How to Think Brilliantly and Creatively In Mathematics

FIVE PRINCIPLES FOR **EVERYONE!**

~~K-12 STUDENTS AND TEACHERS~~

James Tanton

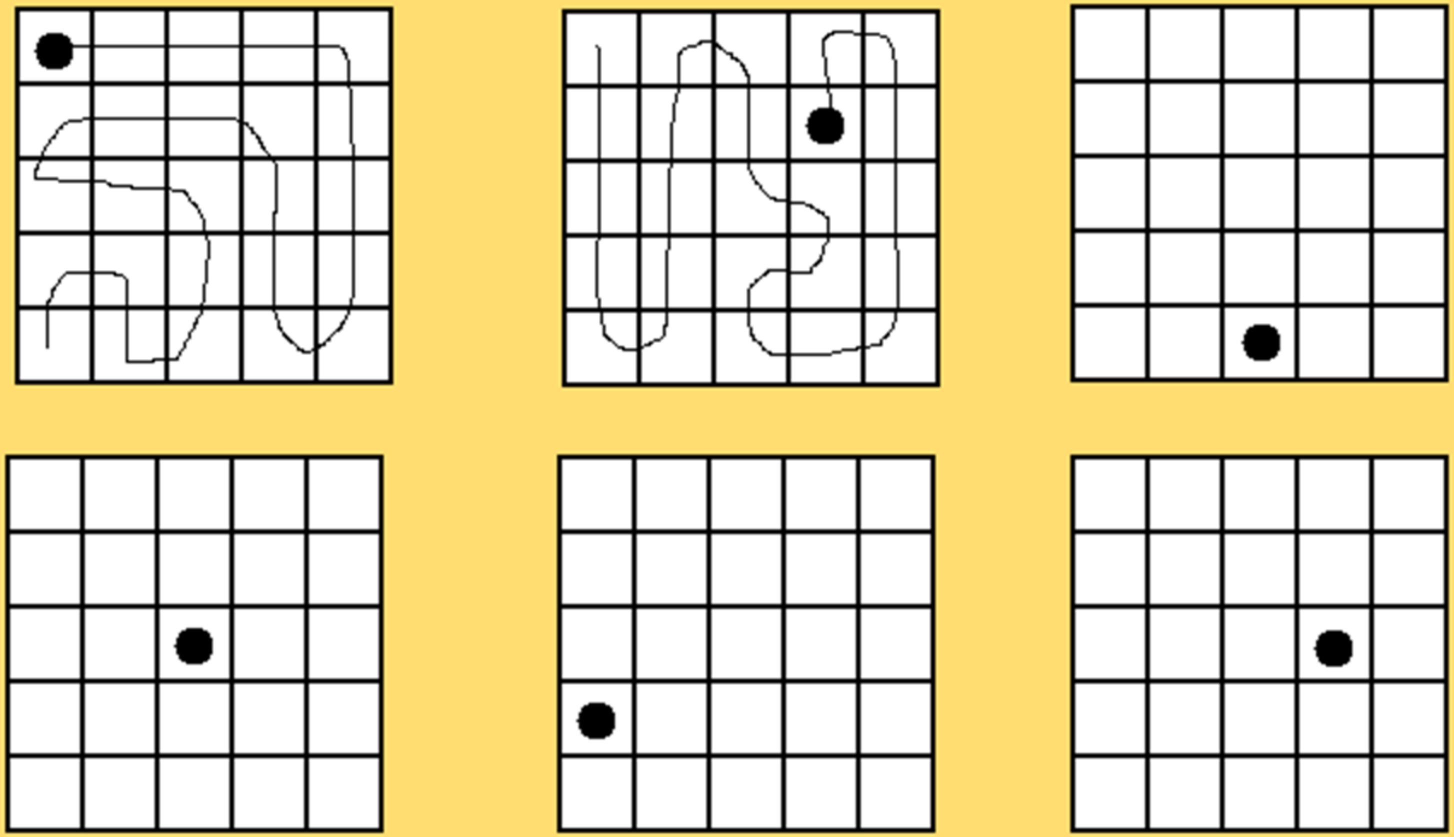
MAA, Mathematician-at-Large

www.maa.org/ci

Who I am

– and the power of a pressed-tin ceiling.

Designed by Macgregor Campbell @mainsequence



What do I mean by “brilliant” math thinking?

