Some FIBONACCI SURPRISES
The Power of a Picture

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Most everyone is familiar with the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ...$$

and many of their delightful properties and unexpected appearances in mathematics.

TOUGH QUESTION : *Are* 1 *and* 144 *the only square Fibonacci numbers?*

Each number beyond the initial pair of 1s is the sum of the two entries preceding it.

QUESTION: *If this pattern persists, what number should precede the pair of* 1*s? Extend the Fibonacci sequence towards the left. What pattern of numbers appears?*

In this essay we present a few more appearances of the Fibonacci numbers (some of which are possibly new to the world). We do this first as a series of puzzlers to mull upon, which is Part I of this essay. In Part II we present one final puzzler also intimately connected to Fibonacci numbers, a classic, about path counting, and then show in part III how this visual interpretation of these numbers unites the first twelve puzzles and explains their Fibonacci connection in one fabulous and spectacular fell swoop. Part IV ends the essay with some ideas for research.

Just to be clear with notation we set:

$$F_1 = 1,\ F_2 = 1 \text{ with } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 3.$$

When it is needed we'll also declare $F_0 = 0$.

PART I: THE PUZZLERS

1. PLACING PARENTHESES

There are 13 ways to place parentheses within a string of three symbols avoiding nested of parentheses:

```
                    *   *   *
(*) *  *     (*  *) *     (*)(*) *     (*)(*  *)
 *(*) *       *  (*  *)   (*) * (*)    (*  *)(*)
 *  *(*)     (*  *  *)     *(*)(*)     (*)(*)(*)
```

There are 2 ways to accomplish this feat with a string one symbol long: * and (*); and 5 ways with a string of two symbols: ** (*)* *(*) (**) and (*)(*).

Verify that there are 34 ways to place parentheses in a string of four symbols.

We have so far:

number of symbols	1	2	3	4
count of arrangements	2	5	13	34

Every second Fibonacci number?

2. DOUBLING ONES IN PARTITIONS

There are 4 ways to write the number three as a sum of positive integers, with order of terms considered important. (That is, there are 4 ordered <u>partitions</u> of three.)

$$3 = 2+1 = 1+2 = 1+1+1$$

(ASIDE CHALLENGE: Show, in general, there are 2^{N-1} ways to write a positive integer N into a sum of positive integers with order relevant.)

Suzy is fond of the number 1 and will write it either using her red pen or her blue pen. She writes all other numbers with black pen.

Given that there are now two different types of number "1" we find that there are 13 ways to partition the number three:

$$\begin{aligned} 3 &= 2+\underline{1} = 2+\bar{1} = \underline{1}+2 = \bar{1}+2 \\ &= \underline{1}+\underline{1}+\underline{1} = \underline{1}+\underline{1}+\bar{1} = \underline{1}+\bar{1}+\underline{1} \\ &= \bar{1}+\underline{1}+\underline{1} = \underline{1}+\bar{1}+\bar{1} = \bar{1}+\underline{1}+\bar{1} \\ &= \bar{1}+\bar{1}+\underline{1} = \bar{1}+\bar{1}+\bar{1} \end{aligned}$$

Thirteen is a Fibonacci number. Coincidence?

3. ABEEBA
The language of ABEEBA contains only three letters, A, B and E, and most any combination of letters is a word in this language. The only stipulation is that the letter A can never be immediately followed by the letter E.

For example, AABBBEEEBBBE and AABEA are words in ABEEBA, as is the name of the language itself, but AAABBAE and BEEEAEAAA are not.

There are 3 one-letter word (A, B, and E), 8 two-letter words (AA, AB, BA, BB, BE, EA, EB, EE) and there are 21 three-letter words (all 27 strings of three letters but with words of the form AE* and *AE deleted).

Prove that there are F_{2N+2} N-letter words in ABEEBA.

4. ABOOBA
The language of ABOOBA contains only three letters, A, B and O. Any word in this language begins and ends with A and between any pair of As there are is even number of Os.

For example, there is 1 two-letter word (AA), 2 three-letter words (AAA, ABA) and 5 four-letter words (AAAA, ABAA, AABA, ABBA, AOOA).

a) Verify that there are 13 five-letter words.

b) Prove that there are F_{2N-3} N-letter words for $N \geq 2$.

c) How many N-letter words DON'T begin with AA?

5. ORDERED PARTITION PRODUCTS
This puzzler was inspired by Sam Vandervelde.

If we list all the ordered partitions of the number 3, multiply the terms of each partition together and sum the products, we obtain the answer 8. If we do the same for the number 4, we obtain the answer 21.

3	= 3
1x2	= 2
2x1	= 2
1x1x1	= 1
	8

4	= 4
1x3	= 3
3x1	= 3
2x2	= 4
1x1x2	= 2
1x2x1	= 2
2x1x1	= 2
1x1x1x1	= 1
	21

8 and 21 are Fibonacci numbers. Coincidence?

6. PRODUCTS OF DIFFERENCES
Write the numbers 1 through N in a list and circle two or more of the numbers. Compute the differences between consecutive circled numbers and multiply them together.

For example, with $N = 3$, we obtain the following possibilities:

$$
\begin{array}{lll}
1\ 2\ 3 \rightarrow (2-1) & = 1 \\
1\ 2\ 3 \rightarrow (3-1) & = 2 \\
1\ 2\ 3 \rightarrow (3-2) & = 1 \\
1\ 2\ 3 \rightarrow (2-1)(3-1) & = 1 \\
\hline
 & \ \ 5
\end{array}
$$

The sum of all products is 5.

a) Verify that the sum of all possible products of differences for $N = 4$ is 17.

