Math Wrangle Solutions: Set I

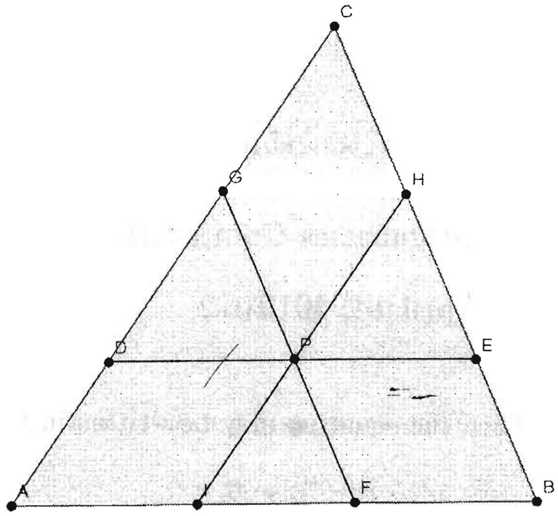
American Mathematics Competitions

April 14, 2012

1. (337) Let $y = \sqrt[4]{x}$. Then the equation may be written in the form
$$y^2 - 7y + 12 = 0,$$
whose roots are $y = 3$ and $y = 4$. Consequently, we obtain the x-values of 3^4 and 4^4, whose sum is 337.

2. (033) Let k be the number of the page that was counted twice. Then, $0 < k < n+1$, and $1+2+\cdots+n+k$ is between $1+2+\cdots+n$ and $1+2+\cdots+n+(n+1)$. In other words, $n(n+1)/2 < 1986 < (n+1)(n+2)/2$; i.e.,
$$n(n+1) < 3972 < (n+1)(n+2).$$
By trial and error (clearly, n is a little larger than 60) we find that $n = 62$. Thus $k = 1986 - (62)(63)/2 = 1986 - 1953 = 33$.

3. (306) As shown in the figure, $EH = BC - (BE + HC) = BC - (FP + PG) = 450 - d$. In like manner, $GD = 510 - d$. Moreover, from the similarity of $\triangle DPG$ and $\triangle ABC$ we have $DP/GD = AB/CA$. Hence $DP = \frac{AB}{CA} \cdot GD = \frac{425}{510}(510 - d) = 425 - \frac{5}{6}d$. (1) In like manner, since $\triangle PEH$ and $\triangle ABC$ are similar, $PE/EH = AB/BC$. Hence $PE = \frac{AB}{BC} \cdot EH = \frac{425}{450}(450 - d) = 425 - \frac{17}{18}d$.(2) Since $d = DP + PE$, adding (1) and (2) we find that $d = 850 - \frac{16}{9}d$, from which $d = 306$.



More generally, let $AB = c$, $BC = a$, $CA = b$, $EH = x$, $GD = y$, and $IF = z$. Then, with $DE = FG = HI = d$, we find that $DP/y = c/b$, $PE/x = c/a$, and hence $d = DE = DP + PE = c(\frac{y}{b} + \frac{x}{a})$. From this, since $d = c - z$, it follows that $c - z = c(\frac{y}{b} + \frac{x}{a})$; i.e., $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$. Substituting $a - d$, $b - d$, and $c - d$ for x, y, and z, respectively, and rearranging the resulting equation leads to

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = \frac{2}{d}, \tag{3}$$

which may be viewed as the key to setting and/or solving the problem.

To use Equation (3) for the generation of "nice" AIME-like problems, start off with carefully chosen positive integers p, q, r, and their sum, s, and divide through the equation $p + q + r = s$ by the least common multiple of p, q, r, s. To generate the data for the present problem, we started with $15 + 17 + 18 = 50$, divided through by 7650, and found that

$$\frac{1}{510} + \frac{1}{450} + \frac{1}{425} = \frac{2}{306}.$$

Since the triple $(510, 450, 425)$ satisfies the triangle inequality, this choice of data led to a healthy problem. By contrast, $(p, q, r) = (3, 8, 9)$ would not have done so.

A solution is more straightforward using knowledge of *barycentric coordinates*. Every point P in a triangle ABC can be represented by a triple of nonnegative numbers α, β, γ, where α is the ratio of the altitudes of $\triangle PBC$ and $\triangle ABC$, with corresponding definitions for β and γ. Equivalently, α is the ratio of the areas of $\triangle PBC$ and $\triangle ABC$. It follows that $\alpha + \beta + \gamma = 1$. Now, turning to the problem at hand, and using similar triangles and the fact that the lines through P all have length d, we find that $1 - \alpha = dBC, 1 - \beta = dAC, 1 - \gamma = dAB$. Adding these equations, substituting $\alpha + \beta + \gamma = 1$, and dividing by d, we obtain $2/d = 1/BC + 1/AC + 1/AB$, the same Equation (3) we used to solve for d before.