All the things our community practices and/or strives for every day:

* Setting the stage for epiphanies … lots of mulling.

* Using patience and perseverance to just “nut things out”

* Seeing the big picture; making connections

* Developing intellectual agility (which does not equate with intellectual speed)

* Asking “WHY?” and “WHAT ELSE?” and, better yet, “WHAT IF?” questions over “WHAT” questions.

* Engaging in joyful intellectual play – being willing to innovate, to just try things, to flail (and even fail!)

These skills pervade all types of thinking, all subjects of learning.

LIFE SKILLS!

FIVE PRINCIPLES OF MATH THINKING:

I: VISUALIZE: THINK OF A PICTURE

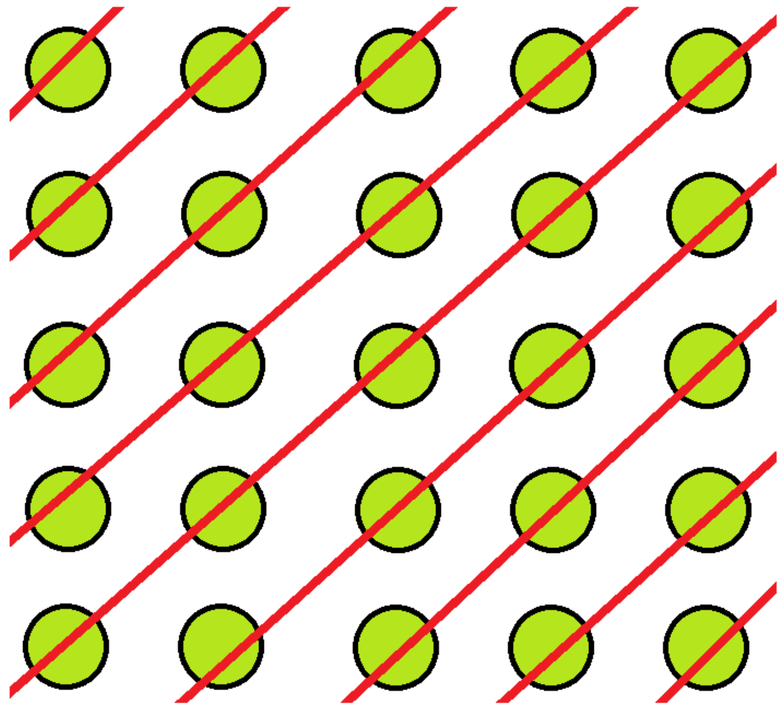
II: USE COMMON SENSE: AVOID GRUNGY WORK

III: ENGAGE IN INTELLECTUAL PLAY

IV: UNDERSTANDING TRUMPS MEMORIZATION

V: MATH PEOPLE ARE CLEAR ON WHAT THEY DON'T KNOW
- and are comfortable admitting that they don't know!

I: VISUALISE: THINK OF A PICTURE



$$1 + 2 + 3 + 4 + 5 + 4 + 3 + 2 + 1$$

I: VISUALISE: THINK OF A PICTURE

$$1+2+3+\ldots+N=\frac{N(N+1)}{2}$$

$$1+2+3+4+5+4+3+2+1=5^2$$

$$1+2+3+4+5+4+3+2+1=5^2$$

$$1+2+3+4+5+4+3+2+1=5^2 \; +5$$

+5

$$1+2+3+4+5=\frac{5^2+5}{2}$$

I: VISUALISE: THINK OF A PICTURE

Multiplication as a geometry problem.

23 x 17

	20	3
10	200	30
7	140	21

$$23 \times 17 = 200 + 30 + 140 + 21$$
$$= 391$$

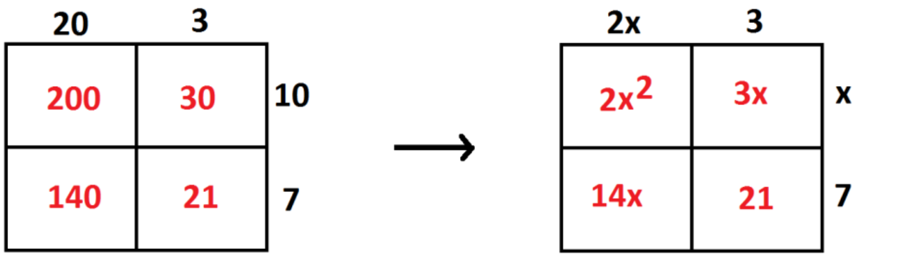
It also explains the traditional algorithm.

$$\begin{array}{r} 23 \\ \times\ \underline{17} \\ 21 \\ 140 \\ 30 \\ \underline{200} \\ = 391 \end{array} \qquad \begin{array}{r} {}^{2}23 \\ \times\ \underline{17} \\ 161 \\ \underline{230} \\ = 391 \end{array}$$

One can almost do this in one's head!

