1. In the Wrangle, 4 kids are on team A, and 5 on team B. How many kids in all?
2. Each card has one of 4 suits and one of 13 ranks. How many cards are in a deck?
3. How many positive integer divisors has 72?
4. How many paths are there from A to B through the grid in Figure 1, traveling only down or to the right?
5. How many paths are there from A to B in the 3×3 grid in Figure 2, traveling only down or to the right?

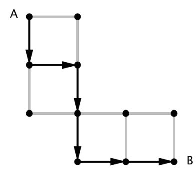


Figure 1

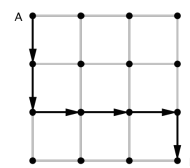


Figure 2

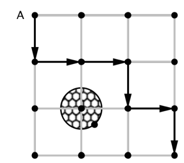
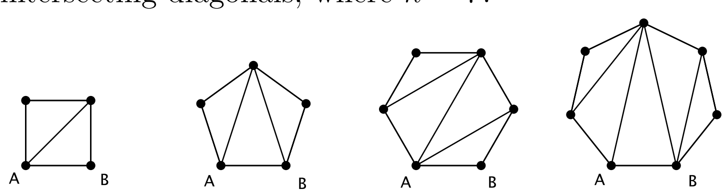
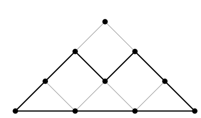


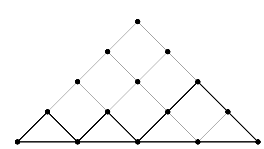
Figure 3

6. How many paths are there from A to B in Figure 3? Watch out! Aggressive bees have taken over one intersection, and must be avoided.
7. How many paths of length 10 are there through a 5×5 grid (not shown) from the upper left corner to the bottom right corner?
8. In how many ways can a 2" $\times$ 10" rectangle be tiled by 2" $\times$ 1" dominoes?
9. In how many ways can a (plane convex) polygon of n sides be divided into triangles by $(n-3)$ non-intersecting diagonals, where $n = 7$?

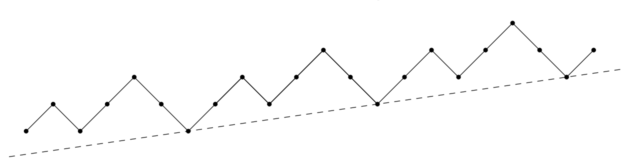


10. In how many ways can you *validly* insert 2 pairs of parentheses into the arithmetic expression $7 - 2 \times 3 - 10$ so that it can unambiguously be evaluated without resorting to PEMDAS? How about 3 pairs of parentheses into $48 - 24 \div 2 + 2 \times 2$? And 4 pairs of parentheses into an expression with 5 operations?
11. How many mountain ranges fit in the $n = 5$ grid?


$n = 3$


$n = 4$

12. How many distinct periodic mountain ranges are there of slope $1/(2n+1)$, where $n = 5$?



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A **Math Circle** is an enrichment activity for K–12 students or their math teachers, which brings them into direct contact with mathematically sophisticated leaders, fostering a passion and excitement for deep mathematics in the participants. Math Circles combine significant discovery and excitement about mathematics through problem solving and exploration.

SIGMAA on Circles

The Special Interest Group of the MAA on Math Circles for Students and Teachers (**SIGMAA MCST**) supports those interested in creating or participating in Math Circles by hosting a number of events at the national meetings each year, such as contributed paper sessions where circle leaders can share about their experiences, demos illustrating how a math circle could be run, a Math Wrangle competition, and in some years, mini-courses to help prepare future Math Circle leaders.

Our SIGMAA webpage is `sigmaa.maa.org/mcst`, and offers math circle notes from past national meetings, math wrangles rules, and further information about our group.

Please join our SIGMAA! If you are a member of the MAA, please call Membership Services Department at 1-800-331-1622 to join the SIGMAA on Circles (SIGMAA MCST). But you must be an MAA member, so visit `www.maa.org/membership/membership-categories` to join.

