Golumb Rulers

Warm up problem.

a. A standard $12''$ ruler is a useful tool, but in a sense it is also inefficient. For example, to measure a length of $6''$ you do not need a $6''$ mark on the ruler. If this mark were removed you could still measure $6''$ by measuring from the $2''$ to the $8''$ mark. And any other whole number of inches from $0''$ to $12''$ can be measured without the $6''$ mark. What is the minimal number of marks on the ruler, and where should they be, if we want to measure every distance from 0 to 12. By the way, we will count the two ends ($0''$ and $12''$) as two of the marks.

b. For the ruler you have designed, are there any distances that can be measured in two or more ways? How many? Can you find an argument to convince people that you have the "most efficient" $12''$ ruler?

c. Suppose you want a 12" ruler that measures any integer distance only once, even if there are some distances it cannot measure. What is the greatest number of different distances that can be measured under these circumstances and what might such a ruler look like?

Definition. A ruler is called a *Golumb ruler* if each distance that it measures can be measured in only one way. The largest distance that can be measured is called the length of the *Golumb ruler.* The number of marks on the ruler (including the two ends) is called the *size of the ruler*. We will use the notation $\{a_1, a_2, \ldots, a_m\}$ with

$$0 = a_1 < a_2 < \cdots < a_m = L$$

to describe a Golumb ruler of length L and size m (i.e., with m marks.)

1. What is the maximal possible number of marks on a Golumb ruler of length 12? Where should the marks be? How many such rulers are there? Answer the same question for rulers of length 5, 6, 8, 13.

2. A Golumb ruler of length N is called perfect if it can be used to measure each of the distances $1, 2, 3, \ldots, N$. How many perfect Golomb rulers can you find? Can you find all perfect rulers? Why or why not?

Definition. Now suppose that we wish to make a Golomb ruler with m marks. There are many such rulers but we are interested in the ruler of minimal length. We denote the length of this ruler by $G(m)$.

3. Calculate $G(2)$, $G(3)$, $G(4)$, and $G(5)$.

4. For a positive integer $m > 5$ what can you say about $G(m)$?

Definition. A set $A = \{a_1, a_2, \ldots, a_N\}$ of natural numbers, where $a_1 < a_2 < \cdots < a_N$ is called a *Sidon set* if the sum of no two elements of the set is equal to the sum of two other elements in the set. That is, if a_i, a_j, a_k, a_ℓ are elements of A, then $a_i + a_j \neq a_k + a_\ell$. (Note it is possible that, say, $i = k$ here, but then we must have $j \neq \ell$.) Given a positive integer n, $S(n)$ is the number of elements in the largest subset of $\{0, 1, 2, \ldots, n\}$ that forms a Sidon set.

5. Find the value of $S(n)$ for $n = 2, 3, 5, 6, 8, 12, 13$.
6. Discuss your results and the Sidon sets you found in doing Problem 1. What conjectures come to mind?
7. Prove that every Sidon set is a Golomb ruler. Is the converse also true, that is, is every Golomb ruler a Sidon set?

Definition. A ruler is a *sparse ruler* if some of the distance marks are missing but you can still measure any (integral) distance up to the full length. More formally, a sparse ruler of length L with m marks is a sequence of integers

$$0 = a_1 < a_2 < \cdots < a_m = L$$

such that for each k, $1 \leq k \leq L$ there are marks a_i, a_j such that $a_j - a_i = k$.

A sparse ruler of length L with m marks is called *minimal* if there is no sparse ruler of length L and with $m - 1$ marks. The sparse ruler is called *maximal* if there is no sparse ruler of length $L + 1$ with m marks.

8. Find the maximum possible length of a sparse ruler with 2, 3, 4, 5 or 6 marks.
9. Suppose we want to design a sparse ruler with m marks. What can you say about the maximum possible length L for such a ruler? When can this maximum be realized?