I: VISUALISE: THINK OF A PICTURE

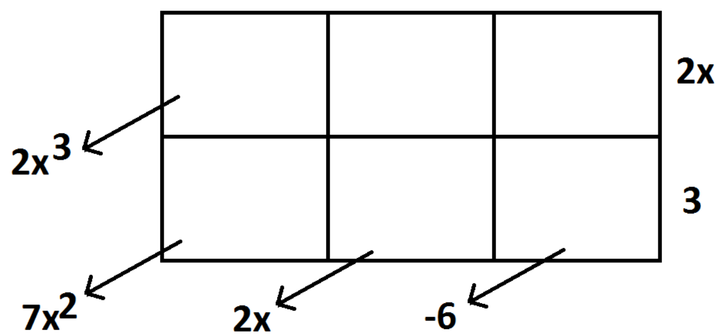
This area model provides a wonderful story-line throughout all of K-12.



$$23 \times 17 = 200 + 30 + 140 + 21 = 391$$

$$(2x+3)(x+7) = 2x^2 + 3x + 14x + 21 = 2x^2 + 17x + 21$$

Compute: $\dfrac{2x^3 + 7x^2 + 2x - 6}{2x+3}$.



See www.gdaymath.com for the full power of area throughout the curriculum.

I: VISUALISE: THINK OF A PICTURE

Obviate common student mistakes.

e.g.

"Square" - a geometry word!

$$(a+b)^2 = a^2 + b^2 + \underline{2ab}$$

	a	b
a	a^2	ab
b	ba	b^2

No need to memorize. No “FOIL.”

A picture makes it clear what to do!

$(a+b+c)^2$

9 terms

$(x+y+w+5)(x+a+b)$

12 terms

A PICTURE → INSIGHT and DEEP UNDERSTANDING

II: USE COMMON SENSE: AVOID GRUNGY WORK

Avoid tedious grungy work!

Mathematicians will work hard to avoid hard work!

This means:

*** Pausing before leaping into action**

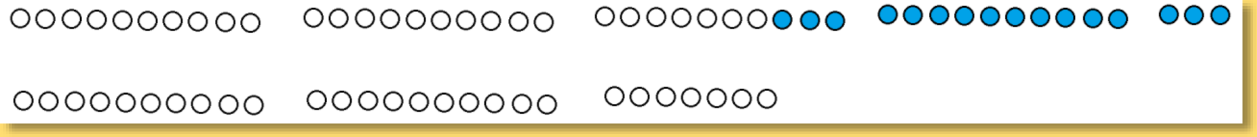
*** Being confident to rely on your wits,**
to "nut your way" through things

Applies even to simple arithmetic.

e.g. What's 43 take away 27?

$$43 - 27 = 3 + 10 + 3 = 16$$

$$\begin{array}{r} \overset{3}{\cancel{4}}\,\overset{1}{3} \\ -\ \underline{27} \\ =16 \end{array}$$



II: USE COMMON SENSE: AVOID GRUNGY WORK

$$\begin{array}{r} 1005 \\ -\ 168 \\ \hline \end{array}$$

$$\begin{array}{r} 1000 \\ -\ 163 \\ \hline \end{array}$$

$$\begin{array}{r} 999 \\ -\ 162 \\ \hline \end{array}$$

II: USE COMMON SENSE: AVOID GRUNGY WORK

e.g. What is 15% of 62?

<u>Answer</u>: 6.2 + 3.1 = 9.3

e.g. What is 17½ % of 80?

e.g. What is 815 divided by 5?

<u>Answer</u>: Double and divide by 10: 1630 / 10 = 163

e.g. One kilogram is equivalent to 2.2 pounds.

So 34 kilograms is equivalent to ……. 68 + 6.8 = 74.8 pounds.