Let $D(N)$ denote the sum of all possible products of differences starting with the numbers 1 through N. (So $D(3) = 5$ and $D(4) = 17$.) In general, $D(N)$ is not a Fibonacci number.

b) Prove that $D(N) = F_{2N} - N$

7. ODD PARTITIONS
There are 8 ordered partitions of the number six into odd parts:

$$
\begin{aligned}
5+1 = 1+5 = 3+3 & \\
&= 3+1+1+1 = 1+3+1+1 \\
&= 1+1+3+1 = 1+1+1+3 \\
&= 1+1+1+1+1+1
\end{aligned}
$$

If $P_{odd}(N)$ denotes the count of odd ordered partitions of N, prove: $P_{odd}(N) = F_N$.

8. ONE-LESS PARTITIONS
There are 5 ordered partitions of the number six that avoid use of the number 1:

$$6 = 4+2 = 2+4 = 3+3 = 2+2+2$$

If $P_{\hat{1}}(N)$ denotes the count of one-less ordered partitions of N, prove: $P_{\hat{1}}(N) = F_{N-1}$

While we're at it, we should include …

9. Let $P_{1,\,2}(N)$ denote the number of ordered partitions of N using only the numbers 1 and 2. Show that $P_{1,\,2}(N) = F_{N+1}$

10. TRICKY FORMULAS
Consider every second Fibonacci number (the ones with odd index): 1,2,5,13 34, ..
Create triangular arrays with the first few terms of this sequence and add the entries in the array:

```
1 2 5 13
1 2 5
1 2        SUM = 34 -1
1
```

```
1 2 5 13 34
1 2 5 13
1 2 5            SUM = 89 - 1
1 2
1
```

```
1 2 5
1 2      SUM = 13 - 1
1
```

Prove: $N \cdot F_1 + (N-1)F_3 + (N-2)F_5 + \cdots + 1 \cdot F_{2N-1} = F_{2N+1} - 1$

For the even-indexed terms: 1, 3, 8, 21, 55,… prove:

$$N + (N-1)F_2 + (N-2)F_4 + \ldots + 1 \cdot F_{2N-2} = F_{2N}$$

While we're at it, we should include …

11. Prove: $F_a F_b + F_{a+1} F_{b+1} = F_{a+b+1}$

and

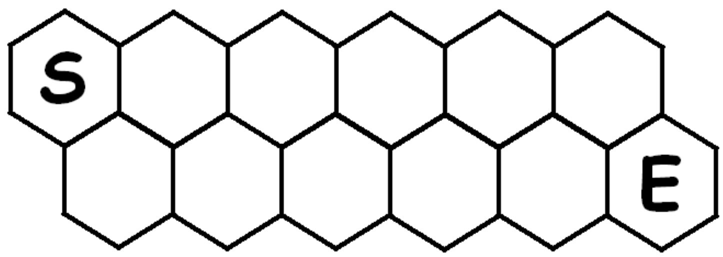
12: Prove that if N is a multiple of a, then F_N is a multiple of F_a.

CHALLENGE: Find formulas for the quotient and remainder upon dividing F_N by F_a even if N is not a multiple of a.

PART II: THE UNITING PUZZLER

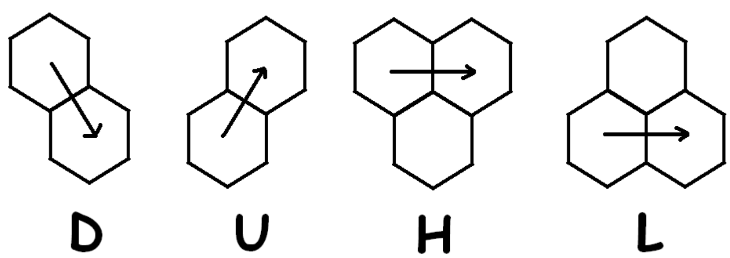
The Fibonacci numbers make an appearance in a classic path counting puzzle:

13. HONEYCOMB WALKS
A bee, starting at cell marked S wants to move to the cell marked E in a honeycomb design consisting of two rows of hexagonal cells.



Each step must move to the right and to a neighbouring cell. This gives four possible types of steps:

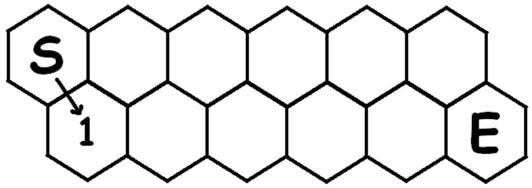
D = downward diagonal
U = upward diagonal
H = high horizontal step
L = lower horizontal step



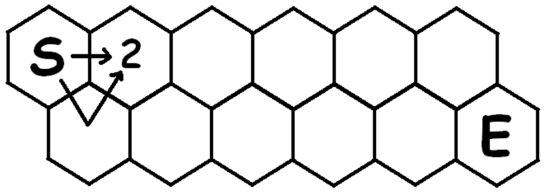
How many different routes are there from S to E using these four types of steps?

As with any complex problem it is often easier to start by examining smaller cases.