Barycentric coordinates are often used in advanced mathematics (e.g., topology and the study of convex sets in many dimensions) where they are usually defined using linear algebra. For more about their use in classical geometry, see Dan Pedoe, "Geometry, a Comprehensive Course," Dover Publications.

4. (061) First we show that S contains at most 5 elements. Suppose otherwise. Then S has at least $\binom{6}{1} + \binom{6}{2} + \binom{6}{3} + \binom{6}{4}$ or 56 subsets of 4 or fewer members. The sum of each of these subsets is at most 54 (since $15 + 14 + 13 + 12 = 54$), hence, by the Pigeonhole Principle, at least two of these sums are equal. If the subsets are disjoint, we are done; if not, then the removal of the common element(s) yields the desired contradiction.

   Next we attempt to construct such a 5-element set S, by choosing its elements as large as possible. Including 15, 14 and 13 in S leads to no contradiction, but if 12 is also in S, then (in view of $12 + 15 = 13 + 14$) the conditions on S would be violated. Hence we must omit 12. No contradiction results from letting 11 be a member of S, but then $10 \notin S$ since $10 + 15 = 11 + 14$, and $9 \notin S$ since $9 + 15 = 11 + 13$. So we must settle for 8 as the fifth element of S. Indeed, $S = \{8, 11, 13, 14, 15\}$ satisfies the conditions of the problem, yielding $8 + 11 + 13 + 14 + 15$ or 61 as the candidate for its solution.

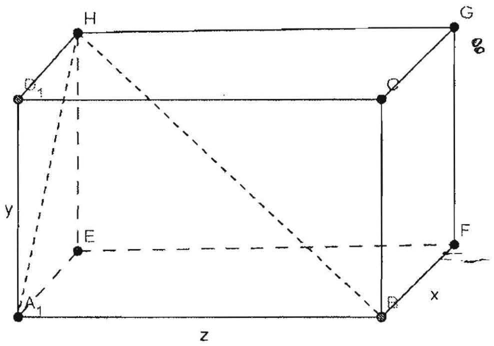
   Finally, to show that the maximum is indeed 61, suppose that the sum is more for another choice of S. Observe that this set must also contain 15, 14 and 13, for if even the smallest of them (13) is omitted, the maximum possible sum (62) is achievable only by including 10, 11 and 12, but then $15 + 11 = 14 + 12$. Having chosen 15, 14 and 13, we must exclude 12, as noted before. If 11 is included, then we are limited to the sum of 61 as above. If 11 is not included, then even by including 10 and 9 (which we can't) we could not surpass 61 since $15 + 14 + 13 + 10 + 9 = 61$. Consequently, 61 is indeed the maximum sum one can attain.

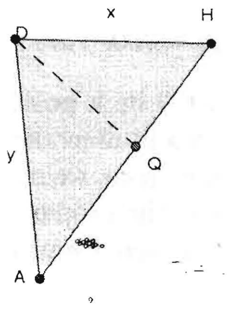
In the original formulation of the problem, the largest element of S was restricted to be no greater than 25. The problem was posed with this data in the USA Mathematical Talent Search in the Summer 1990 issue of *Consortium*. The interested reader may also wish to consult George Berzsenyi's article, entitled "A Pigeonhole for Every Pigeon", on page 40 in the September–October 1990 issue of *Quantum* for further comments on this problem.

5. (750) In order to find the volume of P, we will determine its dimensions, x, y and z. To this end, consider the rectangular parallelepiped shown in the first figure below, with vertices and side lengths as indicated. For definiteness, we labeled the box so that

$$\begin{aligned} d(BH, CD) &= 2\sqrt{5}, \\ d(BH, AE) &= \frac{30}{\sqrt{13}} \\ \text{and } d(BH, AD) &= \frac{15}{\sqrt{10}}. \end{aligned} \tag{1}$$

where, in general, $d(RS, TU)$ denotes the distance between lines RS and TU.





