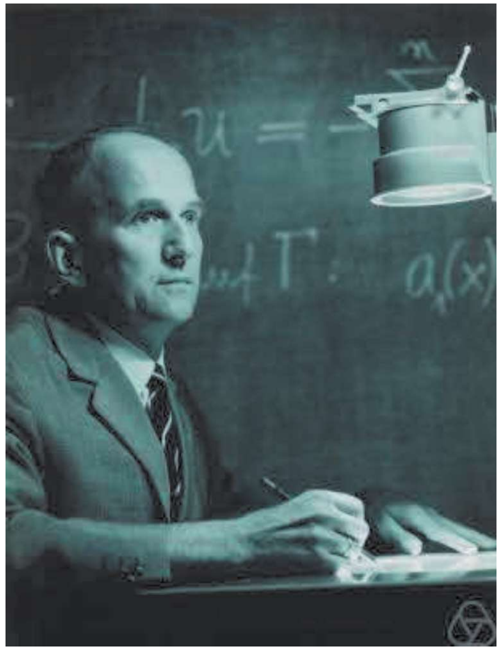
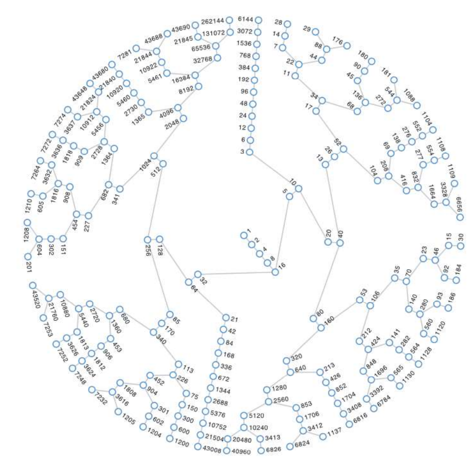


What is the Collatz Conjecture?

Russell Coe and Alexander Atwood
Suffolk County Community College
MAA MathFest Conference (virtual)
August 7, 2021

Proposed by Lothar Collatz in the 1930's, the Collatz Conjecture is one of the most difficult open problems in mathematics. We will describe the conjecture, demonstrate how it works, talk about why proving it is so difficult, and describe recent significant work by Terence Tao on this subject.



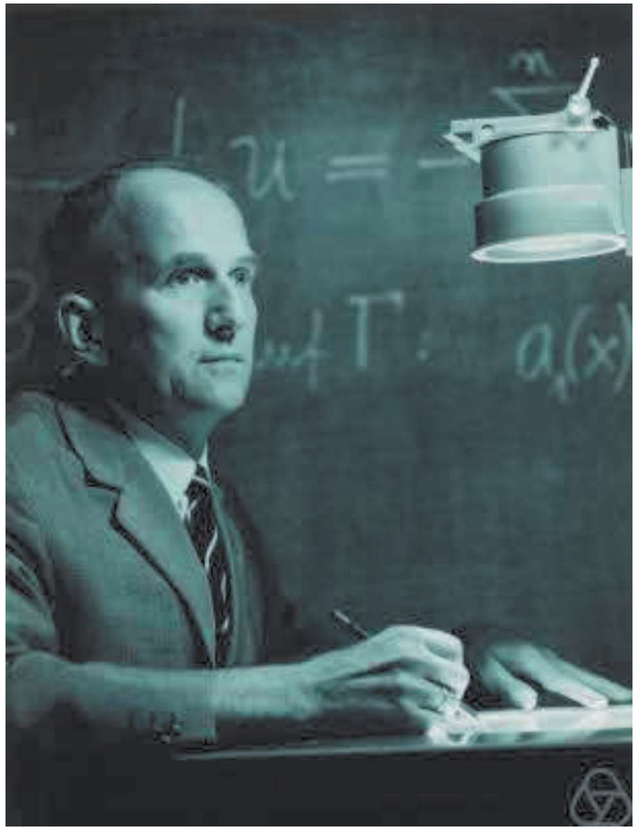
Overview of Presentation

- Mathematical Statement of the Collatz Conjecture
- Illustration of the Collatz Conjecture
- Brief history of the Conjecture
- Why the Conjecture is interesting for our students
- How one might try to prove the Conjecture
- Why the Conjecture is so difficult to prove
- The important work of Terence Tao (2019 paper)
- Variations on the Collatz Conjecture
- References