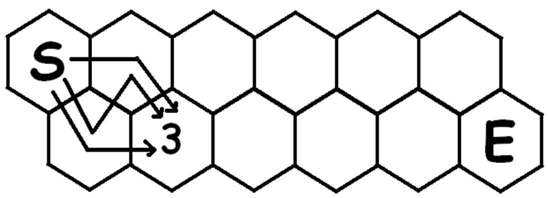
Clearly there is only one route from S to the cell just below it: Take a single D step.



And there are two routes from S to the cell directly to its right: DU or H.



One can check that there are three routes to the next cell along the honeycomb:
DL, DUD. HD.

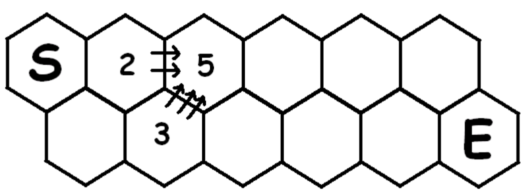


To reach the fifth cell of the honeycomb (the third cell along the top row) one has two options: Either

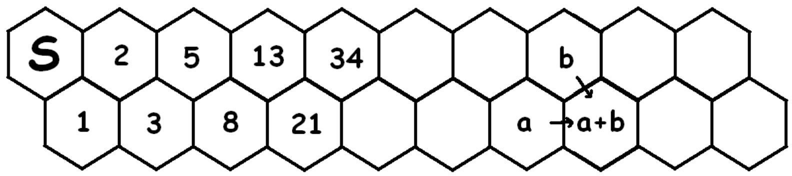
> Reach the cell directly to its left and then take a step H to the right.
> (There are two ways to accomplish this,)

or

> Reach the cell diagonally to its left and then take a diagonal step U.
> (There are three ways to accomplish this.)



In general, the number ways to reach any particular cell of the honeycomb must be the sum of the two counts of paths to each of the two cells to its left.

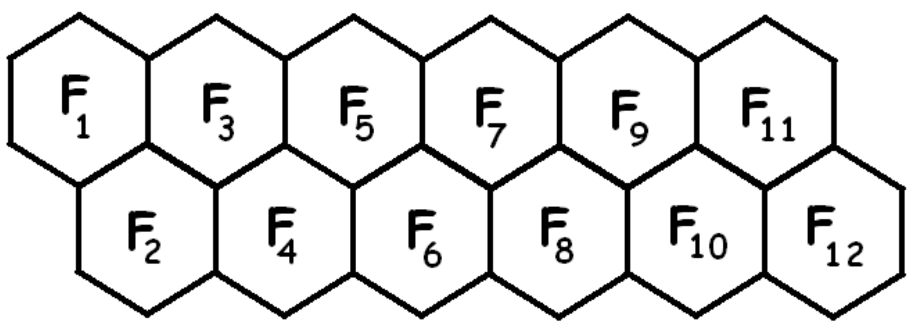


This matches the construction of the Fibonacci numbers: each value at one level is the sum of the values at the two previous levels. (And since our path counting pattern starts with the numbers 1, 2, 3 ,5 , the same as the Fibonacci numbers, we will indeed continue to match the Fibonacci numbers forever more.)

It is compelling to say that there is 1 way to walk to cell S starting at cell S. (This matches the Fibonacci numbers precisely.)

If we number the cells 1, 2, 3, 4, … in a zig-zag pattern from left to right, then we can say:

The number of paths from the leftmost cell to the Nth cell is F_N.

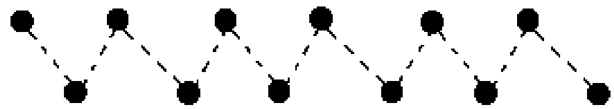


The count of paths that end on an upper cell is a Fibonacci number with an even index.
The count of paths that end on a lower cell is a Fibonacci number with an odd index.

In particular there are $F_{12} = 144$ paths from S to E in puzzle 12.

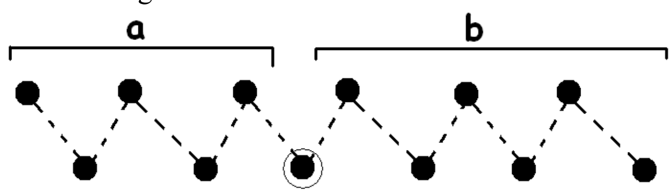
PART III: INTERPRETING PATHS

For convenience, let's draw dots instead of hexagonal cells and denote the zigzag of cells with dotted line segments,



If we are walking to the N th dot, there are $N-1$ line segments along this zigzag line.
There are F_N honeycomb paths to that N th dot.

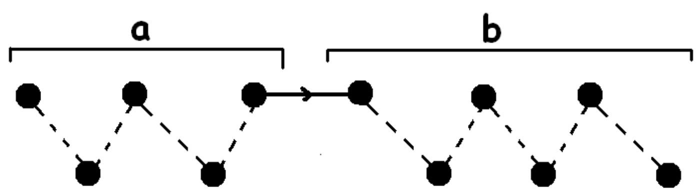
ASIDE: Consider the $(a+1)$th dot along the zigzag line and assume that $N = a+b+1$ so that there are a dots to its left and b dots to its right.



Any one of the F_N honeycomb paths in this diagram must either got through the $(a+1)$th dot or avoid it.

To walk a path that goes through this $(a+1)$th dot we must first walk to it in one of F_{a+1} ways, and then walk the a path from it through the remaining b dots, which can be done in one of F_{b+1} ways.

