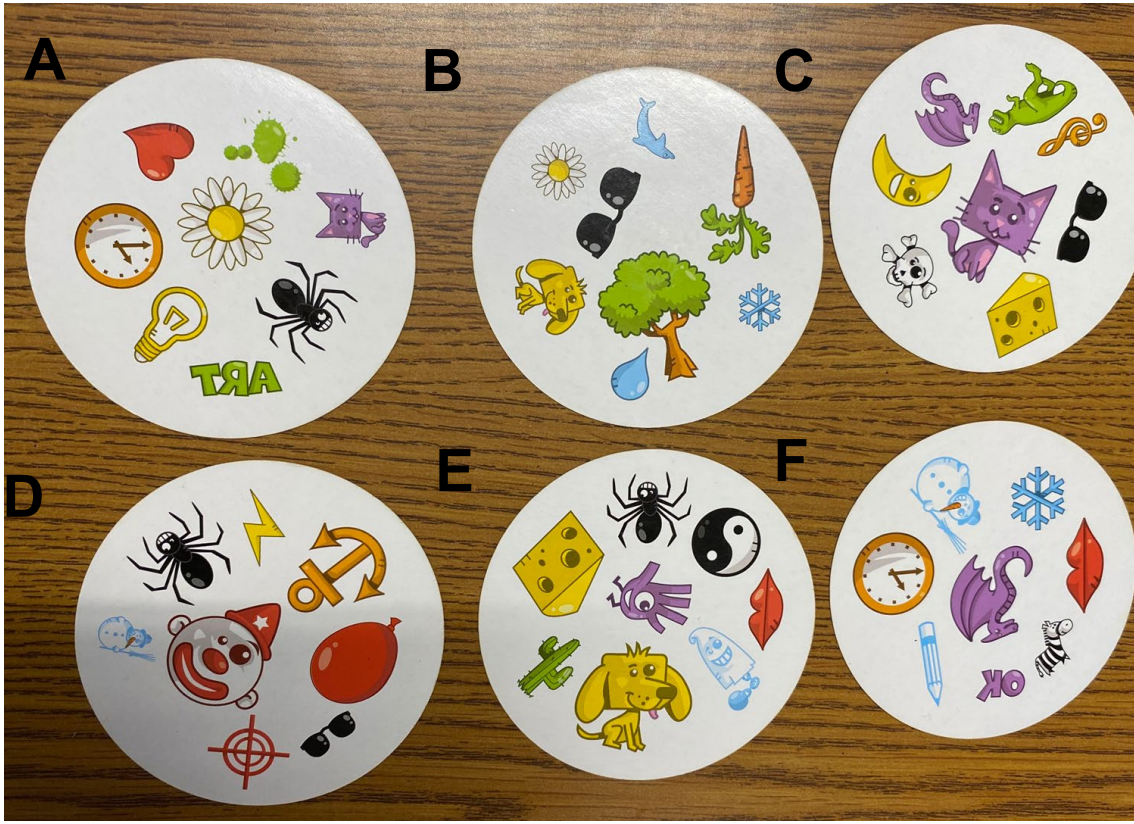
Variation: Spot it Triplets

A **triplet** is three cards that share the same symbol.

Are there any triplets in these cards?

Mathematical Questions?

