Mathematics of MTC's Spanning the K-20 Spectrum

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A Math Teachers' Circle problem is readily accessible, engaging, and challenging for a wide audience including:

- Students from late elementary, middle, and high school
- K-12 Teachers
- Undergraduate and graduate mathematics students
- Mathematicians

We will examine two Math Teachers' Circle Problems, highlighting their origin in elementary and secondary school mathematics and following them through into important university and research level mathematics questions

Problem 1: Using Parentheses

A fifth grade standard about Operations and Algebraic Thinking reads as follows:

5.OA.1 Use parentheses, brackets, or braces in numerical expressions, and evaluate expressions with these symbols.

Here is a task from Illustrative Mathematics written for this standard:

What numbers can you make with 1, 2, 3, and 4? Using the operations of addition, subtraction, and multiplication, we can make many different numbers. For example, we can write 13 as

$$13 = (3 \times 4) + 1.$$

You can use parentheses as many times as you like and each of the numbers 1, 2, 3, and 4 can be used *at most once*.

a. Find two different ways to make 9.

b. Find two different ways to make 7.

c. Find two different ways to make 11.

d. Can you make 26?

A more open ended version of this problem would be:

> What is the smallest whole number you can *not* make with 1, 2, 3, and 4 in this way?

- Still accessible to any audience
- Develops arithmetic fluency and strategic thinking about operations
- Justifying answer requires careful reasoning

Many variants of all levels of difficulty abound! Here are a few:

- How many *different* ways can you make 9 (or some other number)? Show your list is complete.
- Which number(s) can be made in the *most* different ways?
- What is the *smallest* number that can only be made in *one* way?
- What is the *largest* number you can make?
- How many different *integers* can you make and what is the *smallest*?

In a different direction:

> What if we work with 1, 2, 3, 4, *and* 5. What is the smallest whole number that you can *not* make?

This is a substantially more challenging problem to which I do not know the answer!

Making prime numbers (or numbers with a small number of prime factors) tends to be challenging: work on this problem leads to important thinking about factoring whole numbers.

Finally, a very difficult open question: suppose $f(n)$ is the smallest whole number which can *not* be made with 1, 2, 3, 4, . . . , n. What can we say about $f(n)$?

- Is there an explicit formula for $f(n)$??????
- Is $f(n)$ always at least $n! = n(n-1)(n-2)\ldots 1$?
- How "often" is $f(n)$ a prime number?
- Create and explore your own problem!

Problem 2: Triangles in the Coordinate Plane

Is there an equilateral triangle in the plane whose vertices all have integer x and y coordinates?

Different level audiences tend to apply different approaches:

- Elementary school teachers and students tend to use concrete models and measurement
- Middle school teachers and students often apply the Pythagorean Theorem and algebraic equations
- High School teachers can apply rigid transformations or more complex algebraic techniques
- Mathematicians can use Pick's Theorem

What do these two problems have in common?

a. Answer to question is not provided or likely to be known

b. Experimentation is invaluable

c. Problems are readily approached (if not solved!)

d. Mathematics involved is not specified or clear

These are all hallmarks of the work mathematicians do, work which is fun, exciting, at times frustrating, and readily available for all to enjoy!!