To walk a path that avoids this $(a+1)$th dot, we must take a horizontal step opposite it. This means first walking one of F_a ways to the a th dot, taking a horizontal step, and walking a path along the remaining b dots in one of F_b ways. There are $F_a F_b$ such paths in all.



This proves puzzle 11:

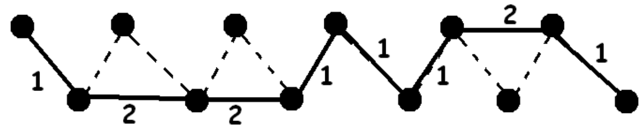
11. $F_a F_b + F_{a+1} F_{b+1} = F_{a+b+1}$

Let's now interpret honeycomb paths along the zigzag line in different ways.

PATH INTERPRETATION 1

Think of the zigzag line on N dots as a string of $N-1$ line segments.

Each U or D step in a path to the N th dot traces a single line segment and each L or H step skips over two line segments. We thus see that each path corresponds to a way of writing $N-1$ as a sum of 1s and 2s. (And conversely, each partition of $N-1$ into 1s and 2s corresponds to a path.)



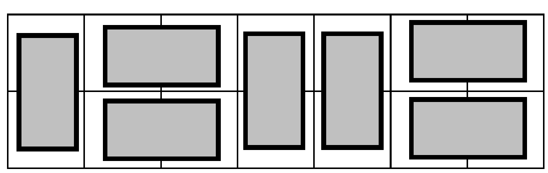
$1+2+2+1+1+1+2+1 = 12$

As there are F_N paths, there are this many partitions of $N-1$. If we adjust the index by one, we have explained puzzle 9:

9. There are F_{N+1} ways to partition N using the numbers 1 and 2 with order considered relevant.

QUESTION: How many ways are there to partition N into 1s and 2s if order is NOT considered relevant?

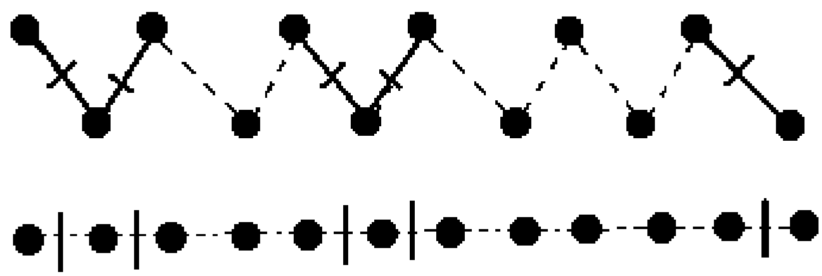
QUESTION: In how many ways can one tile a $2\times N$ strip of square cells with 1×2 dominos?



PATH INTERPRETATION 2

Think of the zigzag line on N dots as a string of N dots with the $N-1$ line segments as the spaces between the dots.

For any path to the N th dot interpret a U or a D step as a "separator."



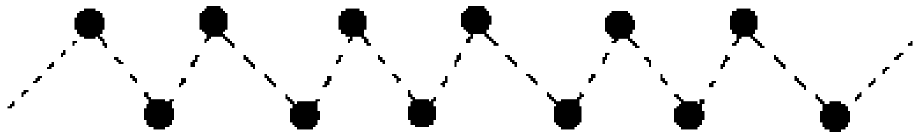
Since U and D steps move the path from the top to bottom row, or vice versa, there are an odd number of dots between each separator. Thus we see that honeycomb paths to the N th dot correspond to partitions of N into odd parts. (And, conversely, any ordered partition of N into odd parts can be interpreted as a path to the N th dot.)

As there are F_N paths in all, we have established puzzler 7:

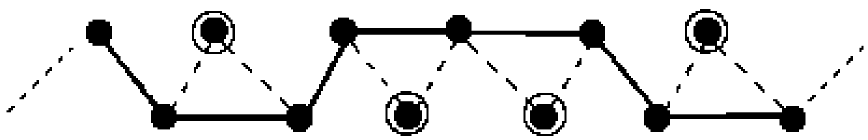
7. $\boxed{P_{odd}(N) = F_N}$

PATH INTERPRETATION 3

Place false line segments at the beginning and end of a zigzag line of $N-1$ dots and think of each dot as a space between the $\left(N-2\right)+2=N$ line segments.

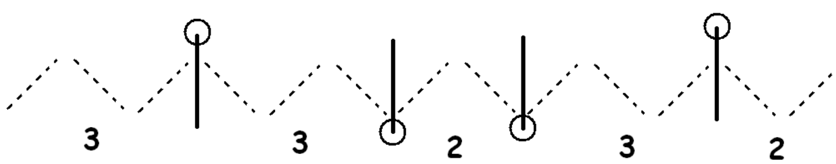


For any honeycomb path from the first to the $\left(N-1\right)$th dot circle the dots it misses.



The first dot will not be circled, the final dot will not be circled, and no two dots that are consecutive along the zigzag line will both be circled.

The circled dots thus partition the N line segments into parts containing 2 or more segments each. This represents a partition of the number N that avoids use of 1.