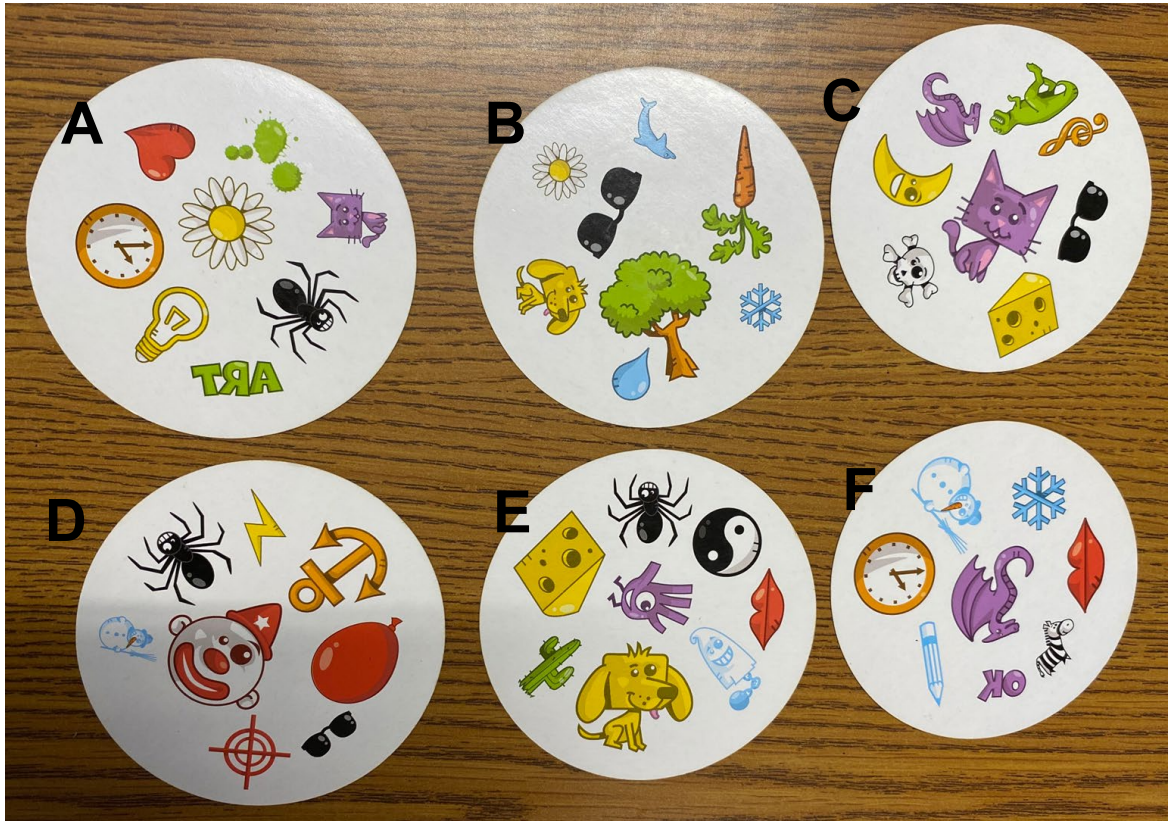
1. What is the probability that 3 cards contain a triplet?
2. How many cards must you lay out to guarantee a triplet?
3. What about quadruples, etc?



Avoiding Triples: Caps

A **cap** is a collection of cards that does not contain a triplet.

What is the largest cap in this layout?



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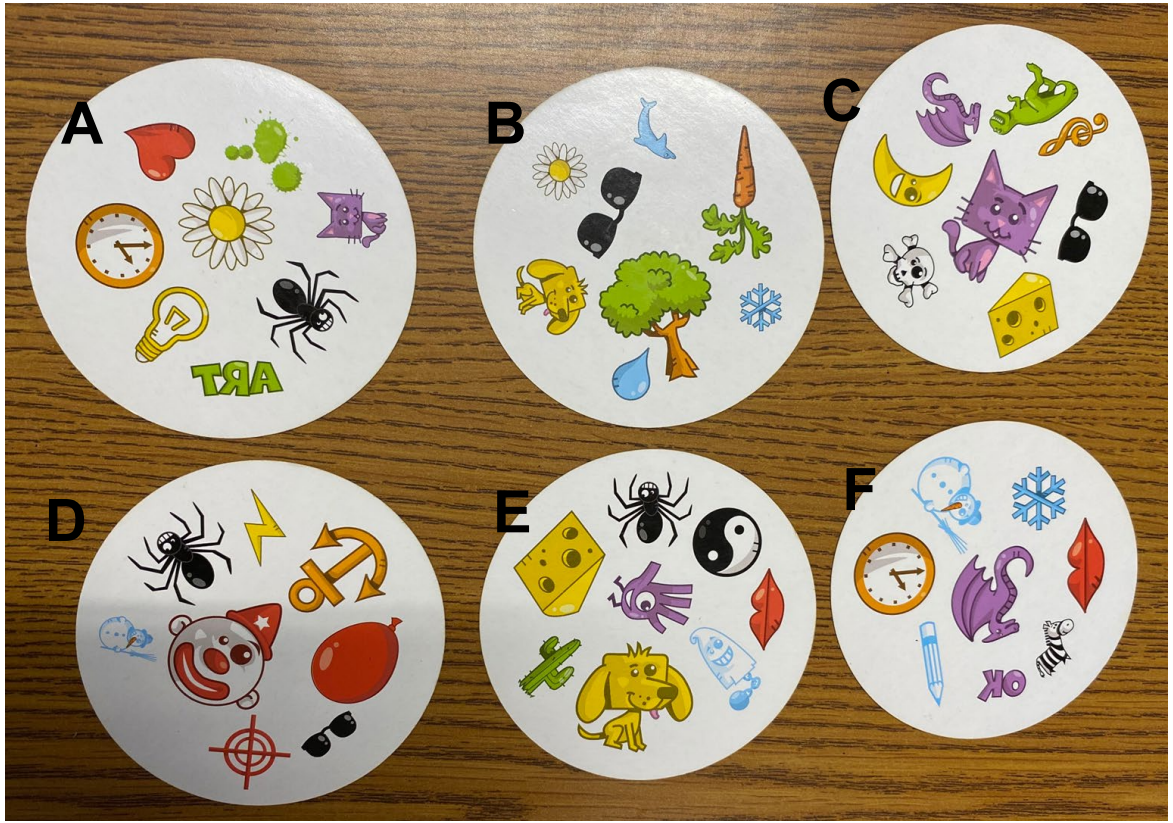
What is the largest cap in this layout?