Mathematical Statement of the Collatz Conjecture

Let n be a positive integer. Then a sequence of integers can be defined as follows:

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases}$$



Lothar Collatz (1910-1990)

(proposed in 1937)

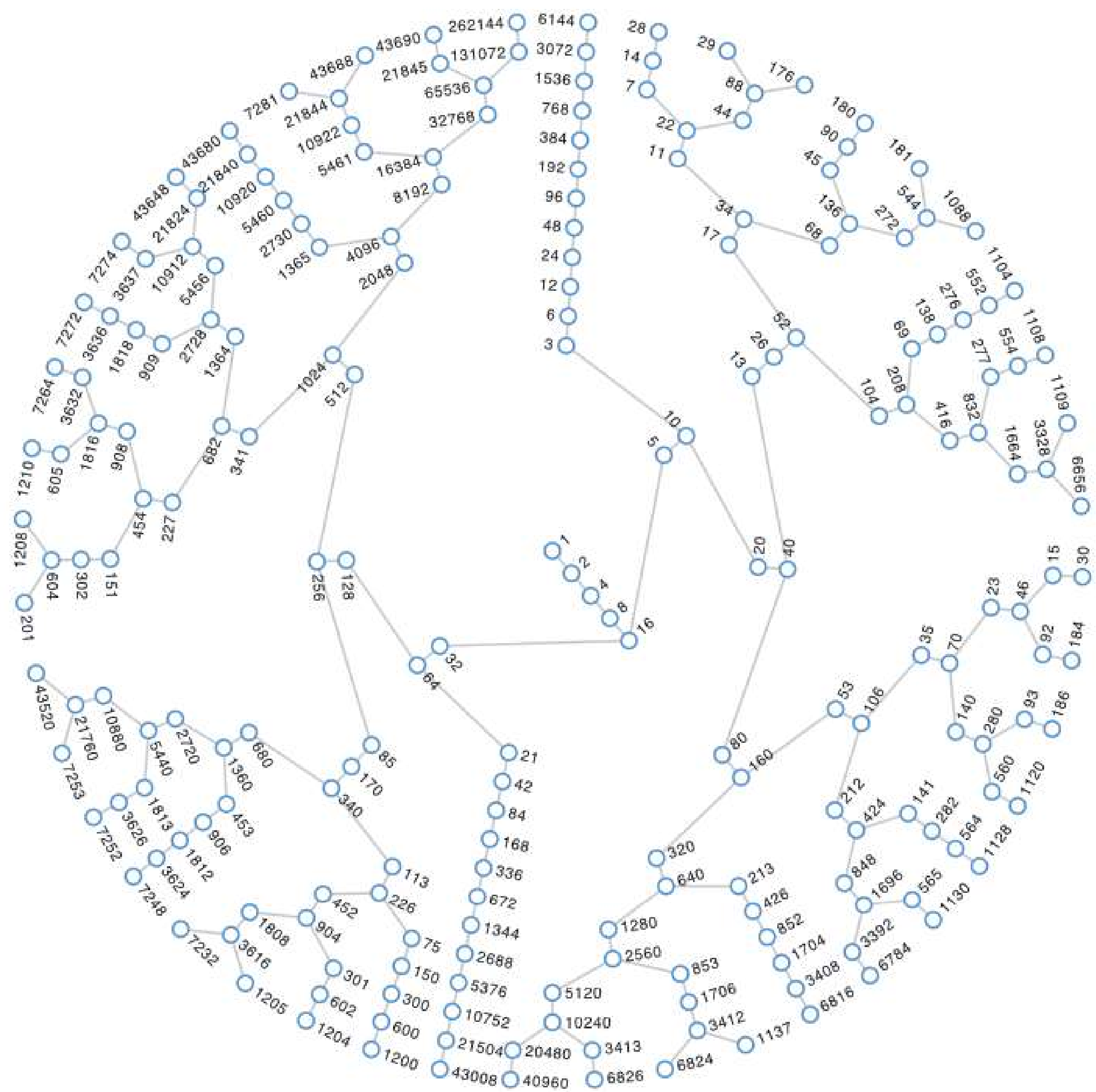
The Collatz Conjecture says that this sequence always leads to the number 1 when one starts with any positive integer.

