# Math Wrangle Solutions: Set II

American Mathematics Competitions

April 14, 2012

1. (890) By division we find that

$$n^3 + 100 = (n + 10)(n^2 - 10n + 100) - 900.$$

Thus, if $n + 10$ divides $n^3 + 100$, then it must also divide 900. Moreover, since $n$ is maximized whenever $n + 10$ is, and since the largest divisor of 900 is 900, we must have $n + 10 = 900$. Therefore, $n = 890$.

2. (560) Think of such sequences of coin tosses as progressions of blocks of T's and H's, to be denoted by (T) and (H), respectively. Next note that each HT and TH subsequence signifies a transition from (H) to (T) and from (T) to (H), respectively. Since there should be three of the first kind and four of the second kind in each of the sequences of 15 coin tosses, one may conclude that each such sequence is of the form

$$\text{(T)(H)(T)(H)(T)(H)(T)(H).} \qquad (1)$$

Our next concern is the placement of T's and H's in their respective blocks, so as to assure that each sequence will have two HH and five TT subsequences. To this end, we will assume that each block in (1) initially contains only one member. Then, to satisfy the conditions of the problem, it will suffice to place two more H's into the (H)'s and five more T's into the (T)'s. Thus, to solve the problem, we must count the number of ways this can be accomplished.

Recall that the number of ways to put $p$ indistinguishable balls (the extra H's and T's in our case) into $q$ distinguishable boxes (the (H)'s and (T)'s, distinguished by their order in the sequence) is given by the formula $\binom{p+q-1}{p}$. (Students who are not familiar with this fact should verify it.) In our case, it implies that the two H's can be placed in the four (H)'s in $\binom{2+4-1}{2}$ or 10 ways, and the five T's can be placed in the four (T)'s in $\binom{5+4-1}{5}$ or 56 ways. The desired answer is the product, 560, of these numbers.

**Note.** Such problems as this (only harder) often occur in the study of DNA molecules.

3. (400) More generally, we will show that if $\triangle ABC$ is a right triangle with right angle at $C$, if $AB = 2r$ and if the acute angle between the medians emanating from $A$ and $B$ is $\theta$, then

$$\text{area}(\triangle ABC) = \frac{4}{3}r^2 \tan\theta. \tag{1}$$

In our case, $r = 60/2 = 30$ and $\tan\theta = 1/3$ (determined either by the standard formula for the tangent of the angle between two given lines, or by glancing at the first accompanying figure), so the answer to the problem is $(4/3)(30)^2(1/3)$ or 400.

To establish (1), first note that

$$\text{area}(\triangle ABC) = r^2 \sin\psi, \tag{2}$$

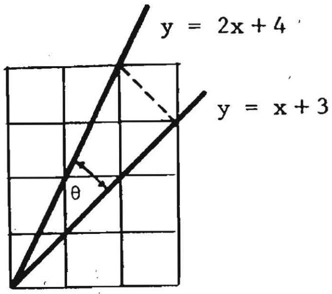
where $\psi = \angle AOC$, as shown in the second figure below, where $O$ is the midpoint of $AB$, $D$ is the centroid of $\triangle ABC$ and $CO = r$ (since $\triangle ABC$ is a right triangle). Consequently, to prove (1), it suffices to verify that $\sin\psi = (4/3)\tan\theta$, which is equivalent to establishing that

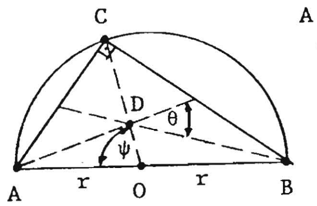
$$\cos\phi = -(4/3)\tan\theta, \tag{3}$$

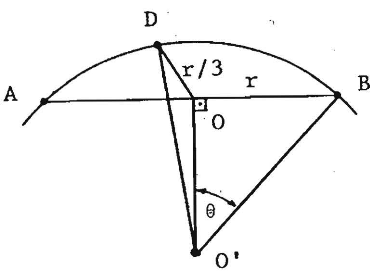
where $\phi = \psi + 90°$. This consideration leads us to the third figure below, where $O'$ is the center of the circle through $A$, $D$ and $B$, and $\angle DOO' = \phi$.