And one more question for you to think about:

10. You have shown that there are only finitely many perfect Golomb rulers, those of length 1, 3, 6. Is there a perfect Golomb ruler of infinite length? That is, is there an unending sequence of integers $0 = a_1 < a_2 < a_3 < \cdots$ such that for any positive integer k there is exactly one pair a_i, a_j in the sequence so that $a_j - a_i = k$?

Golumb Rulers—Notes.

The inspiration (and much of the material) for this circle came from the article *Golomb Rulers* by Roger C. Alperin and Vladimir Drobot that appeared in the February 2011 issue of *Mathematics Magazine.* I have used this circle with a group of fifth through tenth graders. It seemed to work well, kept them engaged and generated a lot of interesting discussions. I have yet to try it with teachers but am pretty sure it would work well with that group too.

Warm up problem a. took about 10 minutes, but by that time students had produced all of the most efficient *sparse* 12" rulers. They also pointed out that any reflection of a workable ruler gave another workable ruler.

Parts b. and c. are related. The main focus of the work was this: we have constructed several 6 mark workable 12" rulers; can we get by with just 5 marks? Students explored this by trying many examples and ideas for the construction. I was hoping that eventually they would think to investigate "how many distances can I measure with m marks?" I finally suggested that, as mathematicians do, they look at a simpler problem...say a 4" ruler and asked with just three marks (0, 4, and one more) can they measure every distance from 1 to 4? Once they heard this question they quickly were able to show that with m marks you can measure at most $m(m-1)/2$ distances. I was pleasantly surprised by their work on this: one group noted by considering one point at a time that the number of distances was $(m-1)+(m-2)\cdots+2+1$ while another noted that each of the m points can measure to each of the $m-1$ others and that this argument counts each pair twice, hence $(m-1)m/2$. As a teachable moment I pointed out that they had proved that

$$1+2+\cdots+(m-1)=\frac{(m-1)m}{2}.$$

From here the discussion can go many ways. Parts a. and b. talk about *sparse rulers* while part c. concerns Golomb rulers. Each of these can be further developed. I did Golomb rulers next and did not get back to sparse rulers until near the end of the two-and-a-half hour session.

After discussing the definition of Golomb ruler the students were able to come up with rulers of various lengths. They then looked into perfect rulers and found as examples, rulers of length 1, 3, 6. After the work with the warm up problems they quickly conjectured and proved that any perfect ruler has a triangular number as length. They then worked for a while trying to construct perfect rulers of length 10, 15, 21, etc. After some time a couple of student suggested there were no others. To help direct their investigation I asked them to consider a ruler of length 28 and asked where must marks be if we want to measure a length of 27. After possible reflection, we see that there must be a mark at 1. Having done this, what to you need to measure 26? etc. Sifting through the small set of possibilities for each case students found the construction was impossible (e.g., distances are repeated) by the time we consider measuring distances of 23. The discussion of $G(m)$ followed. Students quickly noted the lower bound based on the previous triangular number considerations.The determination of $G(m)$ is a very difficult problem (NP hard). The

Wikipedia page on Golomb rulers (just google Golomb rulers) has a table showing what is presently known about $G(m)$ (only up to $m = 26$). A great way to expose students to problems that are easy to understand but unsolved.

It is not hard to prove that every Sidon set is also the set of marks for a Golomb ruler:

$$a_i + a_j = a_k + a_\ell \qquad \text{if and only if} \qquad a_j - a_k = a_\ell - a_i.$$

Thus the Sidon sets they find in Problem 5 will match some of the sets found in Problem 1. Once students observed these similar sets they quickly conjectured Sidon and Golomb sets are the same. They needed a bit of help in organizing the proof of the "if and only if" statement however.

There are still open questions about the construction of sparse rulers. Again just google sparse rulers and look at the Wikipedia site for information about sparse rulers constructions.

Finally, yes there are Golomb rulers of infinite length and there is an algorithm (due to Erdös) for construction of such rulers. I did not talk about this during the circle but left it as a problem to think about during the week. See the Drobot, Alperin article cited earlier for more about this.

Also see the reference http://cgm.cs.mcgill.ca/ athens/cs507/Projects/2003/JustinColannino/ for more about Golomb rulers and related topics.