Illustration of How the Collatz Conjecture Works

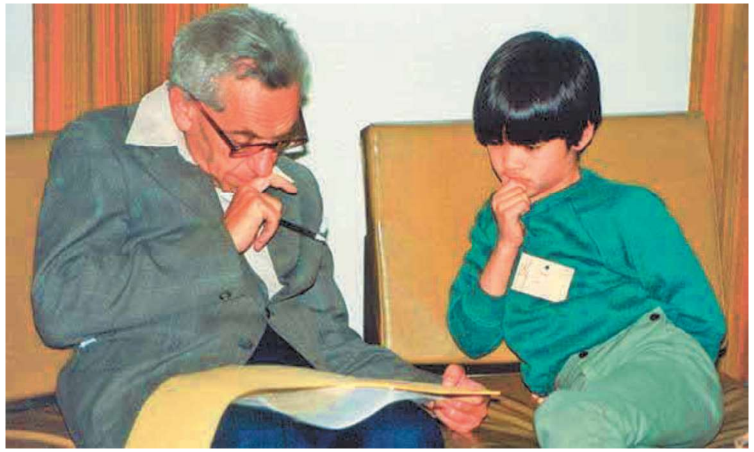
$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n + 1 & \text{if } n \text{ is odd} \end{cases}$$

- 1
- 2 → 1
- 3 → 10 → 5 → 16 → 8 → 4 → 2 → 1
- 4 → 2 → 1
- 5 → 16 → 8 → 4 → 2 → 1
- 6 → 3 → 10 → 5 → 16 → 8 → 4 → 2 → 1

Each sequence of integers is an “orbit”



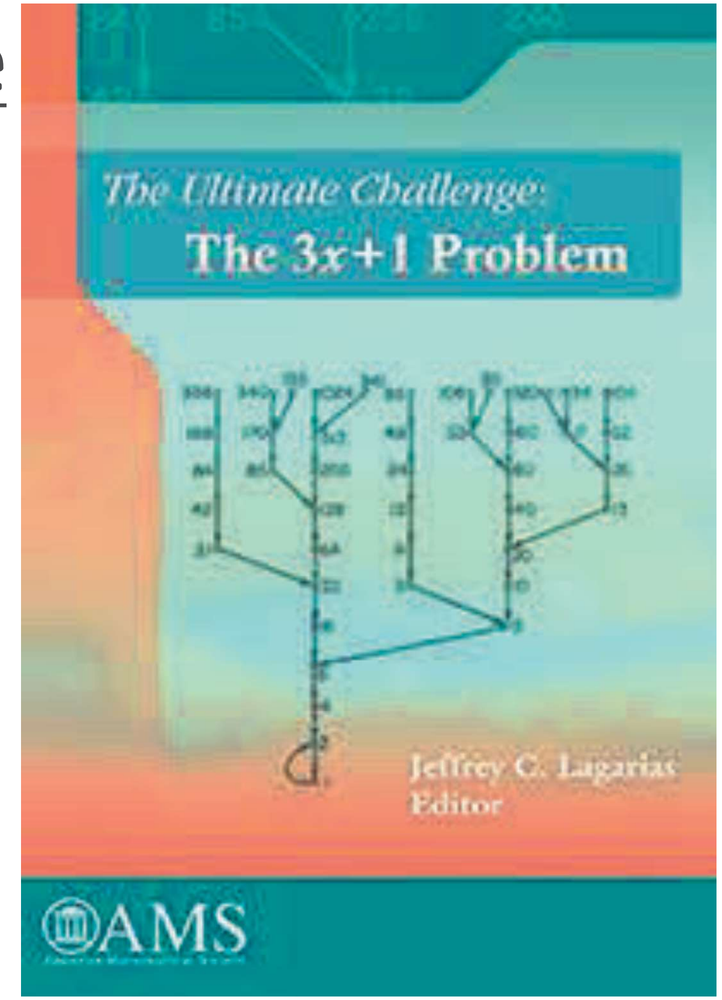
To this date nobody has proven that the Collatz conjecture is true for all positive integers. Using computers, it has been verified to to work for every positive integer less than 2^{68} (approximately 2.9×10^{20}).