To prove (3), we will apply the Law of Cosines to $\triangle DOO'$. (The fact that $\angle OO'B = \theta$ comes from the observation that $\angle ADB = 180° - \theta$, and hence arc $ADB$ has central angle $2\theta$; $DO = r/3$ is a well-known fact concerning the centroid.) Noting that $DO' = BO' = r\csc\theta$ and $OO' = r\cot\theta$ (from $\triangle BOO'$), indeed,

$$\cos\phi = \frac{(r/3)^2 + (r\cot\theta)^2 - (r\csc\theta)^2}{2(r/3)(r\cot\theta)} = -\frac{4}{3}\tan\theta,$$



as was to be shown.

*Second solution.* Based on a suggestion of Noam Elkies (then a student at Stuyvesant High School in New York — later the youngest ever Full Professor at Harvard University), one can show more directly that if the hypotenuse of right $\triangle ABC$ is $c$, then its area is $(c^2 \tan\theta)/3$, where $\theta$ is the angle enclosed by its medians emanating from $A$ and $B$, as shown in the figure on the right.

To this end, note that $\theta + \pi/2 = \angle AA'C + \angle BB'C$, and hence

$$\begin{aligned}\tan\theta &= -\cot(\theta+\pi/2) = -\cot(\angle AA'C + \angle BB'C) \\ &= \frac{(\tan\angle AA'C)(\tan\angle BB'C) - 1}{\tan\angle AA'C + \tan\angle BB'C}.\end{aligned}$$

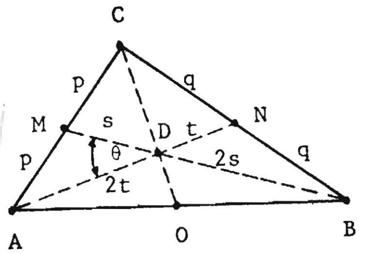
Since $\tan\angle AA'C = 2b/a$, $\tan\angle BB'C = 2a/b$, and $a^2+b^2=c^2$, the last expression above can be simplified to $3ab/2c^2$. From this, in view of the fact that area$(\triangle ABC) = ab/2$, the desired result follows.

Moreover, by the Pythagorean Theorem, we find that

$$\begin{aligned}4p^2+4q^2 &= AB^2 = 3600,\\ p^2+4q^2 &= 9s^2,\\ 4p^2+q^2 &= 9t^2.\end{aligned}$$

With the help of these expressions, from (1) and (2) it follows that

$$\begin{aligned}(\text{area}(\triangle ABC))^2 &= \frac{18}{5}s^2t^2\\ &= \frac{18}{5}\cdot\frac{p^2+4q^2}{9}\cdot\frac{4p^2+q^2}{9}\\ &= \frac{2}{45}((2p^2+2q^2)^2+9p^2q^2)\\ &= \frac{2}{45}((3600/2)^2+\frac{9}{4}(\text{area}(\triangle ABC))^2)\\ &= 144,000+\frac{1}{10}(\text{area}(\triangle ABC))^2.\end{aligned}$$



Consequently, $\frac{9}{10}(\text{area}(\triangle ABC))^2 = 144,000$, from which area$(\triangle ABC) =$ 400.

*Third solution.* Since $y = x + 3$ and $y = 2x + 4$ intersect at $(-1, 2)$, in the diagram on the right the coordinates of $A$ and $B$ will be of the form $(-1+a, 2+2a)$ and $(-1+b, 2+b)$, respectively. Moreover, since $-1$ and 2 are the averages of the coordinates $A, B$, and $C$, it is not difficult to show that the coordinates of $C$ are given by $(-1-a-b, 2-2a-b)$. We will find the area of $\triangle ABC$ by computing $\frac{1}{2}AC \cdot BC$ in terms of $a$ and $b$, without actually determining them.