A,B,C,E,F



Complete Caps and Maximal Caps

- A cap is **complete** if adding any card results in a triple
- A **maximal cap** is the largest cap possible in a given deck
- Any maximal cap is complete
- Not all complete caps are maximal

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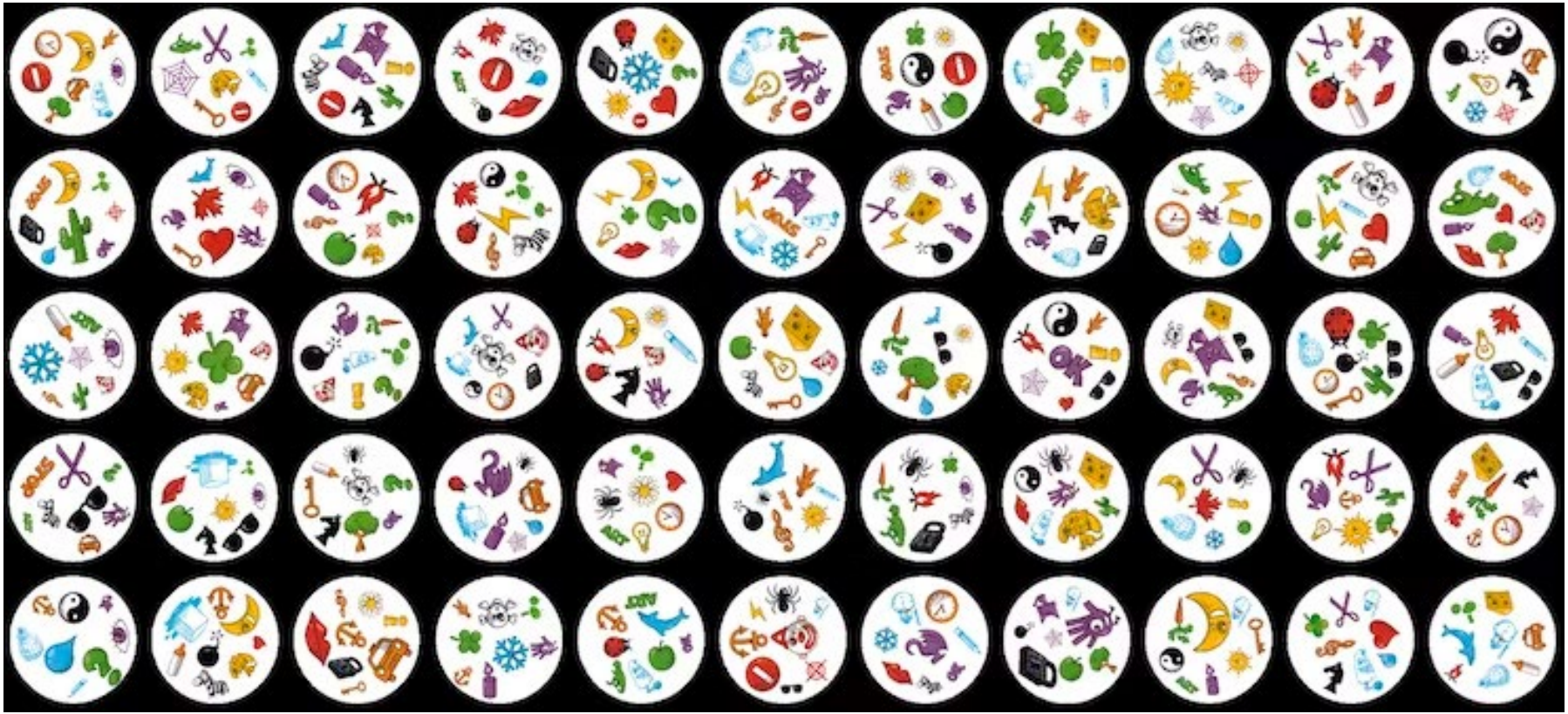
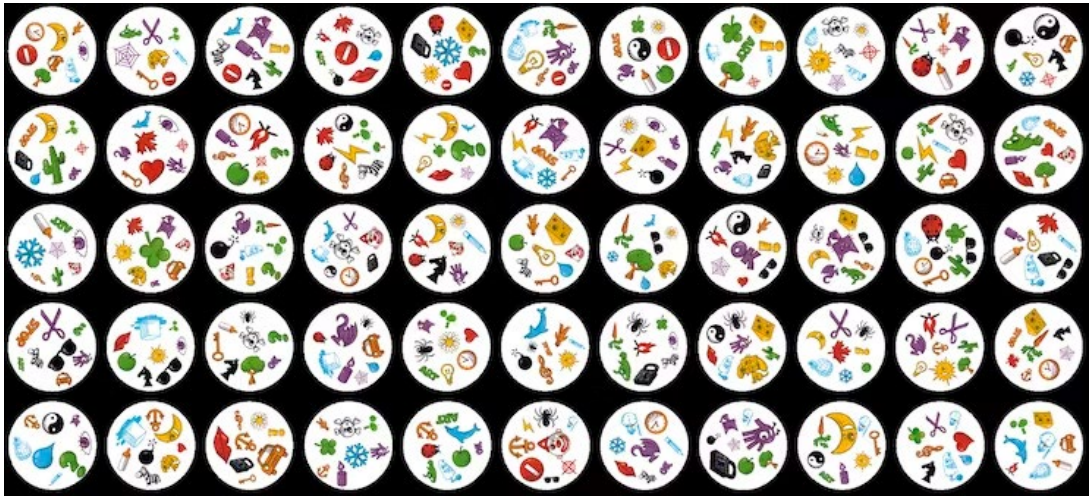
Q: How do we address finding caps in Spot It?