Now the distance from BH to CD is equal to the distance from CD to the plane ABH, which is the same as the length of the perpendicular from D to the diagonal AH of rectangle $AEHD$. To see this, note that this perpendicular is also perpendicular to the plane ABH and the line CD. If one "slides" it over so that its top moves from D towards C, it eventually intersects line BH. This gives an equally long segment, which is perpendicular to both CD and BH, so its length is indeed the distance $d(BH, CD)$. This distance is found via similar triangles, as shown in the second figure above, in which $DQ/DA = DH/AH$, and hence $DQ = xy/\sqrt{x^2+y^2}$. Treating the other distances similarly, in view of (1), this leads to the equations

$$\frac{xy}{\sqrt{x^2+y^2}} = 2\sqrt{5}, \quad \frac{yz}{\sqrt{y^2+z^2}} = \frac{30}{\sqrt{13}} \text{ and } \frac{zx}{\sqrt{z^2+x^2}} = \frac{15}{\sqrt{10}}. \tag{2}$$

Upon squaring each equation in (2), taking reciprocals and simplifying one arrives at the system

$$\frac{1}{x^2} + \frac{1}{y^2} = \frac{1}{20}, \quad \frac{1}{y^2} + \frac{1}{z^2} = \frac{13}{900} \quad \text{and} \quad \frac{1}{z^2} + \frac{1}{x^2} = \frac{2}{45}. \tag{3}$$

We solve (3) by first adding the equations therein, and then subtracting from 12 times the resulting equation each of the original equations in (3). Thus we find that $1/x^2 = 1/25$, $1/y^2 = 1/100$ and $1/z^2 = 1/225$. From these, $x = 5$, $y = 10$, and $z = 15$. Consequently, the volume of P is $xyz = 750$.

6. (182) Let k be a positive integer, and let $1, d_1, d_2, \ldots, d_{n-1}, d_n, k$ be its divisors in ascending order. Then $1 \cdot k = d_1 \cdot d_n = d_2 \cdot d_{n-1} = \cdots$. For k to be nice, we must have $n = 2$. Moreover, d_1 must be prime, for otherwise, the proper divisors of d_1 would have appeared in the listing above between 1 and d_1. Similarly, d_2 is either a prime or the square of d_1, for otherwise d_1 could not be the only divisor between 1 and d_2. Therefore, k is either the product of two distinct primes or – being the product of a prime and its square – is the cube of a prime.

   In view of the above, one can easily list the first ten nice numbers. They are: 6, 8, 10, 14, 15, 21, 22, 26, 27 and 33. Their sum is 182.

7. (486) To solve the problem, we must find integers n and k such that n is non-negative, k is as large as possible, and

$$3^{11} = (n+1) + (n+2) + \cdots + (n+k). \tag{1}$$

   Noting that

$$\begin{aligned}(n+1) + (n+2) + \cdots + (n+k)& \\ &= [1 + 2 + \cdots + (n+k)] - [1 + 2 + \cdots + n] \\ &= \frac{(n+k)(n+k+1)}{2} - \frac{n(n+1)}{2} \\ &= k(k + 2n + 1)/2,\end{aligned}$$

   it follows that (1) is equivalent to

$$k(k + 2n + 1) = 2 \cdot 3^{11}. \tag{2}$$

   In solving (2), we must ensure that the smaller factor, k, is as large as possible, and that n is a non-negative integer. These conditions lead to $k = 2 \cdot 3^5 = 486$, $n = 121$ and $3^{11} = 122 + 123 + \cdots + 607$. Let m be the average of the k consecutive integers. If k is odd, then m must be the middle integer, and $km = 3^{11}$. Now $k = 3^5$ and $m = 3^6$ is the best we can do, for if $k = 3^6$ then $m - (k-1)/2$, the smallest summand, is negative. But if k is even, then m lies halfway between the middle two integers in the sum. Thus $(2m)k = 2 \cdot 3^{11}$ and now the largest even divisor of $2 \cdot 3^{11}$ which does not give rise to a negative first summand is $2 \cdot 3^5 = 486$. This is the answer.

8. (019) We solve the equivalent problem of finding the smallest positive integer n for which

$$n^3 + 1 < (n + 10^{-3})^3. \tag{1}$$

This is equivalent to the given problem because

$$n < \sqrt[3]{m} < n + 10^{-3} \iff n^3 < m < (n + 10^{-3})^3,$$

and because if some integer m satisfies the double inequality on the right above, then $n^3 + 1$ is the smallest such m.

Rewriting (1) in the form

$$\frac{1000}{3} < n^2 + \frac{n}{1000} + \frac{1}{3,000,000}, \tag{2}$$

we observe that n^2 must be near $1000/3$, for the contributions of the other two terms on the right side of (2) are relatively small. Consequently, since $18^2 < 1000/3 < 19^2$, we expect that either $n = 18$ or $n = 19$. In the first case, (2) is not satisfied; this can be verified by an easy calculation. It is even easier to show that $n = 19$ satisfies (2), so it is the smallest positive integer with the desired property. The corresponding $m = 19^3 + 1 = 6860$ is the smallest positive integer whose cube root has a positive decimal part which is less than $1/1000$.