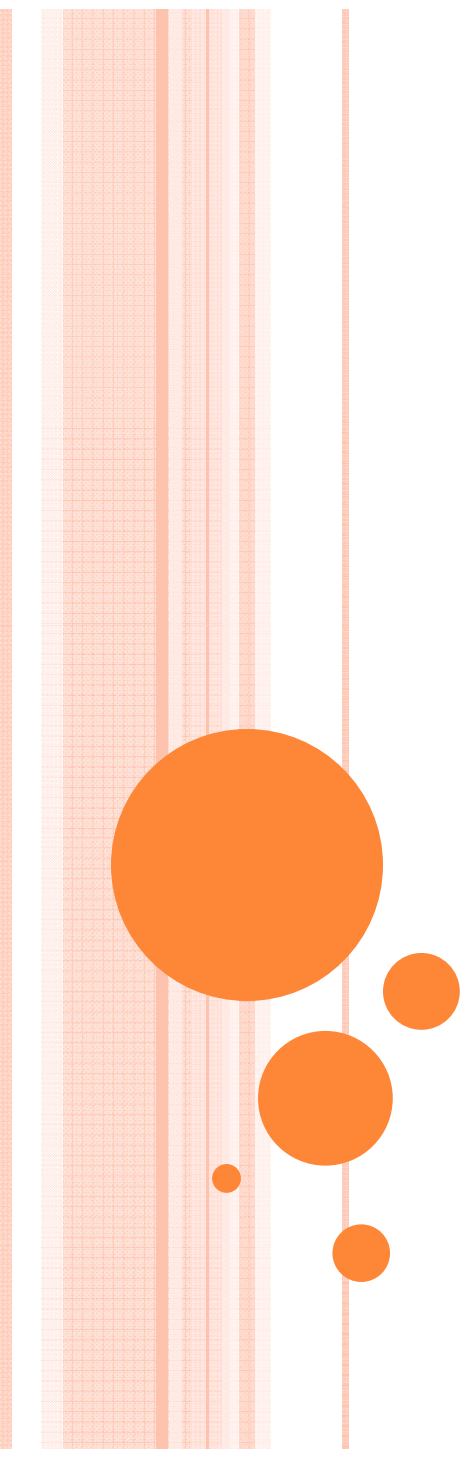




GEOMETRIC COMBINATORICS – A TREASURE TROVE OF MATH CIRCLE PROBLEMS

Tatiana Shubin

tatiana.shubin@sjsu.edu

I. BORSUK'S PROBLEM

- 1. Prove that:
 - (a) diameter of a polygon is equal to the length of its longest diagonal or its longest side;
 - (b) diameter of a convex polyhedron is equal to the length of its longest diagonal or its longest edge.
- 2. Suppose that the area of a plane quadrilateral is 1. What are possible values of its diameter?



- 3. Diameter of (a) a triangle; (b) a quadrilateral; (c) a convex quadrilateral is 1. What are possible values of its (i) area; (ii) perimeter?
- 4. Given a set of n points in the plane such that the distance between any two of these points is at least 1, let D denote the diameter of this set. Prove that:
 - (a) $D \geq 1$ if $n = 2$ or $n = 3$;
 - (b) $D \geq \sqrt{2}$ if $n = 4$;
 - (c) $D \geq (1 + \sqrt{5})/2$ if $n = 5$.

Also, construct sets for which the equalities hold in parts (a), (b), and (c) above.



- 5. Prove that every plane figure of diameter 1 can be completely covered by:
 - (a) a square of side 1;
 - (b) a circle of radius $\sqrt{3}/3$;
 - (c) a regular hexagon with a side $\sqrt{3}/3$.

- 6. Prove that:
 - (a) Every finite set of points in the plane can be split into 3 parts each of which has the diameter smaller than that of the original set.
 - (b) Every convex polygon can be split into 3 polygons each of which has the diameter smaller than the original polygon.



(c) Every finite set of points in 3-space can be split into 4 parts each of which has the diameter smaller than that of the original set.

(d) Every convex polyhedron can be split into 4 polyhedra each of which has the diameter smaller than the original polyhedron.

Borsuk's Conjecture: *In the Euclidean d -dimensional space, every set of a finite positive diameter can be partitioned into at most $d + 1$ sets of smaller diameter.*



60 years after Karol Borsuk's paper, the conjecture was **disproved** in 1993. For large dimensions, the possible number of partitioning subsets with smaller diameters is bounded below by a much larger than $d + 1$ number which is, in essence, $(1.2255\dots)^{\sqrt{d}}$; an explicit counterexamples for the Borsuk's Conjecture are currently known for dimensions $d \geq 561$.



II. KISSING NUMBERS

- 1. Let c be a unit circle. Find the largest number k of unit circles $c_1, c_2, \dots, c_k$ such that c_i kisses circle c for every i , and any two circles c_i, c_j have at most one common point.
- 2. How many billiard balls can be arranged around a central billiard ball so that the outer balls “kiss” the center ball?



The kissing numbers are known only for dimensions

1, 2, 3, 4, 8, and 24;

these numbers are

2, 6, 12, 24, 240, and 196,560, respectively; surprisingly, the latest one found was the kissing number for $d = 4$.



III. CHROMATIC NUMBERS

- 1. Is it possible to paint every point of the plane using exactly three colors so that every line contains points of exactly two colors?
- 2. Is it possible to color each point on a circle either red or blue in such a way that no three points of the same color form an isosceles triangle? What if instead of just two colors you can use three different colors? Four colors? 1,000,000 colors?



VAN DER WAERDEN'S THEOREM:

For any given positive integers r and k , there is some number N such that if the integers $\{1, 2, \dots, N\}$ are colored, each with one of r different colors, then there are at least k monochromatic integers forming an arithmetic progression.



The least such N is the *Van der Waerden Number* $W(r, k)$.

It's not too hard to find that

$$W(2, 3) = 9$$

$$W(3, 3) = 27.$$

The current record for an upper bound belongs to Timothy Gowers (a Fields medallist); he proved that

$$W(r, k) \leq 2^{2^{r^{2^{2^{k+9}}}}}$$



- 3. How many colors are needed to paint the real number line so that no two points a unit distance apart are painted the same color?
- 4. How many colors are needed to paint the plane so that no two points a unit distance apart are painted the same color?

