Remodeling the Floor

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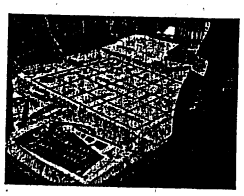
There are many wonderful mathematical problems and results related to tiling. This circle addresses one idea that I will call *remodeling.* It may be combined with a variety of sessions related to tiling problems. The standard tiling question asks if it is possible to completely fill a given region with non-overlapping tiles from a given set. The classic example asks if it is possible to tile a chess board after two opposite corners are removed using only dominos. See page 4 problem 1.

This session has been prepared to illustrate a large group of separate math circle sessions that are related to math circle leaders.

Few students would be able to discover the standard coloring argument for the classic tiling problem without seeing something similar first. However, it is possible to bring students to the solution with a series of warm-up questions: Can you tile a 3×3 rectangle with dominos? a 4×4, a 5×5? If one square was to be removed from a 3×3 grid and the result tiled with dominos, what are the possibilities for that square? What about a 5×5 grid? Via these examples students will discover **coloring arguments.**

A different idea that can be explored is the idea of substitution tilings. If one can tile scaled copies of a tile set with the tiles, the one can tile the plane or increasing larger regions with tiles from the tile set. Page 3 problem 4 and page 4 problem 3 are problems of this type. An example that would be appropriate for page 2 problem 7 would be to start with the letter A, and replace every A by AB and every B by A - then repeat. Matthias Kawski suggests using such problems to introduce proofs by induction. It is a nice way to demonstrate induction without the "just replace n by $n+1$."

It is possible to make nice tiles and grids using foam core. It is also possible to use the environment. Many buildings have square tiles on the floor or elsewhere. Grid areas may be marked off using masking tape or wet-erase markers. Shoes may be used as tiles...



See Chapter 6 of Tanton's *Solve This* [7] for some introductory tiling problems. See [3] for an introduction to substitution tilings. See [4] for many interesting ideas related to Polyminos. Some webpages related to Math Circle tiling sessions are listed below. See

http://www.mathcircles.org/node/748

for a list of tiling handouts for math circles and further links. (You will need to log into www.mathcircles.org to see all of the files.)

http://juliarobinsonmathfestival.org/problems/TilingTorment.pdf

for a Julia Robinson Math Festival hand out on tilings and

https://www.mathcircles.org/files/tiling-handout.pdf

for the handout that Kawski used with the foam core tiles and grids.

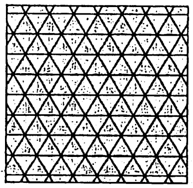
The 2015 JMM circle: After introducing tiling problems in a math circle there are many different related sessions one can do. The demonstration session for the 2015 JMM is on *remodeling.* It is a good idea to give participants one-sided copies of pages 3, 4, 5, and 6. Problem 4 on page 6 is a good puzzle for people familiar with tiling problems who wish to understand the remodeling idea. The basic idea is to deform one grid into a different grid. There are several ways to make such deformations. The easiest way is to collapse an edge in each of the cells. One can also replicate tiles. Using these two ideas the problems on page 4 may be converted into the problems on page 3, the problems on pages 5 and 6 may be converted into the problems on page 4.

The pattern here is that page 3 problems are related to triangular grids, page 4 problems are related to square grids, page 5 problems are related to Cairo Pentagonal grids, and page 6 problems are related to hexagonal grids. Furthermore problem N on page P is essentially the same as problem N on any other page. A *semi-regular* grid is a tiling of the plane by regular polygons so that there is a symmetry of the grid taking any one vertex to any other vertex. The *dual* of a grid is created by placing a vertex at the centroid of each polygon in a grid, and connecting two centroids by an edge whenever the corresponding polygons share an edge http://en.wikipedia.org/wiki/Tiling_by_regular_polygons. In fact it is interesting to consider tiling problems based on any semi-regular grid or its dual.

It is very important for mathematicians to create new problems. However we rarely show students how to do so. An easy way to get started is to just modify existing questions. This is the main point for students of the 2015 JMM demo. Participants should be able to fill in the blank problems on each page by modifying problems from other pages.

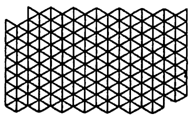
> Give someone a math problem and they'll have fun for a while. Show them how to make up questions and they will have fun for a lifetime.

This handout is available at http://www.mathcircles.org/node/748.



Simple Tiling

1. (a) Tile Set: two tri-octaminos 

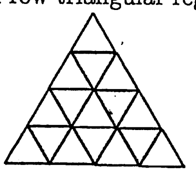
   Region: 

   (b)

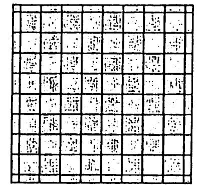
2.

3.

4. Tile Set: All orientations of the tri-triomino 

   Region: 64 row triangular region with the top triangle removed. (The 4 row region is shown.) 

5.

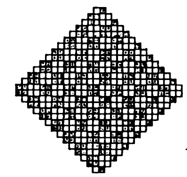


Simple Tiling

1. (a) Tile Set: Dominos Region: 8×8 grid with opposite corners removed.