1. Brute force: lay out cards and look for caps.
2. Find mathematical properties of the game.
3. Make the problem **SMALLER** (thank you Paul Zeitz and Japheth Wood).

1. Brute Force



2. Spot it! Math Facts



1. A deck of 57* cards, 57 symbols
2. Each card has 8 symbols
3. Each symbol appears on exactly 8 cards.
4. Each pair of cards shares exactly one symbol.
5. Each pair of symbols appears on exactly one card.

* The official game only has 55 cards, which messes up the math!

Spot it Axioms

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2. 57 symbols
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Finite Projective Plane Axioms

1. n^2+n+1 lines
2. n^2+n+1 points
3. Each line has n+1 points
4. Each points lies on exactly n+1 lines.
5. Each pair of lines intersect in exactly one point.
6. Each pair of points lie on exactly one line.

2. Go Smaller: Set n=2

Spot it Axioms

1. 7 cards
2. 7 symbols
3. Each card has 3 symbols
4. Each symbol lies on exactly 3 cards.
5. Each pair of cards shares exactly one symbol.
6. Each pair of symbols appears on exactly one card.

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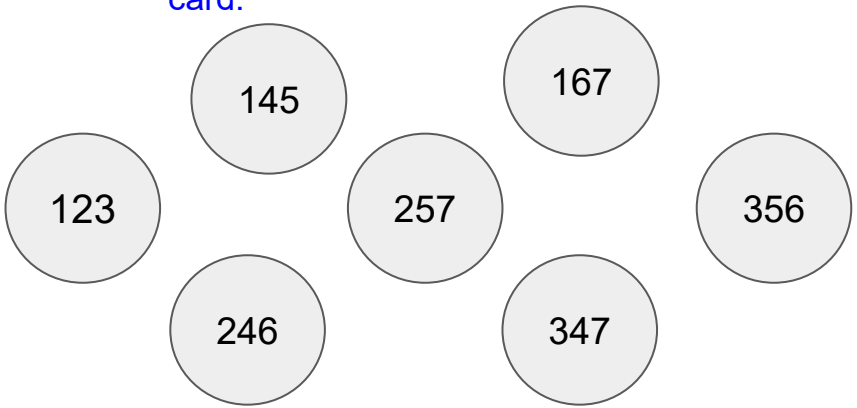