II: USE COMMON SENSE: AVOID GRUNGY WORK

Teach this principle within basic algebra too:

e.g. Solve $(x+2)^2 = 25$

Rote way:

$$(x+2)^2 = 25$$

$$x^2 + 4x + 4 = 25$$

$$x^2 + 4x - 21 = 0$$

$$\vdots$$

Math way:

$$(x+2)^2 = 25$$

$$x + 2 = 5 \text{ or } -5$$

$$x = 3 \text{ or } -7$$

Solve: $(22x-21)^{20} = 1.$

II: USE COMMON SENSE: AVOID GRUNGY WORK

e.g. Solve $\frac{x}{7} = \frac{2}{3}$

Many students are “programmed” to cross-multiply no matter what.

$$3x = 14 \longrightarrow x = \frac{14}{3}$$

A math people wonder: Why do work you are only later going to undo?

$$x = \frac{2 \cdot 7}{3} = \frac{14}{3}$$

II: USE COMMON SENSE: AVOID GRUNGY WORK

e.g. Expand and simplify: $(x-a)(x-b)(x-c)(x-d)\cdot\cdots\cdot(x-y)(x-z)$

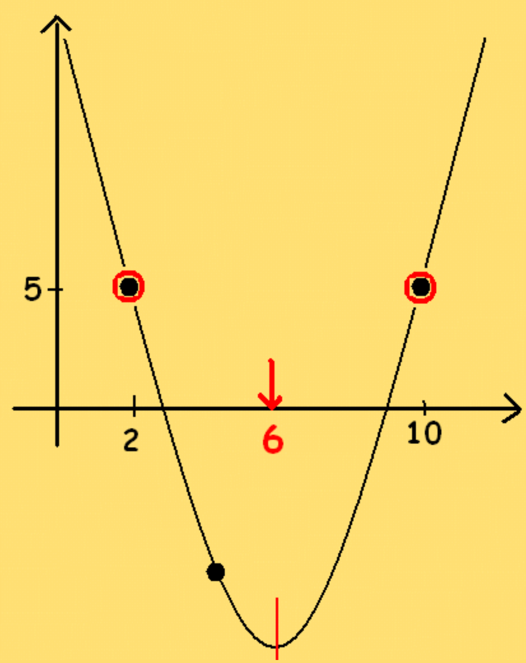
$(x-x) = 0$

II: USE COMMON SENSE: AVOID GRUNGY WORK

A parabola passes through the points (2,5) and (3,-6) and (10,5). What is the x-coordinate of its vertex?

Avoiding Hard Work Answer:

We have two symmetrical points: (2,5) and (10,5).
And a parabola has a symmetrical U-shaped graph.



The line of symmetry must be half way between x = 2 and x = 10.

Vertex is at x = 6.

II: USE COMMON SENSE: AVOID GRUNGY WORK

Sketch a graph of the parabola $y = x^2 - 4x + 7$ **.**

From the previous example:

If we find two symmetrical points on a symmetrical graph all becomes pretty clear.

What x-values are interesting for the given equation?

$$y = x^2 - 4x + 7$$
$$y = x(x - 4) + 7$$

We see putting

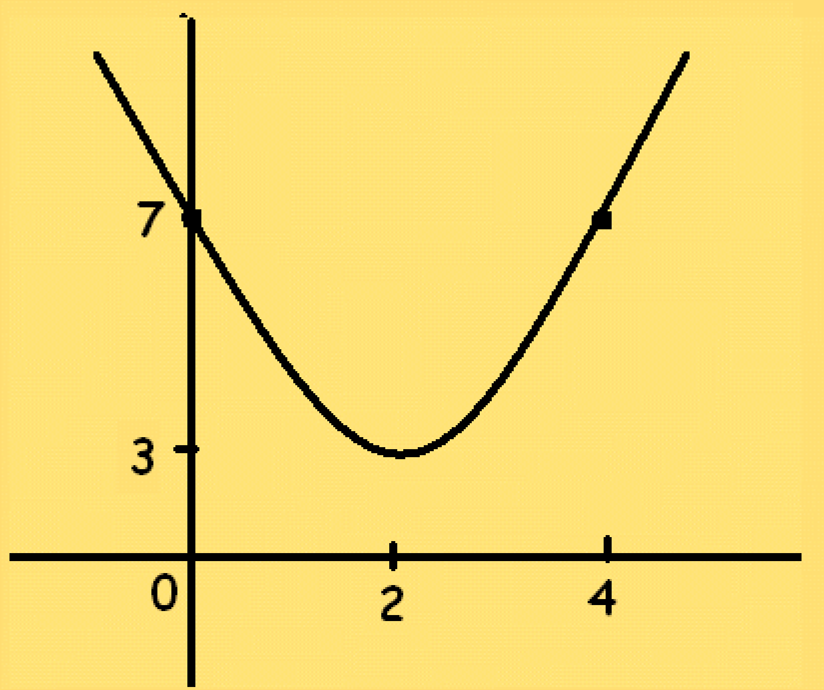
$x = 0$ and $x = 4$

both give $y = 7$.

