Lewis Carroll's Barbershop Puzzle

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Russell on Carroll

Here's what Bertrand Russell said about Lewis Carroll's work on logic:

> *I think he was very good at inventing puzzles in pure logic. When he was quite an old man, he invented two puzzles he published in a learned periodical,* Mind, *to which he didn't provide answers. And the providing of answers was a job, at least so I found it.*

Russell on Euclid . . .

For a comparison, here's what he said about Euclid:

> *His definitions do not always define, his axioms are not always indemonstrable, his demonstrations require many axioms of which he is quite unconscious. A valid proof retains its demonstrative force when no figure is drawn, but very many of Euclid's earlier proofs fail before this test . . . [T]he value of his work as a masterpiece of logic has been very grossly exaggerated.*

. . . And Aristotle

And Aristotle:

> *I conclude that the Aristotelian doctrines with which we have been concerned in this chapter are wholly false, with the exception of the formal theory of the syllogism, which is unimportant. Any person in the present day who wishes to learn logic will be wasting his time if he reads Aristotle or any of his disciples.*

Lewis Carroll's Work in Logic

- Lewis Carroll was the pseudonym of mathematician Charles Dodgson. He generally published his mathematical work under the latter name, and his more popular work under the former name.
- Later in his career, he published two books and two papers on logic, with further work left unpublished when he died in 1898.
- Interestingly, he published his scholarly work in logic under the Carroll name.

What Research Papers *Used* to Look Like

"What, nothing *to do?" said Uncle Jim. "Then come along with me down to Allen's. And you can just take a turn while I get myself shaved."*

"All right," said Uncle Joe. "And the Cub had better come too, I suppose?"

The "Cub" was me, *as the reader will perhaps have guessed for himself. I'm turned* fifteen *more than three months ago; but there's no sort of use in mentioning* that *to Uncle Joe: he'd only say, "Go to your cubbicle little boy!" or "Then I suppose you can do cubbic equations?" or some equally vile pun. He asked me yesterday to give him an instance of a Proposition in A. And I said, "All uncles make vile puns". And I don't think he liked it.*

The Puzzle Without the Melodrama

Caroll's paper was called, "A Logical Paradox." Here's a succinct formulation of the puzzle that captures Carroll's intent:

- We imagine a barbershop staffed by three people: Allen, Brown, and Carr.
- At all times during business hours, at least one of the three must be in the store.
- Thus, if Allen and Carr are out of the shop, then Brown must be in.
- But Allen prefers never to be alone outside the shop. So, when he goes out he always takes Brown with him.

The Puzzle Without the Melodrama

We now reason as follows: We start with the rules

(1) If Carr and Allen are out, then Brown is in.

(2) If Allen is out, then Brown is out.

But item 1 is equivalent to

(3) If Carr is out, then (if Allen is out, then Brown is in).

The Puzzle Without the Melodrama

Common sense tells us that "If Allen is out, then Brown is out," and "If Allen is out, then Brown is in," are contraries. They cannot both be true.

Since the first of these statements is given as (2), we conclude that:

(4) It is not the case that: If Allen is out, then Brown is in.

But now we apply modus tollens to statements (3) and (4) to get

(5) Carr is not out.

This seems to show that rules (1) and (2) imply that Carr can never be out. This is plainly false, because if either Allen or Brown is in, then Carr can go out without any difficulty. And that's the paradox!

How do we Interpret Conditionals?

Carroll intended his puzzle to illustrate difficult questions in the interpretation of conditionals. In particular, he wondered if a conditional whose antecedent was false could be regarded as "legitimate."

In the following issue of *Mind*, Warren Johnson wrote:

> *I presume Mr. Lewis Carroll's position to be that the problem raised by him in the last number of* Mind *presents a conflict between common sense and the rules of logic. It appears to me certain that the rules of logic when properly applied agree with the results of common sense.*

The Johnson/Russell Solution

Johnson (1894) now presented the solution that has, no doubt, already occurred to you. Specifically, it is not true that $p \rightarrow q$ and $p \rightarrow \neg q$ are contraries. Under the rules for the material conditional, both can be true if p is false.

In Carroll's puzzle, the conditionals "If Allen is out, then Brown is in," and "If Allen is out, then Brown is out," imply that Allen must be in. Therefore, taking all of our premises together, the conclusion is, "If Carr is out then Allen is in," and "If Allen is out then Carr is in."

Johnson further noted that proofs by contradiction require that we regard conditionals with false antecedents as legitimate.