Finite Projective Plane Axioms

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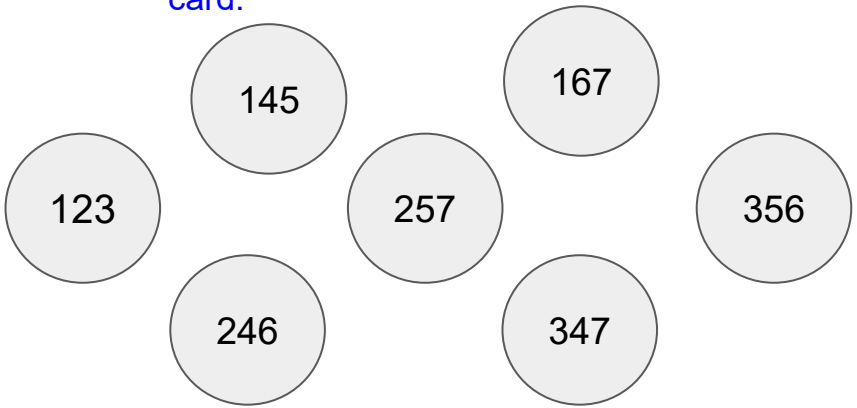

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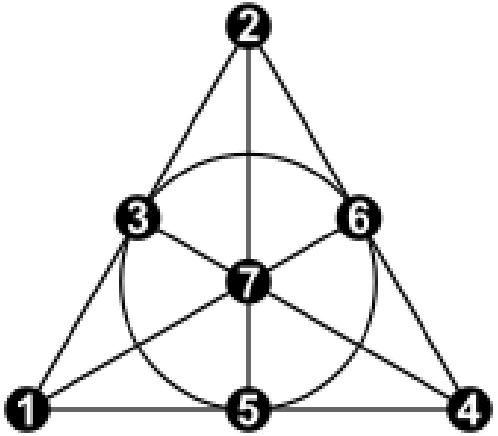

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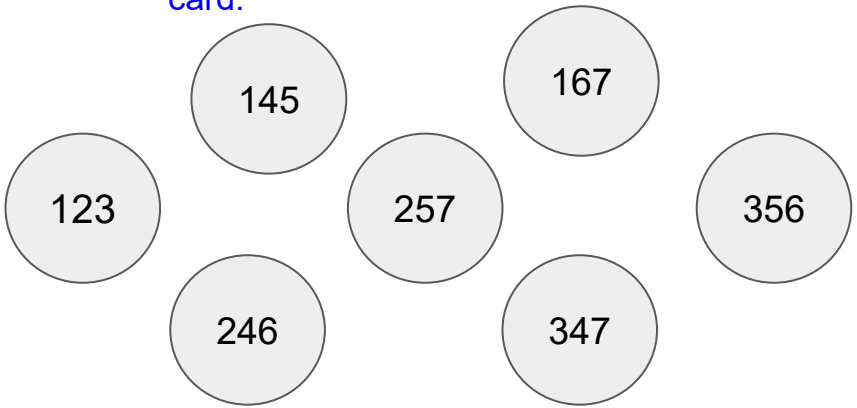

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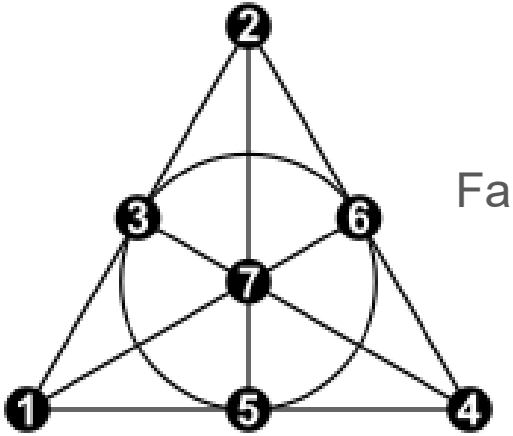