II: USE COMMON SENSE: AVOID GRUNGY WORK

$$y = x^2 - 4x + 7 \; = x(x-4)+7$$

x = 0 gives y = 7
x = 4 gives y = 7



Vertex at x = 2

When x = 2 we have y = 3.

II: USE COMMON SENSE: AVOID GRUNGY WORK

YOUR TURN!

Find a value *k* so that $y = 3(x-5)(x-7)+k$ **just touches the *x*-axis.**

II: USE COMMON SENSE: AVOID GRUNGY WORK

i) Show that $x^5 - 1$ is divisible by $x - 1$.

ii) Is $2^{100} - 1$ prime?

Answer: $x^5 - 1 = (x - 1)(stuff)$

$$2^{100} - 1 = \left(2^{20}\right)^5 - 1 = \left(2^{20} - 1\right)(stuff)$$

II: USE COMMON SENSE: AVOID GRUNGY WORK

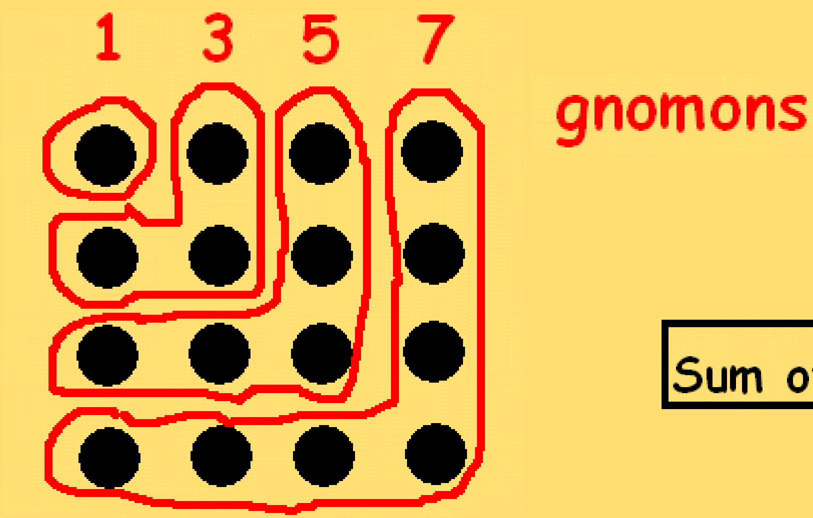
Find a fraction that lies between $\frac{1}{11}$ and $\frac{1}{12}$.

<u>Answer</u>: $\dfrac{1}{11\frac{1}{2}}$

III: ENGAGE IN INTELLECTUAL PLAY

This is the heart of the mathematical enterprise.

A classic Greek example:



Sum of the first N odd numbers is N^2

sum of first four odds

$1+3+5+7 = 4 \times 4 = 16$

III: ENGAGE IN INTELLECTUAL PLAY

$$1+3+5+7+9+11+13+15+17+19 = 10\times10 = 100$$

+1 +1 +1 +1 +1 +1 +1 +1 +1 +1 +10

$$2+4+6+8+10+12+14+16+18+20 = 100 + 10$$

Sum of the first N even numbers is $N^2 + N$

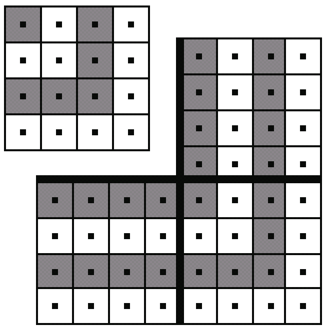
Divide by two:

$$1+2+3+4+5+6+7+8+9+10 = \frac{100 + 10}{2}$$

Sum of the first N numbers is $\frac{N^2 + N}{2}$

III: ENGAGE IN INTELLECTUAL PLAY

High School students (2009):