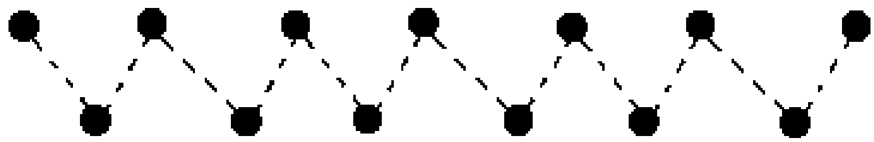
3+3+2+3+2 = 13

And, conversely, any ordered partition of the number $N+1$ into parts of size greater than 1 corresponds to a honeycomb path to the $\left(N-1\right)$th dot. This correspondence between partitions and paths solves puzzler 8:

$\boxed{\textbf{8. } P_{\hat{1}}\left(N\right)=F_{N-1}}$

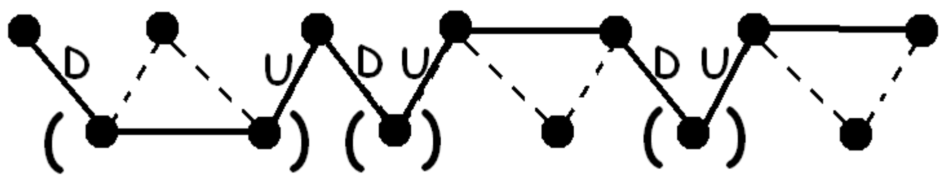
PATH INTERPRETATION 4

Consider the set of honeycomb paths that end on a given odd-numbered dot $2N+1$.



There are N dots along the bottom row.

If we regard a D step as a left parenthesis and a U step as a right parenthesis, then any honeycomb path to the $(2N+1)$th dot corresponds to a set of non-nested parentheses about the N symbols along the bottom row:



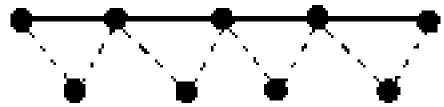
This solves puzzle 1:

1. The number of ways to place non-nested parentheses about some or all of the elements of a string of N symbols is F_{2N+1}, the number of paths to the $(2N+1)$th dot in the honeycomb.

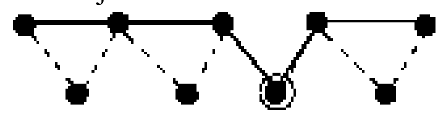
QUESTION: How many ways are there to place non-nested parentheses among N dots if the last dot must be within a parenthesis pair?

Let's go further and examine honeycomb paths touch certain dots on the bottom row.

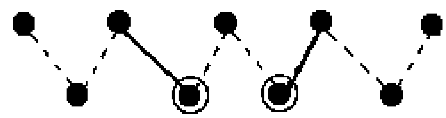
There is one honeycomb path that touches no dots on the bottom row.



There are N honeycomb paths that touch just one dot:

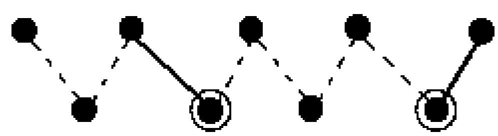


Suppose a path touches just two consecutive dots on the bottom row:



There are $(N-1)$ choices for which pair of points this could be, and between those pairs of points there are F_3 ways to connect them with a honeycomb path. (All other steps must be Hs.) Thus there are $(N-1)F_3$ honeycomb paths that touch two consecutive dots on the bottom row.

Suppose a path touches dots on the bottom row within a span of three dots:



There are $(N-2)$ possible locations for this span of dots and there are F_5 honeycomb paths connecting the leftmost and rightmost bottom dots in this range. Thus there are $(N-2)F_5$ honeycomb paths in all with this property.

Similarly, there are $(N-3)F_7$ honeycomb paths that touch a range of dots on the bottom row 3 places apart, and $(N-4)F_9$ paths that touch a range of dots on the bottom row 4 places apart, and so on.

We have now categorized all F_{2N+1} honeycomb paths in this diagram, and so we conclude:

$$1+N+(N-1)F_3+(N-2)F_5+\cdots+1\cdot F_{2N-1}=F_{2N+1}$$

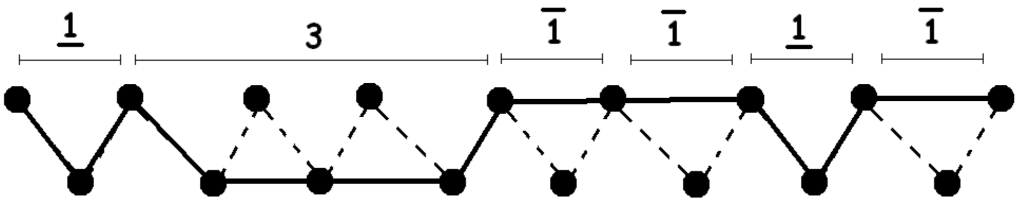
This establishes the first part of puzzle 10:

10 a) $N\cdot F_1+(N-1)F_3+(N-2)F_5+\cdots+1\cdot F_{2N-1}=F_{2N+1}-1$

PATH INTERPRETATION 5

Again consider honeycomb paths to a given odd numbered dot $2N+1$.

There are N spaces between the $N+1$ dots of the top row. Any honeycomb path to the $(2N+1)$th dot matches a partition of N with two types of "one:"



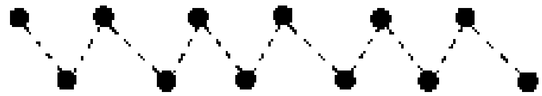
$$\underline{1}+3+\overline{1}+\overline{1}+\underline{1}+\overline{1} = 8$$

We have:

2. The number of ordered partitions of N with two types of 1 is F_{2N+1}.

PATH INTERPRETATION 6

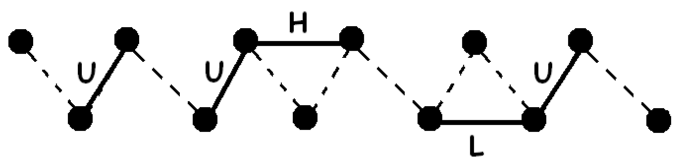
Consider the set of all honeycomb paths that end on a given even numbered dot $2N+2$.