Since $AC$ and $BC$ are perpendicular, their slopes, $(4a + b)/(2a + b)$ and $(2a + 2b)/(a + 2b)$, are negative reciprocals of one another. This fact leads to the equation

$$10a^2 + 15ab + 4b^2 = 0.$$

Now note that the vector (or change) from $A$ to $C$ is $(2a+b, 4a+b)$, and that from $B$ to $C$ is $(a+2b, 2a+2b)$. Since $\triangle ABC$ is a right triangle with $AB = 60$, by the Pythagorean Theorem we also conclude that

$$\begin{aligned}(AC)^2 + (BC)^2 &= \left[(2a + b)^2 + (4a + b)^2\right] + \left[(a + 2b)^2 + (2a + 2b)^2\right] \\ &= 25a^2 + 24ab + 10b^2 = 3600.\end{aligned}$$

From these two equations one can determine that $ab = -800/3$, and then use this fact to ease our computations as follows:

Since $(10a^2+15ab+4b^2)^2 = 100a^4+300a^3b+305a^2b^2+120ab^3+16b^4 = 0$, we find that

$$\begin{aligned}(\text{area}(\triangle ABC))^2 &= \frac{1}{4}(AC)^2(BC)^2 \\ &= \frac{1}{4}[(2a+b)^2 + (4a+b)^2][(a+2b)^2 + (2a+2b)^2] \\ &= \frac{1}{4}(100a^4 + 300a^3b + 314a^2b^2 + 120ab^3 + 16b^4) \\ &= \frac{1}{4} \cdot 9a^2b^2 = \frac{9}{4}(-800/3)^2 = 400^2.\end{aligned}$$

Hence the answer to our problem is 400.

*Fourth solution (sktech).* This approach uses an idea from the paired AHSME problem, #26 of 1986, to wit, if a right triangle is "level" (has legs parallel to the $x$ and $y$ axes) then the slope of the medians are $m/2$ and $2m$, where $m$ is the slope of the hyptenuse. The median slopes in the AIME problem are 1 and 2 (not in the ratio 1:4), so the triangle is not level. If we can rotate it to make it level, we can find $m$. Then it is not hard to show that the area is $L^2|m|m^2+1$, where $L = 60$ is the length of the hypotenuse.

So let $\theta$ be the angle of a rotation that makes the triangle level, and let $\alpha$ and $\beta$ be the angles of elevation of the medians in the leveled triangle, so that $\tan\beta = 4\tan\alpha = 2m$. Then

$$\tan(\beta-\theta) = 2, \tan(\alpha-\theta) = 1.$$

Using the tangent difference formula, and setting $z = \tan\theta$, one obtains two algebraic equations in $m$ and $z$, from which one finds that $m = (9 \pm \sqrt{65})/4$ and thus Area $= 400$.

4. (300) Since there is no carrying in the addition, the ones column must add to 2, the tens column to 9, the hundreds column to 4 and the thousands column to 1 for each simple ordered pair of non-negative integers summing to 1492. To get a single digit $d$ as the sum of two digits, there are $d+1$ ways:

$$0+d,\ 1+(d-1),\ 2+(d-2),\ldots,d+0.$$

Thus the number of simple ordered pairs of non-negative integers that sum to 1492 is $(1+1)(4+1)(9+1)(2+1) = 300$.

5. (480) First we note that the graph of the given equation is symmetric with respect to the $x$-axis, since the replacement of $y$ by $-y$ does not change the equation. Consequently, it suffices to assume that $y \geq 0$, find the area enclosed above the $x$-axis, and then double this area to find the answer to the problem.

By sketching both $y = |x/4|$ and $y = |x-60|$ on the same graph, as shown in the first figure below, we note that $y = |x/4| - |x-60| \geq 0$ only if the graph of $y = |x/4|$ lies above that of $y = |x-60|$. This occurs only between $x = 48$ (where $x/4 = 60 - x$) and $x = 80$ (where $x/4 = x - 60$). In this interval, the graph of $y = |x/4| - |x - 60|$ with the $x$-axis forms $\triangle ABC$, as shown in the second figure. The base of this triangle is $80 - 48 = 32$, its altitude (at $x = 60$) is 15, hence its area is 240. Kite $ABCD$ is the graph of $|x - 60| + |y| = |x/4|$; the area enclosed by it is $2 \cdot 240 = 480$.

6. (588) Rewrite the given equation in the form

$$(y^2 - 10)(3x^2 + 1) = 3 \cdot 13^2,$$

and note that, since $y$ is an integer and $3x^2 + 1$ is a positive integer, $y^2 - 10$ must be a positive integer. Consequently, $y^2 - 10 =$ 1, 3, 13, 39, 169 or 507, implying that $y^2 =$ 11, 13, 23, 49, 179 or 517. Since the only perfect square in the second list is 49, it follows that $y^2 - 10 = 39$, implying that $3x^2 + 1 = 13$, $x^2 = 4$ and $3x^2y^2 = 12 \cdot 49 = 588$.