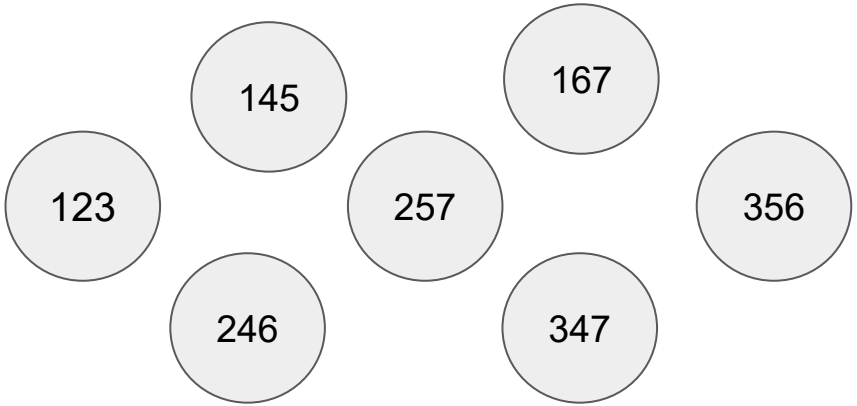
Fano Plane

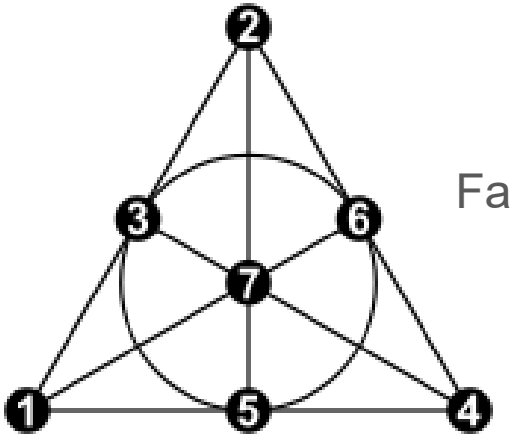
2. Go Smaller: Set n=2

Spot it Deck of Order 2

Finite Projective Plane of Order 2

Q: What is the maximal cap size?





Fano Plane

2. Go Smaller: Set n=2

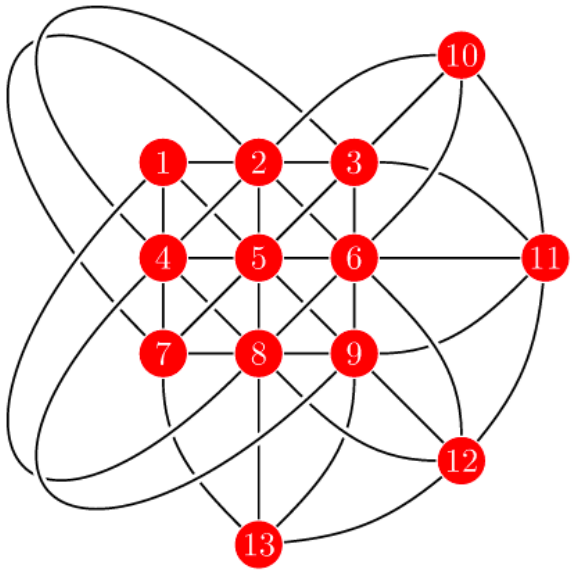
Spot it Deck of Order 3

Finite Projective Plane of Order 3

Q: What is the maximal cap size?

[0, 1, 2, 12]
[0, 3, 6, 9]
[0, 4, 8, 10]
[0, 5, 7, 11]
[1, 3, 8, 11]
[1, 4, 7, 9]
[1, 5, 6, 10]

[2, 3, 7, 10]
[2, 4, 6, 11]
[2, 5, 8, 9]
[3, 4, 5, 12]
[6, 7, 8, 12]
[9, 10, 11, 12]