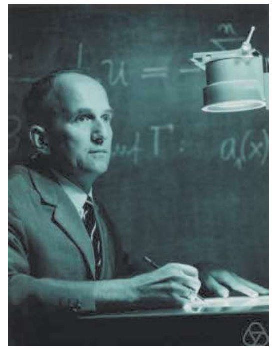
The prolific mathematician Paul Erdos said that "Mathematics is not ready for such problems."

Erdos offered a $500 prize for the proof or disproof of the Collatz Conjecture.

History of the Collatz Conjecture

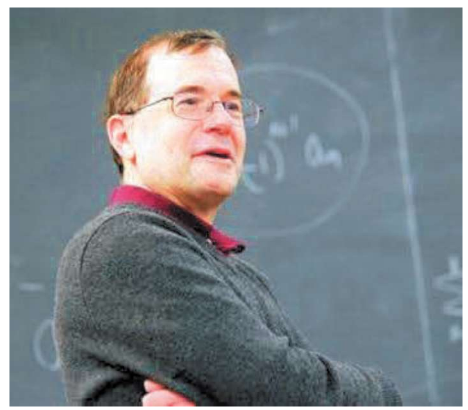


- Lothar Collatz
  (proposed in 1937)



- Stanislaw Ulam (1940's)

- Jeffrey Lagarias
  (about 1967 to present)
  The Ultimate Challenge:
  The 3x+1 Problem (2010 book)



- Terence Tao (2006 Fields Medalist)
  "Almost all orbits of the Collatz map attain almost bounded values" (2019 Paper)

Why the Collatz Conjecture is Interesting for Our Students

- Suitable to discuss in a Discrete Math Class:

  - Methods of Proof, i.e. Mathematical Induction

  - Examination of Cases is very important

- Possible to discuss in a Math for Liberal Arts Class:

  - Simple to state conjecture and illustrate with examples.

  - Some Math Problems are Unsolved!

- Patrick Honner, who wrote “The Simple Math Problem We Still Can’t Solve,” in Quanta Magazine, defined the following function to help explain the difficulty in proving the Collatz conjecture.

- g(n) =n + 1 if n is odd and n/2 if n is even. The following are three sample orbits.

- 10, 5, 6, 3, 4, 2, 1, 2, 1, 2, 1, 2, …

- 11, 12, 6, 3, 4, 2, 1, 2, 1 ,2, …

- 27, 28, 14, 7, 8, 4, 2, 1, 2, 1, …

Nollatz Conjecture: Each orbit, regardless of the initial value, will reach 1.

- If n is even, the next number is $n/2 < n$.

- For half of the natural numbers n, $g(n) < n$.

- If n is odd, $g(n) = n + 1$ is even, and $g(g(n)) = (n+1)/2 = n/2 + \frac{1}{2}$, which is less than n if n is at least 3.