Any honeycomb path is certainly described by a sequence of moves given by the letters D, U, L or H.

In fact, one need not mention the letter D when listing the letters of a path!

For example, from UUHLU and knowing that we are ending on a dot on the bottom row, we deduce that the path must contain the segments as shown below:



> [The segment for the very first U must be at a position as to the far left as possible, otherwise a letter L or H must have appeared before it. In fact each line segment for each letter mentioned must appear as far to the left as it can.]

We see that UUHLU must be the path DUDUHDLUD.

> [No double Ds will appear as they would contain a U between them, but such a U would have been mentioned.]

Thus every path can be described in as a word with the three letters: U, L, and H.

One checks that all pairs UU, UH, UL, HU, HH, HL and LU, LL can appear in a word, but the pair LH cannot. [We need a "U" between them.]

Thus paths can be described as words in the language of ABEEBA with A = L, E = H and B = U!

ONE COMPLICATION: *Do all N-letter words in ABEEBA correspond to paths of the same length?*

Suppose we have an N-letter word with a Us and a total of $N-a$ Ls and Hs. Since our honeycomb paths are ending on the bottom row, each path does wander between top and bottom rows. Each U that appears will be counteracted with one D, plus there will be an additional D to end us on the bottom row. Thus the complete word for the path will contain $a+1$ Ds.

Since each L and H is worth "two steps" along the zigzag line, the word with a Us, $a+1$ Ds and $N-a$ Ls and Hs will be:

$$2(N-a)+a+a+1=2N+1$$

steps long.

Thus all N -letter words correspond to paths ending on cell $2N+1$. As there are F_{2N+1} honeycomb paths to this cell we have:

3. There are F_{2N+1} N -letter words in the language of ABEEBA.

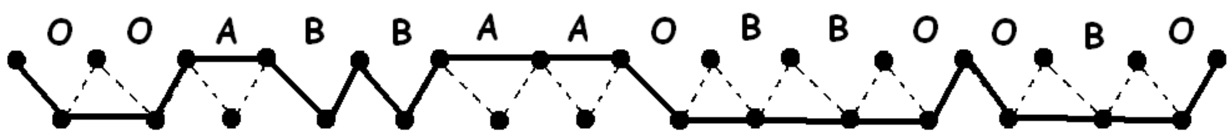
QUESTION: Quentin has a collection of wooden blocks painted red, blue and orange. Each red and each blue block is a $1\times1\times1$ cube but the orange blocks come in all lengths: $1\times1\times k$ for $k=1, 2, 3, 4, \ldots$.

Quentin likes to line up his blocks in long rows to make "trains." How many trains can he make of length N ? How many trains of length N can he make if a red block is never to be immediately followed by a blue block? [Assume Quentin has an infinite supply of each coloured black.]

PATH INTERPRETATION 7

The language of ABOOBA is easier to analyse.

Consider honeycomb paths that end of a cell in the top row and suppose there are $N+1$ dots along that top row (with N spaces between them). Label a path with the letters A, B and O above the spaces between the dots of the top row as suggested by the diagram:



Precisely:

> Each space covered by an H step is labeled A. Each space with one D or one U step underneath it is labeled O. Each space with two or zero diagonal steps below it is labeled B.
>
> OR … going back to ordered partitions with two types of one … set

$$\bar{1} = A$$
$$\underline{1} = B$$
$$2 = OO$$
$$3 = OBO$$
$$4 = OBBO$$
$$5 = OBBBO$$
$$\vdots$$

There must be an even number of Os between any two As, as well as to the left of the leftmost A and to the right of the rightmost A. Thus the number of N-letter words with the letters A, B and O, with the Os satisfying this condition is the number of honeycomb paths to this $(N+1)$th dot on the top row. There are F_{2N+1} such paths.

This almost solves the ABOOBA problem. An N-letter word in ABOOBA contains an irrelevant beginning A and an irrelevant ending A: only the middle $N-2$ letters are of interest to us for paths. Thus we must adjust our index by 2.

4. There are $F_{2(N-2)+1} = F_{2N-3}$ N-letter words in the language of ABOOBA.

CHALLENGE: Answer part b) of the puzzler by looking at paths that end on the bottom row. (Place an imaginary additional dot to the left of the bottom row.)

PATH INTERPRATION 8

Consider honeycomb paths that end on a lower cell, cell number $2N$, and the possible placement of the U steps in these paths.

Between any two Us, and to the left of the leftmost U and to the right of the rightmost U, there is a single D.

In the above diagram with $N = 12$ (a zigzag line of 24 steps) there are 2 possible of choices for the placement of a D in the leftmost section, 4 choices for the placement of D in the second section, 1 in the third, 2 in the fourth and 3 in the final section. Thus there are

$$2 \times 4 \times 1 \times 2 \times 3$$

possible honeycomb paths with Us in these given positions.

For a honeycomb path possessing every possible U step (with $N = 12$ still) there is just $1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1 \times 1$ option for what that path could be and for a honeycomb path containing no U steps there are 12 options. (There still must be a single D at some location.) Every partition of the number 12 corresponds to a count of honeycomb paths with Us in specific positions.