The Galilean Ratios:

$$\frac{1}{3}=\frac{1+3}{5+7}=\frac{1+3+5}{7+9+11}=\frac{1+3+5+7}{9+11+13+15}=\cdots$$

$$\frac{1}{2}=\frac{2+3+4}{5+6+7}=\frac{3+4+5+6+7}{8+9+10+11+12}=\cdots$$

$$\frac{1}{3}=\frac{1-3+5}{7-9+11}=\frac{1-3+5-7+9}{11-13+15-17+19}=\cdots$$

$$\frac{1}{3+5}=\frac{1+3}{(5+7)+(9+11)}=\frac{1+3+5}{(7+9+11)+(13+15+17)}=\cdots=\frac{1}{8}$$

III: ENGAGE IN INTELLECTUAL PLAY

A playful question …

Is today opposite day?

What must the answer be?

This type of self-reference often appears in mathematics.

e.g.

What is $-x$ if $x=-50$?

What is $7^{\log_7 3}$?

What is $\sin\left(\sin^{-1} 0.85\right)$?

Consider the set of all elements that are not in their own assigned sets ….

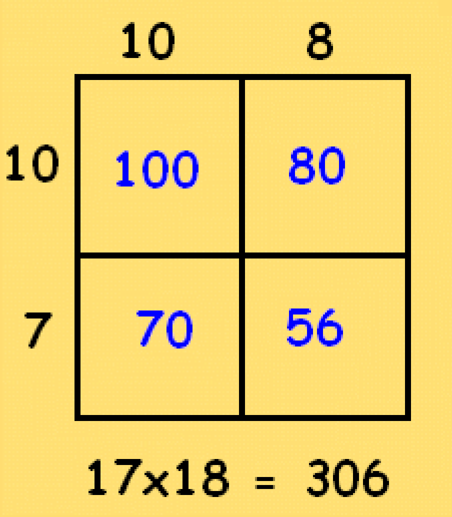
III: ENGAGE IN INTELLECTUAL PLAY

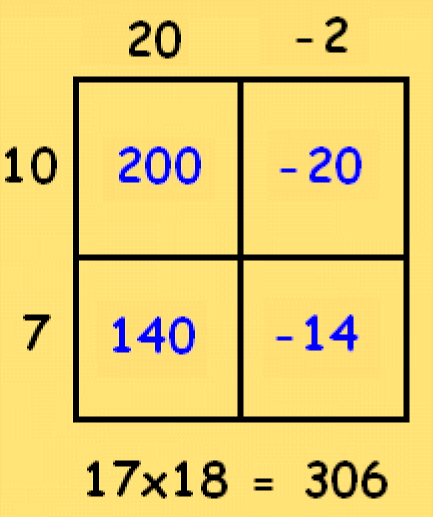
Why is negative times negative positive?

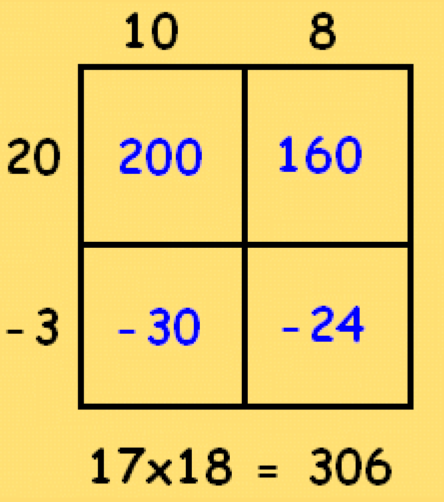
Multiplication = Areas of rectangles.

Wild idea … Allow rectangles to have negative side lengths!

Consider 17x18







III: ENGAGE IN INTELLECTUAL PLAY

	20	-2
20	400	-40
-3	-60	??