- Thus the orbit will always trend downwards (except when the initial value is 1.) Once the sequence hits 1, one gets 1,2,1,2,….

- With the Collatz function, however, we don't know that a sequence will trend downward.

$$f(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \\ 3n+1 & \text{if } n \text{ is odd} \end{cases}$$

- For even natural numbers n, the next number will be half of n.

- If n is odd, the next number will be 3n+1 which is even. The next number after that will be (3n+1)/2 = 1.5n + 0.5, which is larger than n.

- If (3n+1)/2 is even, then the next number will be (3n+1)/4 = 3n/4 +1/4, which is less than n if n > 1. For half of the odd numbers, the number will be less than n after three steps.

- Terence Tao wanted to find an appropriate sample of numbers and prove that if almost all (close to 100%) of the numbers in the sample end up at 1 or close to 1, then almost all numbers would end up at 1 or close to 1.

- One difficulty was trying to pick an appropriate sample of numbers that would represent all numbers.

- Different numbers have different properties (even, odd, multiples of 3, etc.) Some numbers differ from each other in only subtle ways.

- Terence needed to weight his sample to reflect the proportions of numbers with different properties. For example, if 90 percent of numbers shared a particular property, then he would want a sample where about 90% had that property.

- The difficulty in choosing such a sample is that after choosing a sample and applying the conjecture a few times to each member of the sample, the resulting sample may no longer have the desired distribution of properties.

- Terence decided to avoid multiples of 3, as a hailstone sequence quickly loses all multiples of 3. (One more than a multiple of 3 has no factor of 3 in it. Dividing it by 2 wouldn't change that. If a number is an even multiple of 3, after applying the function by the number of times the number is divisible by 2, it would be multiplied by 3 and increased by 1.)

- Terence also chose to use more numbers that when divided by 3 gave a remainder of 1 and not very many numbers that when divided by 3 gave a remainder of 2. A number with a remainder of 2 would eventually lead to a number with a remainder of 1.

- The sample he chose maintained its character after the Collatz algorithm is repeatedly applied.
- Using this sample he was able to prove that almost all (99% or more) of all starting values eventually reach a value below 200.
- As it was known that each starting value up to 200 eventually leads to 1, the conjecture is proved to be true for at least 99% of all numbers.

- I decided to investigate what would happen if instead of adding 1 when n is odd, subtracting 1. (Of course, it would be naïve of me to assume that others who have studied the conjecture had not looked into this.)
- I found what others found. Any initial positive integer eventually leads to one of the following loops.
- f(n) = 3n-1 if n is odd or n/2 if n is even.
- 2,1,2,1,2,1….
- 5, 14, 7, 20, 10, 5,…
- 17, 50, 25, 74, 37, 110, 55, 164, 82, 41, 122, 61, 182, 91, 272, 68, 34, 17….

I also considered what would happen if we used the function $h(n) = 5n+1$ if n is odd or $n/2$ if n is even. Some numbers lead to one of the following loops, where other numbers appeared to not lead to a loop.

- 1, 6, 3, 16, 8, 4, 2, 1
- 5, 26, 13, 66, 33, 166, 83, 416, 208, 104, 52, 26, 13, 66, 33
- 17, 86, 43, 216, 108, 54, 27, 136, 68, 34, 17

Starting with the number 7 appeared to not lead to a loop.

In Wikipedia it was pointed out that if we allow any integer, positive, negative, or 0, to be the initial value in a sequence, it appears (not proven) that one will always end up in one of the four following loops.

- 1, 4, 2, 1,…
- -1, -2, -1,…
- -5, -14, -7, -20, -10, -5, …
- -17, -50, -25, -74, -37, -110, -55, -164, -82, -41, -122, -61, -182, -91, -272, -136, -68, -34, -17, …

References

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- Collatz conjecture, published in Wikipedia.