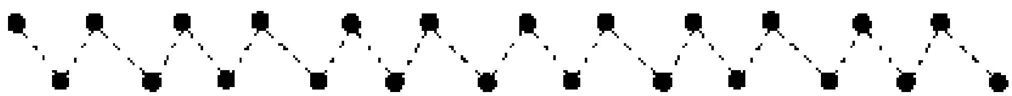
In general, as there are a total number of honeycomb paths to cell $2N$, the sum of all possible partitions of N with terms multiplied together must give F_{2N}.

This solves puzzle 5.

5. Multiplying the terms of each partition of N and summing all the products yields F_{2N}.

PATH INTERPRETATION 9

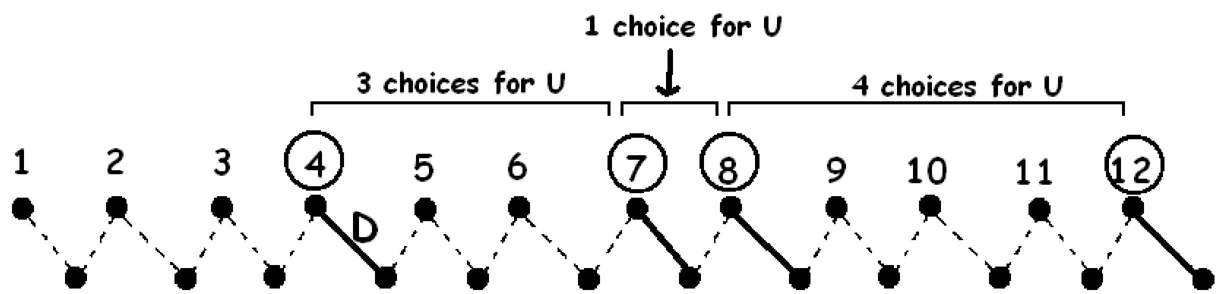
Consider honeycomb paths that end on a lower cell, cell number $2N$, and the possible placement of the D steps in these paths. Each path has at least one D.



There are N paths with exactly one D.

If a path has two or more Ds, then there needs to be exactly one U between each pair of neighbouring Ds.

Now, if we number of the dots on the top row 1 through N, choosing a D corresponds to circling one of these numbers. For two neighbouring Ds, the possible number of placements of a U between them is the difference of the two circled numbers for those Ds.



All other steps in the path once the locations of Ds and Us has been set must be Ls and Hs. Thus the number of honeycomb paths with two or more Ds fixed in position is given by the product of the differences between neighbouring circled numbers.

This accounts for all F_{2N} possible paths and we have established puzzle 6:

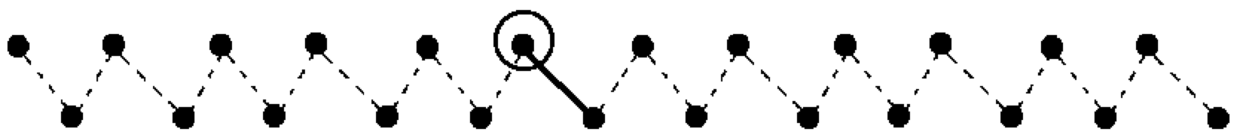
6. $F_{2N} = N +$ sum of all possible products of differences from circling two or more terms in the list of numbers 1 through N.

QUESTION: Suppose we list all the numbers 1 through N, circle two or more numbers, and sum all possible products of differences of consecutive circled terms. Suppose we insist that the number 1 always be circled. Show that this sum of products of differences is sure to be $F_{2N-1} - 1$.

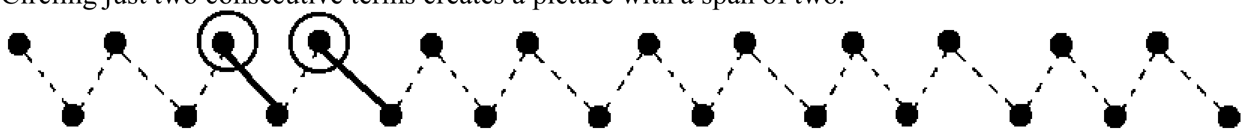
Going further …

Consider the span of possible circled numbers on the top row – the number of dots from the leftmost circled dot to the rightmost circled dot.

As we have seen, there are N honeycomb paths containing exactly one D. That is, there are N paths with a span of one.

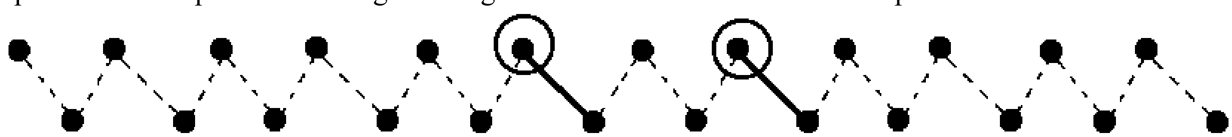


Circling just two consecutive terms creates a picture with a span of two.



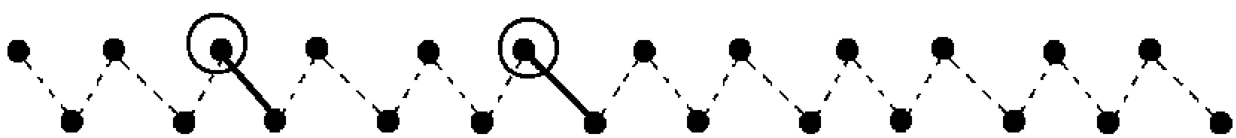
There are $\left(N-1\right)$ paths with just two Ds positioned this way.