National Association of Math Circles

The National Association of Math Circles (`www.mathcircles.org`) provides resources and support for Math Circles and other similar informal mathematics education programs. They aid in the creation and sustainability of Math Circle programs, through the development and sharing of resources for Math Circle leaders, as well as the development of a national network of Math Circles.

Math Teachers' Circle Network

The Math Teachers' Circle Network (`www.mathteacherscircle.org`) is a project of the American Institute of Mathematics that links together Math Teachers' Circles around the country. Their goals are to encourage teachers as mathematicians, connect mathematics professors with K–12 education, and build a K–20 community of mathematics professionals committed to fostering a love and understanding of mathematics among all students. The MTC Network is committed to helping start new Teachers' Circles and also to the continued support of existing Circles in ways that contribute to their long-term sustainability, including through a Circle Mentor program.

Answers, Solutions, and More Information

Answers

1. 9   2. 52   3. 12   4. 9   5. 20   6. 11   7. 252   8. 89   9. 42   10. 42   11. 42   12. 42

Solutions and Commentary

1. Add: $4 + 5 = 9$. Since teams A and B are disjoint, we use the Addition Principle.
   Addition Principle: If sets A and B are disjoint, then $\#(A \cup B) = \#(A) + \#(B)$.

2. Multiply: $4 \times 13 = 52$ (assuming no jokers). This uses the Multiplication Principle.
   Multiplication Principle: If A and B are sets, then $\#(A \times B) = \#(A) \times \#(B)$.

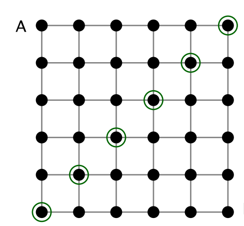
3. The number $72 = 2^3 3^2$, so each positive integer divisor has form $2^a 3^b$ where $a \in A = \{0, 1, 2, 3\}$ and $b \in B = \{0, 1, 2\}$. Thus, there are $|A| \times |B| = 4 \times 3 = 12$ divisors.

4. Label the juncture point C. Every path from A to B must pass through C. There are 3 paths from A to C and 3 paths from C to B. Thus, by the multiplication principle, there are $3 \times 3 = 9$ paths from A to B.

5. Each path from A to B passes through exactly one of the four points along the diagonal from the bottom left corner to the top right corner. There are 1×1, 3×3, 3×3 and 1×1 such paths, for a total of 20.

6. Call a path from A to B "bad" if it passes through the swarm of bees, and "good" if it avoids them. By the addition principle, the total number of paths (20, by problem #5) is the sum of the good paths and the bad paths. By problem #4, there are 9 bad paths. Therefore, there are 11 good paths.

7. This is similar to problem #5. Each path must pass through exactly one of the 6 special points along the diagonal from the bottom left to the top right corner, and the number of paths through any one of these points can easily be computed by the multiplication principle. The total number of paths from A to B is $1 \times 1 + 5 \times 5 + 10 \times 10 + 10 \times 10 + 5 \times 5 + 1 \times 1 = 252$.

   Bonus: Can you prove that $\binom{n}{0}^2 + \binom{n}{1}^2 + \cdots + \binom{n}{n}^2 = \binom{2n}{n}$?

8. Generalize: let F_n denote the number of ways a 2" $\times n$" rectangle can be tiled by 2" $\times$ 1" dominoes. We are asked to find F_{10}. It's easy to enumerate that $F_1 = 1$, $F_2 = 2$, and $F_3 = 3$. For larger n, note that the set of all valid tilings of the 2" $\times n$" rectangle is the disjoint union of those tilings whose leftmost tile is vertical, and those tilings whose leftmost tiles are horizontal. What remains is a 2" $\times (n-1)$ or a 2" $\times (n-2)$ rectangle.