Russell Concurs

In *The Principles of Mathematics* (1903), Russell agreed with this solution:

> *The principle that false propositions imply all propositions solves Lewis Carroll's logical paradox .... The assertion made in the paradox is that, if p, q, r be propositions, and q implies r, while p implies that q implies not-r, then p must be false, on the supposed ground that "q implies r" and "q implies not-r" are incompatible. But in virtue of our definition of negation, if q be false both these implications will hold: the two together, in fact, whatever proposition r may be, are equivalent to not-q. Thus the only inference warranted by Lewis Carroll's premises is that if p be true, q must be false, i. e. p implies not-q.*

Jones' Solution

E. E. C. Jones (1905) demurred. She notes that $p \to q$ and $p \to \neg q$ do not necessarily imply $\neg p$ because there might be suppressed premises.

She gives as an example:

1. If that is your dog, your dog is a Welsh collie.
2. If that is your dog, your dog is not a Welsh collie.

> *In (1) the suppressed premiss is "That is a Welsh collie", while in (2) it is "That is not a Welsh collie", and clearly (1) and (2) as thus understood cannot be true together. The contradictoriness of the expressed consequents of (1) and (2) could only disprove their common antecedent, if those consequents followed from that expressed antecedent, in which case that expressed antecedent must in itself be contradictory.*

Wilson is Impatient

John Cook Wilson (1905) thought this whole thing was silly:

> *It is difficult to see how this argument, which is no paradox but a paralogism, should have been taken seriously, and that the fallacy in it should not have had its true character made clear nor have received the simple treatment of which it is capable. It is merely a question of language, and hardly needs the technicalities of Logic.*

Wilson's Solution

Wilson argued that the problem was purely linguistic.

Specifically, he argued that the statement "If Carr is out, then if Allen is out, Brown is in," does not mean "If Carr is out, then (if Allen is out then Brown is in)." In other words, this is not a a conditional statement whose consequent is itself a conditional.

Instead it means simply, "If Carr and Allen are out, then Brown is in." From this he concludes, "Thus, the conclusion is that the *conjunction* of Carr's absence and Allen's absence is impossible, not that Carr's absence is impossible. This is the true *reductio ad absurdum* argument.

What's so Great About the Material Conditional?

If someone says, "if p, then q," and someone else says, "If p then (not q)," then they sure *seem* like they are at odds with each other. Burks and Copi (1950) write:

> *[I]f one politician argues that if the Conservatives win the election in 1950 then Britain's economic situation will improve, and another argues that if the Conservatives win in 1950 then Britain's economic situation will not improve, there is genuine disagreement. It would indeed be an overzealous proponent of material implication who would expect the disputants to agree that another labor victory at the polls would make both statements true!*

Causal Implication Versus Material Implication

Burks and Copi suggest construing the conditionals as causal implications. That is, they effectively suggest construing "If p, then q," as meaning "p causes q."

The statements "p causes q," and "p causes not $\neg q$," really are contraries. They cannot both be true, meaning this aspect of Carroll's argument now goes through.

Does this mean we have a paradox after all?

There Still is No Paradox

This time the problem lies elsewhere. Schematically, if p and q together cause r, that does not imply that either p or q individually cause r.

Therefore, the statement, "If Carr and Allen are out, then Brown is in," is no longer logically equivalent to, "If Carr is out, then if Allen is out, then Brown is in," and this dissolves the paradox.

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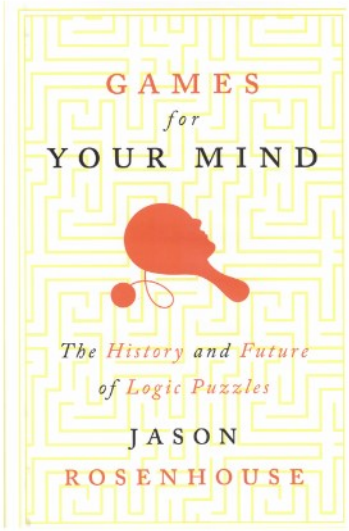

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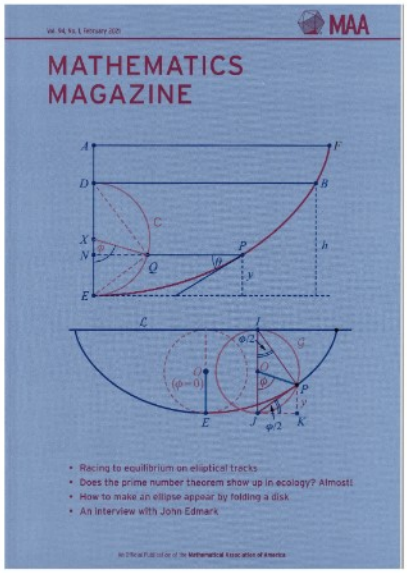

Thanks

Thank You!