   (b)

2. Tile Set: Skew Tetronimos 

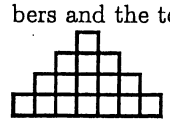
   Region: 

3. Tile Set: L Trionimos 

   Region: 64×64 grid with any one square removed.

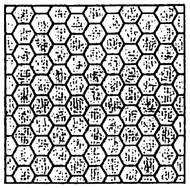
4. Tile Set: Decreasing Trionimos 

   Region: Symmetric stack of 64 rows of contiguous rows with the first 64 odd numbers and the top or far left square removed. (Figure shows top four rows.)

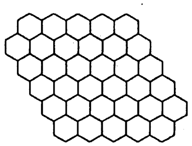
5. Tile Set: L Tetronimos (all orientations) 

   Region: 10×10 grid. Hint: Put a 1 in each square in the odd numbered rows and a 5 in each square of the even numbered rows.



Simple Tiling

1. (a) Tile Set: two hex-dominos 

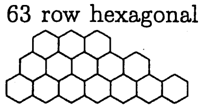
   Region: 

   (b)

2.

3.

4. Tile Set: Translations of the following three hexa-triominos 

   Region: 63 row hexagonal region with 3 on 5 on ... on 127. (The 3 row region is shown.) 

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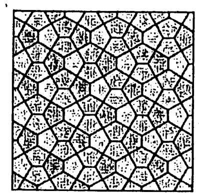
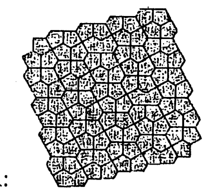
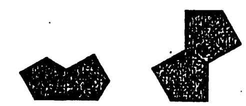


Simple Tiling

1. (a) Tile Set: All orientations of the Cairo Domino 

   Region: 

   (b) Tile Set: All orientations of the two Cairo Dominos 

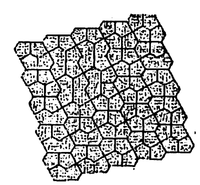
   Region: Same as part (a)

2.

3.

4.

5. Tile Set: All orientations of the following Cairo L tetromino 

   Region: 

Other directions: There are many related ideas that may be explored.

One natural topic is the **classification of semi-regular tilings.** This follows by noticing that the sum of angles in a triangle is 180°, and extending to the sum of angles in an n-gon and the interior angle of a regular n-gon. From here by creating an organized list of all collections of regular polygons so that the sum of the interior angle of one of each is 360° will lead to the classification.

Following in the same line, one may consider regular and semi-regular tilings of the sphere or hyperbolic space. Given Euler's formula as explored in many math circle activities (brussels sprouts, cris-cross, string polytopes, curvature and Gauss-Bonnet) `http://math.berkeley.edu/~giventh/difgem.pdf` one may **classify the Platonic solids** `http://www.mathcircles.org/node/810` and `http://www.mathcircles.org/node/975`. Wall paper groups may be classified in a similar way and the crystallographic point groups follow by adding the crystallographic restrictions (only rotations of orders 2, 3, 4, and 6) to the spherical wall paper patterns [2].

Tiling problems in which there are only a finite number of possible tile placements may be solved by exhaustive search. It is possible to make NP complete examples. In such a tiling question, the certificate to be checked in polynomial time is especially clear – it is just a tiling of the region. Such tiling questions may be phrased as identifying the feasible points in an integer programming problem, so may branch off into this area of mathematics [6].

There are a number of variants on tiling questions, in which one includes anti-tiles and perhaps even allows tiles to go outside of the region. A list of several different variants is listed on the "Basic Questions" page. A number of these reduce to linear algebra, [6]. In fact, when a finite tile homology problem has no solution, one can prove this to be the case with a coloring argument. In this case every cell in the region is labeled with a rational number so that any tile placement covers an integer total. However the total of the labels will fail to be an integer. See problem 5 and [6]. The $\mathbf{Z}_2$ tile homology problem goes under the name **Stomp.** See `http://www.msri.org/attachments/jrmf/activities/Stomp.pdf` and `http://www.msri.org/attachments/jrmf/activities/StompGrids.pdf`

Making the Soma cube uses coloring ideas familiar from tiling and is a nice thing to do after demonstrating the basic tile coloring argument. It is nice to start with the classification question of what are all non-parallelpiped three dimensional ployminos with 3 or four cubes. There are seven and they may fit together to make a $3 \times 3 \times 3$ cube: `http://www.mathcircles.org/files/SomaCube.pdf`

A more sophisticated argument to show that a tiling problem has no solution is given in Propp's pedestrian approach [5] and chapter 19 of [7].

Propp's pedestrian approach is an exposition of a special case of the tile homotopy group due to Conway and Lagarias [1]. Of course the original source is a wonderful paper. We owe many math circle sessions to Conway and his collaborators. Thurston wrote a lovely *Monthly* article on these groups [8]. It is written for undergraduates and introduces ideas of combinatorial group theory. Of course it may be adapted to middle and high school students. The Turning Laughter into Insight activity includes an introduction to tile homotopy for domino tilings, the ABBA language and mystery braid, and the Cayley complex of finitely presented groups. Each one of these could be a separate math circle session. Indeed this activity was designed to demonstrate a collection or related math circles. `https://www.mathcircles.org/content/turning-laughter-insight`.