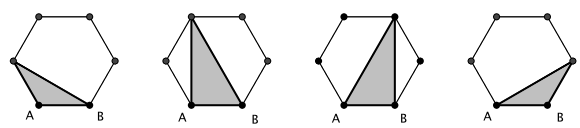
2" $\times n$		2" $\times (n-1)$		2" $\times (n-2)$"
F_n tilings	=	F_{n-1} tilings	+	F_{n-2} tilings

This leads to the recursion $F_n = F_{n-1} + F_{n-2}$, which easily allows us to complete the following table:

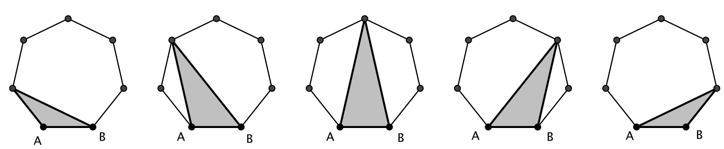
n	1	2	3	4	5	6	7	8	9	10
F_n	1	2	3	5	8	13	21	34	55	89

The remaining problems are all related, in that they are examples of Catalan Numbers. Historically, Leonhard Euler (1707–1783) discussed the Polygon Dissection Problem with Christian Goldbach (1690–1764) in 1751, but this sequence of numbers is named after Eugène Charles Catalan (1814–1894) who studied the Towers of Hanoi puzzle.

9. Let D_n denote the number of ways to triangulate a plane convex polygon of n sides using $(n-3)$ non-intersecting diagonals. It's easy to enumerate that $D_3 = 1$, $D_4 = 2$, and $D_5 = 5$. For larger n, we can use the addition principle by noting that side AB is part of a triangle, and if this triangle is removed from the figure, we must also triangulate (up to) two remaining polygons.



Use this diagram to argue that $D_6 = D_5 + D_3 \times D_4 + D_4 \times D_3 + D_5 = 5 + 2 + 2 + 5 = 14$.



For $n = 7$, this idea yields that $D_7 = D_6 + D_3 \times D_5 + D_4 \times D_4 + D_5 \times D_3 + D_6$, allowing us to complete this table:

n	3	4	5	6	7
F_n	1	2	5	14	42

10. Let P_n denote the number of ways to insert $(n-1)$ pairs of parentheses into an arithmetic expression involving n operations, so that it can be unambiguously evaluated.

For any expression with one operation (like $3 + 4$), there's just one way to insert 0 parentheses: don't insert them! So let's agree that $P_1 = 1$. An expression like $4 + 10 \div 2$ can be evaluated in two ways: $(4 + 10) \div 2$ and $4 + (10 \div 2)$, so $P_2 = 2$. Can you find all 5 ways to evaluate $7 - 2 \times 3 - 10$, showing that $P_3 = 5$?

For larger n, consider the *last* operation to be evaluated. For example, there are four last possible operations to evaluate each possible disambiguation of $48 - 24 \div 2 + 2 \times 2$:

$(48){-}(24{\div}2{+}2{\times}2)$, $(48{-}24){\div}(2{+}2{\times}2)$, $(48{-}24{\div}2){+}(2{\times}2)$, $(48{-}24{\div}2{+}2){\times}(2)$

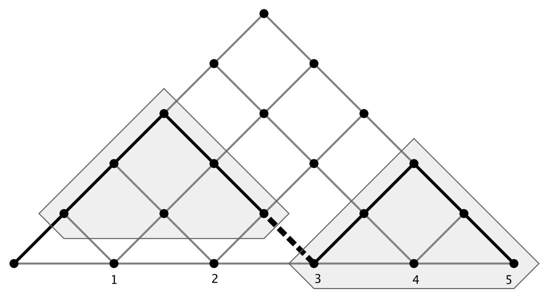
Then the number of ways to complete the disambiguation is the product of the number of ways to disambiguate the sub-expression to the left with the number of ways to disambiguate the sub-expression to the right of this operation.