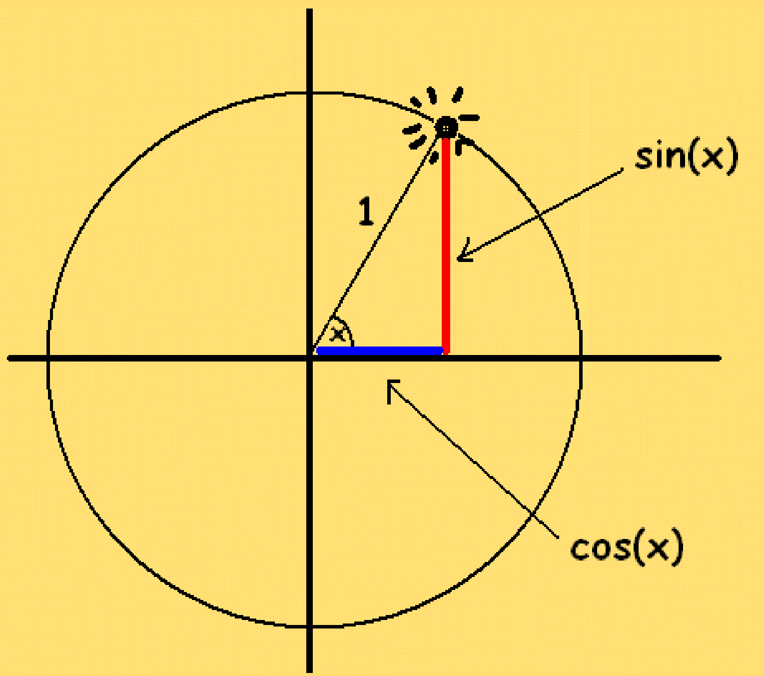
$17 \times 18 = 400 - 40 - 60 + ?? = 306$

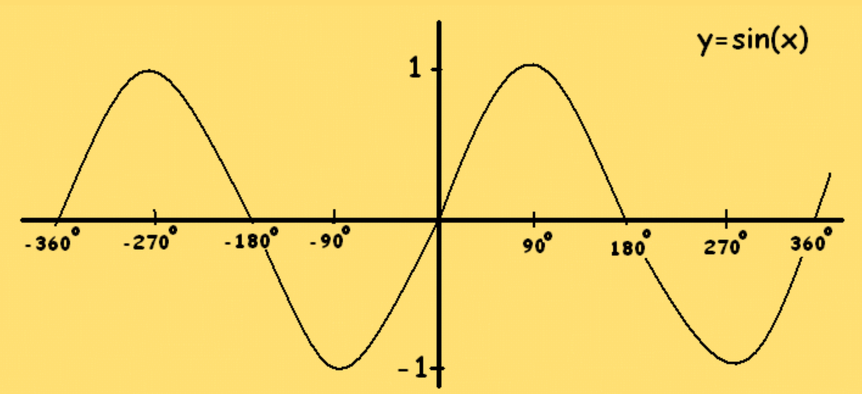
+6

III: ENGAGE IN INTELLECTUAL PLAY

Playing can cement standard ideas.