7. (070) Since both 1000 and 2000 are of the form $2^m 5^n$, the numbers $a$, $b$ and $c$ must also be of this form. More specifically,

$$a = 2^{m_1} 5^{n_1}, \qquad b = 2^{m_2} 5^{n_2}, \qquad c = 2^{m_3} 5^{n_3}, \tag{1}$$

where the $m_i$ and $n_i$ are non-negative integers for $i = 1, 2, 3$. Then, in view of the definition of $[r, s]$, and since

$$[a, b] = 2^3 5^3, \qquad [b, c] = 2^4 5^3, \qquad [c, a] = 2^4 5^3, \tag{2}$$

the following equalities must hold:

$$\max\{m_1, m_2\} = 3, \quad \max\{m_2, m_3\} = 4, \quad \max\{m_3, m_1\} = 4, \tag{3}$$

and

$$\max\{n_1, n_2\} = 3, \quad \max\{n_2, n_3\} = 3, \quad \max\{n_3, n_1\} = 3. \tag{4}$$

To satisfy (3), we must have $m_3 = 4$, and either $m_1$ or $m_2$ must be 3, while the other one can take the values of 0, 1, 2, or 3. There are 7 such ordered triples, namely (0,3,4), (1,3,4), (2,3,4), (3,0,4), (3,1,4), (3,2,4) and (3,3,4).

To satisfy (4), two of $n_1$, $n_2$ and $n_3$ must be 3, while the third one ranges through the values of 0, 1, 2 and 3. The number of such ordered triples is 10; they are (3,3,0), (3,3,1), (3,3,2), (3,0,3), (3,1,3), (3,2,3), (0,3,3), (1,3,3), (2,3,3) and (3,3,3).

Since the choice of $(m_1, m_2, m_3)$ is independent of the choice of $(n_1, n_2, n_3)$, they can be chosen in $7 \cdot 10 = 70$ different ways. This is the number of ordered triples $(a, b, c)$ satifying the given conditions.

8. (120) *First solution.* Let $v_1$ denote Al's speed (in steps per unit time) and $t_1$ his time. Similarly, let $v_2$ and $t_2$ denote Bob's speed and time. Moreover, let $v$ be the speed of the escalator, and let $x$ be the number of steps visible at any given time. Then, from the information given,

$$v_1 = 3v_2, \qquad v_1 t_1 = 150, \qquad v_2 t_2 = 75. \tag{1}$$

From (1) follows that

$$t_2/t_1 = 3/2. \tag{2}$$

We also know that $x = (v_2 + v)t_2 = (v_1 - v)t_1$, from which we have $v = (x - 75)/t_2 = (150 - x)/t_1$, and hence

$$t_2/t_1 = (x - 75)/(150 - x). \tag{3}$$

Therefore, from (2) and (3), upon setting their right sides equal, we find that $x = 120$.

Assume that Al and Bob start at the same time from their respective ends of the escalator. Then the number of steps initially separating them is the same as the number of visible steps on the escalator. Hence, to solve the problem, we must find the number of steps each of them takes until they meet, and then add these two numbers.

Since Al can take $3 \cdot 75 = 225$ steps while Bob takes 75 steps, it follows (from $150 = (2/3) \cdot 225$) that Al walks down the escalator in 2/3 of the time it takes Bob to walk up. Therefore, they meet 2/5 of the way from the bottom of the escalator. To that point, Al takes $(3/5) \cdot 150 = 90$ steps, while Bob takes $(2/5) \cdot 75 = 30$ steps. As indicated above, the sum of these, 120, is the number of visible steps of the escalator.

*Second solution.* The second solution illustrates a general way to find the number of exposed steps on a moving escalator: find a friend to start simultaneously at the opposite end, and simply add the number of steps you have each taken before you meet.

Such escalator problems were rather popular at one time. This one was fashioned after a problem of Henry Dudeney (1857-1930), England's most famous creator of puzzles.