This example shows that $P_4 = P_3 + P_1 \times P_2 + P_2 \times P_1 + P_3 = 5 + 1 \times 2 + 2 \times 1 + 5 = 14$.

Can you explain the equation $P_5 = P_4 + P_1 \times P_3 + P_2 \times P_2 + P_3 \times P_1 + P_4$? This allows us to complete the table:

n	1	2	3	4	5
P_n	1	2	5	14	42

11. Let M_n denote the number of mountain ranges of size n. There are n possible points where the mountain range can first touch the bottom line. For example, when $n = 5$, the picture shows a mountain range that first touches the bottom line at point #3. The number of mountain ranges that *first* touch the bottom line at point #3 is the product of the number of mountain ranges that fit into each of the shaded areas on either side, or $M_2 \times M_2$. Considering all $n = 5$ such points on the bottom line leads to the following identity:



$$M_5 = M_4 + M_1 \times M_3 + M_2 \times M_2 + M_3 \times M_1 + M_4$$

This allows us to complete the table:

n	1	2	3	4	5
M_n	1	2	5	14	42

Amazingly, the last three (very different) problems all yield the same sequence of numbers, the Catalan Numbers.

n	1	2	3	4	5	6	7
D_n			1	2	5	14	42
P_n	1	2	5	14	42		
M_n	1	2	5	14	42		

12. There are several methods to derive a closed formula for the nth Catalan number, and this problem may give the simplest way. First, notice that the periodic mountain ranges of slope $1/(2n+1)$ are essentially the same as the mountain ranges of size n.

Each periodic mountain range of slope $1/(2n+1)$ contains $n+1$ up and n down segments. There are $\binom{2n+1}{n}$ sequences of $n+1$ up and n down segments, that when repeated form periodic mountain ranges of slope $1/(2n+1)$. But this method generates each periodic mountain range $(2n+1)$ times, yielding the formula

$$C_n = \frac{1}{2n+1}\binom{2n+1}{n}.$$

Online References

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2. Tom Davis' Problem Set for the Berkeley Math Circle:
   `http://mathcircle.berkeley.edu/BMC6/pdf0607/catalan.pdf`
3. UCLA Math Professor Igor Pak has amassed a wonderful resource page:
   `http://www.math.ucla.edu/~pak/lectures/Cat/pakcat.htm`
4. Richard Stanley, the Enumerative Combinatorialist, has a very challenging and interesting problem set concerning Catalan Numbers:
   `http://www-math.mit.edu/~rstan/ec/catalan.pdf`
5. *Bijective Proof Problems*, Richard P. Stanley. `www-math.mit.edu/~rstan/bij.pdf`
6. And of course, the Wikipedia page is pretty good!
   `http://en.wikipedia.org/wiki/Catalan_number`

Printed References

For those who still believe in books!

1. *100 Great Problems of Elementary Mathematics, their history and solution*, Heinrich Dörrie, Dover Publications 1965. This includes the Polygon Dissection Problem.
2. *A Dozen Questions about Pascal's Triangle*, James Tanton, Math Horizons, vol. 16, No.2 (Nov. 2008), #5 Parentheses Numbers. A nice collection of questions that includes Catalan Numbers.
3. *The Art and Craft of Problem Solving*, Paul Zeitz, 2e, 2006, John Wiley & Sons. An explicit formula for Catalan Numbers is derived using generating functions. A must read!
4. *A Course in Combinatorics*, J.H. van Lint & R.M. Wilson, Cambridge Univ., 1992.
5. *A Path to Combinatorics for Undergraduates–counting strategies*, Titu Andreescu & Zuming Feng, Birkhäuser, 2004.
6. *Catalan Numbers, Their Generalization, and Their Uses*, Peter Hilton & Jean Pedersen, The Mathematical Intelligencer, vol. 13, No.2 (1991).
7. *The Book of Numbers*, John H. Conway & Richard K. Guy, Springer Verlag, 1996. This is the source for the periodic mountain ranges of slope $1/(2n+1)$ approach.