n=2	n=3	n=4	n=5	n=7
[0, 1, 6]	[0, 1, 2, 12]	[0, 1, 2, 3, 20]	[0, 1, 2, 3, 4, 30]	[0, 1, 2, 3, 4, 5, 6, 56]
[0, 2, 4]	[0, 3, 6, 9]	[0, 4, 8, 12, 16]	[0, 5, 10, 15, 20, 25]	[0, 7, 14, 21, 28, 35, 42, 49]
[0, 3, 5]	[0, 4, 8, 10]	[0, 5, 10, 15, 19]	[0, 6, 12, 18, 24, 26]	[0, 8, 16, 24, 32, 40, 48, 50]
[1, 2, 5]	[0, 5, 7, 11]	[0, 6, 11, 13, 17]	[0, 7, 14, 16, 23, 27]	[0, 9, 18, 27, 29, 38, 47, 51]
[1, 3, 4]	[1, 3, 8, 11]	[0, 7, 9, 14, 18]	[0, 8, 11, 19, 22, 28]	[0, 10, 20, 23, 33, 36, 46, 52]
[2, 3, 6]	[1, 4, 7, 9]	[1, 4, 11, 14, 19]	[0, 9, 13, 17, 21, 29]	[0, 11, 15, 26, 30, 41, 45, 53]
[4, 5, 6]	[1, 5, 6, 10]	[1, 5, 9, 13, 16]	[1, 5, 14, 18, 22, 29]	[0, 12, 17, 22, 34, 39, 44, 54]
	[2, 3, 7, 10]	[1, 6, 8, 15, 18]	[1, 6, 11, 16, 21, 25]	[0, 13, 19, 25, 31, 37, 43, 55]
	[2, 4, 6, 11]	[1, 7, 10, 12, 17]	[1, 7, 13, 19, 20, 26]	[1, 7, 20, 26, 32, 38, 44, 55]
	[2, 5, 8, 9]	[2, 4, 9, 15, 17]	[1, 8, 10, 17, 24, 27]	[1, 8, 15, 22, 29, 36, 43, 49]
	[3, 4, 5, 12]	[2, 5, 11, 12, 18]	[1, 9, 12, 15, 23, 28]	[1, 9, 17, 25, 33, 41, 42, 50]
	[6, 7, 8, 12]	[2, 6, 10, 14, 16]	[2, 5, 13, 16, 24, 28]	[1, 10, 19, 21, 30, 39, 48, 51]
	[9, 10, 11, 12]	[2, 7, 8, 13, 19]	[2, 6, 10, 19, 23, 29]	[1, 11, 14, 24, 34, 37, 47, 52]
		[3, 4, 10, 13, 18]	[2, 7, 12, 17, 22, 25]	[1, 12, 16, 27, 31, 35, 46, 53]
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		[3, 6, 9, 12, 19]	[2, 9, 11, 18, 20, 27]	[2, 7, 19, 24, 29, 41, 46, 54]
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		[4, 5, 6, 7, 20]	[3, 6, 14, 17, 20, 28]	[2, 9, 16, 23, 30, 37, 44, 49]
		[8, 9, 10, 11, 20]	[3, 7, 11, 15, 24, 29]	[2, 10, 18, 26, 34, 35, 43, 50]
		[12, 13, 14, 15, 20]	[3, 8, 13, 18, 23, 25]	[2, 11, 20, 22, 31, 40, 42, 51]
		[16, 17, 18, 19, 20]	[3, 9, 10, 16, 22, 26]	[2, 12, 15, 25, 28, 38, 48, 52]

My students made cards for up to n=7 and looked for patterns.

Their Results:

Theorem: In order *n,* the upper bound on maximal cap size is $n + 2$.

Table 6: Max Cap in Order n

n	Max cap size
2	4
3	4
4	6
5	6
7	8
8	10
16	18

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Conjecture:

1. **When n is odd, max cap size = n+1**
1. **When n is even, max cap size = n+2**

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1. It was difficult to read.

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1. It was difficult to read.
2. It proved it for the dual case
3. We found a more recent exposition.
4. We translated it from the dual to our setting.

Table 8: Found Complete Caps in Order n

n	sizes of complete caps
2	4
3	4
4	6
5	6
7	6, 8
8	6, 10
9	6, 7, 8, 10
11	7, 8, 9, 10, 12
13	8, 9, 10, 12, 14
16	9, 10, 11, 12, 13, 18
17	10, 11, 12, 13, 14, 18
19	10, 11, 12, 13, 14, 20
23	12, 13, 14, 15, 16, 17
29	14, 15, 16, 17, 18
31	15, 16, 17, 18, 19

Open Questions/Future Work

- How many max caps are there?
- Possible sizes of complete caps
- Avoiding Quadruples
- Partitioning a deck into caps?
- Create a game in higher dimensional projective space.

Thanks for listening!

Lauren Rose

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