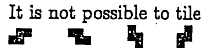
Tile space with circles: Is it possible to cover $\mathbb{R}^2$ with disjoint non-degenerate circles? Is it possible to cover $\mathbb{R}^3$ with disjoint non-degenerate circles? See `https://www.mathcircles.org/problem/508` (You must log in first.)

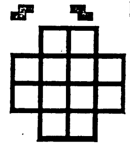
The "Basic Questions" page lists a variety of variations on the basic tiling question. Due to time constraints we will stick to the basic tiling question for this session. However, any of the other variants could be studied in math circle sessions.

It is clear that a solution to a simple tiling problem is also a solution to the rest of the "Basic Questions" tiling problems. Similarly, a solution to an interior homology problem is automatically a solution to all but possibly the simple tiling problem. In addition, a solution to a $\mathbf{Z}_N$ interior homology problem is a solution to a $\mathbf{Z}_N$ homology problem.

Here is an example that shows that solutions to various homology tiling problems need not solve other versions of the "Basic Questions" tiling problems.

It is not possible to tile the following Aztec Diamond, AZ(2), with the Skew Tetronimos

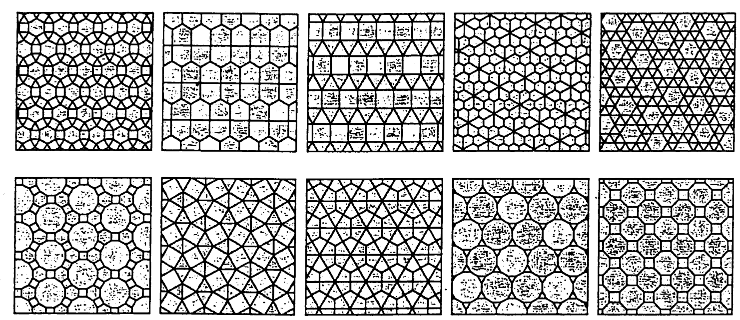




It is possible to cover it so each square is covered by two skew tetronimos. Thus AZ(2) has order two in the skew tetronimo homology group, and there is no solution to the corresponding homology tiling problem. However it is trivial in the $\mathbf{Z}_3$ tile homology group, so one may adjust the three-way lights in skew tetronimo blocks of rooms until all the lights are on dim.

Basic Questions Given a Region and Set of Tiles

Simple Tiling	Can the region be completely filled with tiles from the tile set with no overlaps?
Interior Homology	Imagine that each cell in the region represents a room. When you place a tile in the region you turn the thermostat up one degree in each room covered by the tile. When you anti-place a tile you turn the temperature down one degree. Tiles and anti tiles may overlap. Can you raise the temperature in each room (cell) by one degree?
Homology	Imagine that each cell in the region represents a room. When you place a tile it may be partially in the region and partially out of the region. You turn the thermostat up one degree in each room (cell) of the region covered by the tile. When you anti-place a tile you turn the temperature down one degree. Tiles and anti tiles may overlap. Can you raise the temperature in each room (cell) by one degree (while leaving the thermostats at the same level in all of the rooms outside of the region)?
$\mathbf{Z}_N$ Interior Homology	Imagine that each cell in the region represents a room. When you place a tile in the region you turn an N-way light up one notch in each room covered by the tile. After N clicks a light turns off. Further clicks will turn in on brighter until it goes off again. Tiles may overlap. Can you turn the lights on dim in each room (cell) with all of the lights out of the region off?
$\mathbf{Z}_N$ Homology	Imagine that each cell in the region represents a room. When you place a tile it may be partially in the region and partially out of the region. This turns an N-way light up one notch in each room (cell) covered by the tile. After N clicks a light turns off. Further clicks will turn in on brighter until it goes off again. Can you turn the lights on dim in each room (cell) with all of the lights out of the region off?



References

[1] J. H. Conway and J. C. Lagarias. Tiling with polyominoes and combinatorial group theory. *J. Combin. Theory Ser. A*, 53(2):183–208, 1990.

[2] John H. Conway, Heidi Burgiel, and Chaim Goodman-Strauss. *The symmetries of things.* A K Peters, Ltd., Wellesley, MA, 2008.

[3] Natalie Priebe Frank. A primer of substitution tilings of the Euclidean plane. *Expo. Math.*, 26(4):295–326, 2008.

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[5] James Propp. A pedestrian approach to a method of Conway, or, A tale of two cities. *Math. Mag.*, 70(5):327–340, 1997.

[6] Michael Reid. Tile homotopy groups. *Enseign. Math. (2)*, 49(1-2):123–155, 2003.

[7] James S. Tanton. *Solve This: Math Activities for Students and Clubs.* Mathematical Association of America, Washington, DC, first edition, 2001.

[8] William P. Thurston. Conway's tiling groups. *Amer. Math. Monthly*, 97(8):757–773, 1990.