A picture with a span of three might or might not have the middle dot in the spread circled.



There are F_4 paths between the two outer circled terms and there are $\left(N-2\right)$ possible positions for this span of three. Thus there are $\left(N-2\right)F_4$ honeycomb paths with a span of three.

Similarly there are $\left(N-3\right)F_6$ honeycomb paths with a span of four.



and so on.

As this categorises all F_{2N} paths we have established
$N+\left(N-1\right)F_2+\left(N-2\right)F_4+\cdots+1\cdot F_{2N-2}=F_{2N}$ proving the second part of puzzler 10.

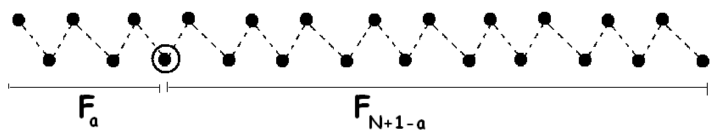
10 b) $N+\left(N-1\right)F_2+\left(N-2\right)F_4+\cdots+1\cdot F_{2N-2}=F_{2N}$

ASIDE: Suppose $N = qa + r$ for some positive integer a and for integer r with $0 \le r < a$. (Thus r is the remainder that appears with N is divided by a and q is the quotient.)

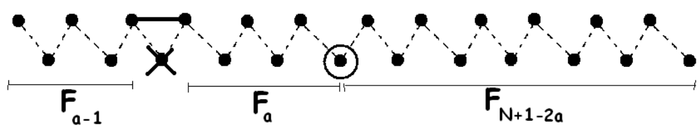
Let's find the remainder and quotient in dividing F_N by F_a.

Consider the F_N honeycomb paths along N dots.

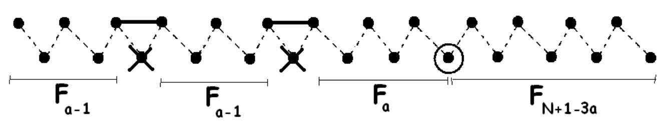
How many paths go through the a th dot?
Answer: $F_a F_{N+1-a}$



How many path don't go through the a th dot but do go through dot $2a$?
Answer: $F_{a-1}F_a F_{N+1-2a}$



How many paths don't go through dots a and $2a$ but do go trhough dot $3a$?
Answer: $\left(F_{a-1}\right)^2 F_a F_{N+1-3a}$



And so on, all the way to …

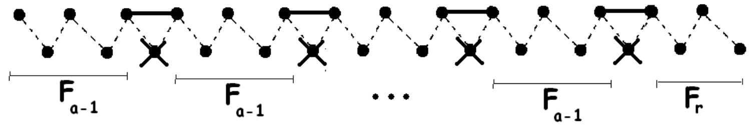
How many paths don't go through dots a, $2a$, $3a$, …, $(q-1)a$ but do go trhough dot qa?

Answer: $\left(F_{a-1}\right)^{q-1} F_a F_{N+1-qa} = \left(F_{a-1}\right)^{q-1} F_a F_{r+1}$.

and

How many paths don't go through dots a, $2a$, $3a$, …, $(q-1)a$, and qa?

Anwser: $\left(F_{a-1}\right)^q F_r$



This accounts for all F_N paths and so:

$$F_N = F_a\left(F_{N+1-a} + F_{a-1}F_{N+1-2a} + \left(F_{a-1}\right)^2 F_{N+1-3a} + \cdots + \left(F_{a-1}\right)^{q-1} F_{r+1}\right) + \left(F_{a-1}\right)^q F_r$$

Since $F_0 = 0$ this establishes puzzle 12:

If N is a multiple of a, then F_N is a multiple of F_a.
(And the quotient is $F_{N+1-a} + F_{a-1}F_{N+1-2a} + \left(F_{a-1}\right)^2 F_{N+1-3a} + \cdots + \left(F_{a-1}\right)^{q-1} F_1$.)

More generally, F_N leaves a remainder of $\left(F_{a-1}\right)^q F_r$ upon division by F_a. (Here $N = qa + r$.) That is:

$$F_N \equiv \left(F_{a-1}\right)^q F_r \mod F_a \ .$$

PART IV:
OTHER INTERPRETATIONS? OTHER IDENTITIES?

Might you be able to interpret honeycomb paths in other ways to establish other curious Fibonacci appearances?

Might you be able to establish new Fibonacci identities a la the identities of puzzler 10 and 12?

Here are some classic Fibonacci identities. Might any of these be proved via path walking?

$$F_1 + F_2 + F_3 + \cdots + F_N = F_{N+2} - 1$$

$$F_1 + F_3 + F_5 + \cdots + F_{2N-1} = F_{2N}$$

$$F_0 + F_2 + F_4 + \cdots + F_{2N} = F_{2N+1} - 1$$

$$F_1 + 2F_2 + 3F_3 + \cdots + NF_N = NF_{N+2} - F_{N+3} + 2$$

$$F_{N-1}F_{N+1} = \left(F_N\right)^2 + \left(-1\right)^N$$

$$F_1^2 + F_2^2 + \cdots + F_N^2 = F_N F_{N+1}$$

$$\gcd\left(F_a, F_b\right) = F_{\gcd(a,b)}$$