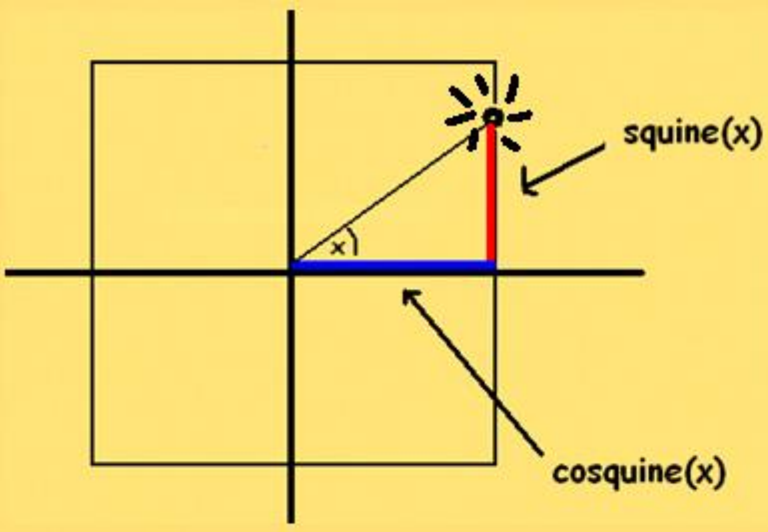
e.g. SINE and COSINE

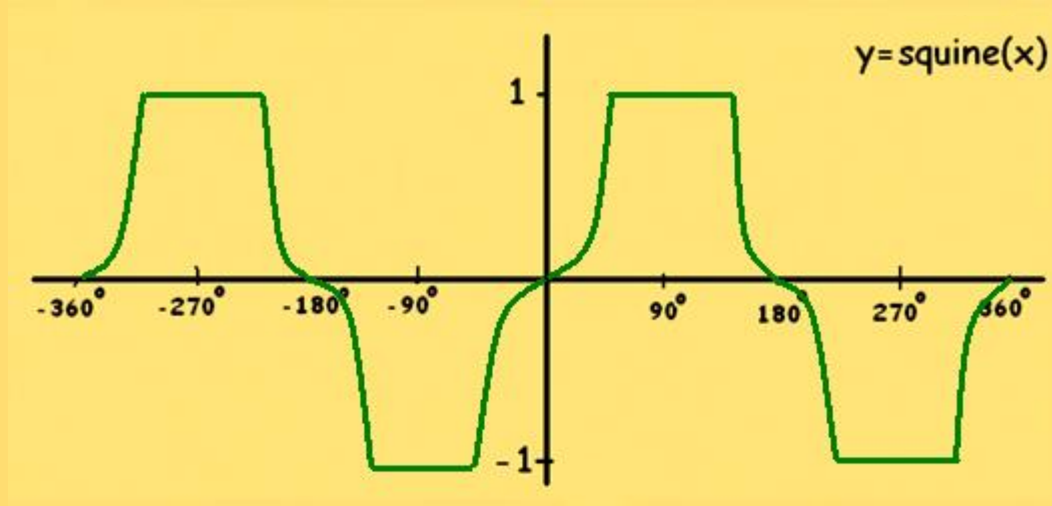




III: ENGAGE IN INTELLECTUAL PLAY

SQUINE and COSQUINE





Graph of $y = \text{tanq}(x) = \dfrac{\text{squine}(x)}{\text{cosquine}(x)}$?

III: ENGAGE IN INTELLECTUAL PLAY

The differential equation $\frac{dy}{dx} = ky$ with $y(0) = 1$ has solution: $y = e^{kx}$.

$$\boxed{\frac{dy}{dx} = iy \;,\; y(0) = 1}$$

$y = e^{ix}$

$y = \cos x + i \sin x$

$$e^{ix} = \cos x + i \sin x \quad ??$$

III: ENGAGE IN INTELLECTUAL PLAY

Create a cultural change in the classroom.

From students asking …

"Am I allowed to do this?"

to students asking …

"Is it helpful to do this?"

for which one answers …

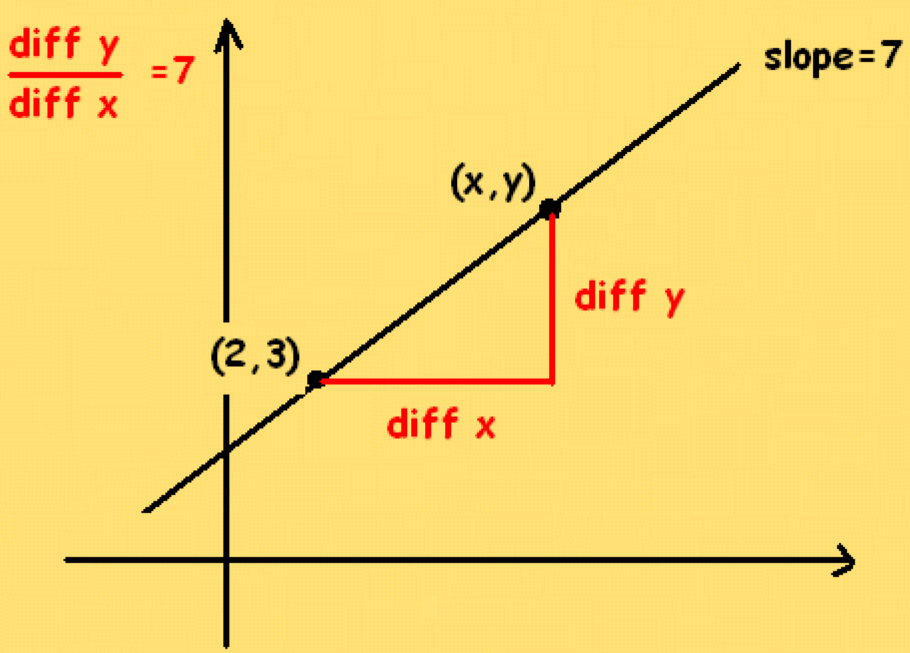
"I don't know. Try it and find out."

IV: UNDERSTANDING TRUMPS MEMORISING

Actually …

Mathematics in the classroom is moving away from memorizing formulas and performing rote “plug-and-chug.” (WooHoo!)

EQUATION OF A LINE:



We’re looking for an equation that must be true for a point (x,y) to be on the line.

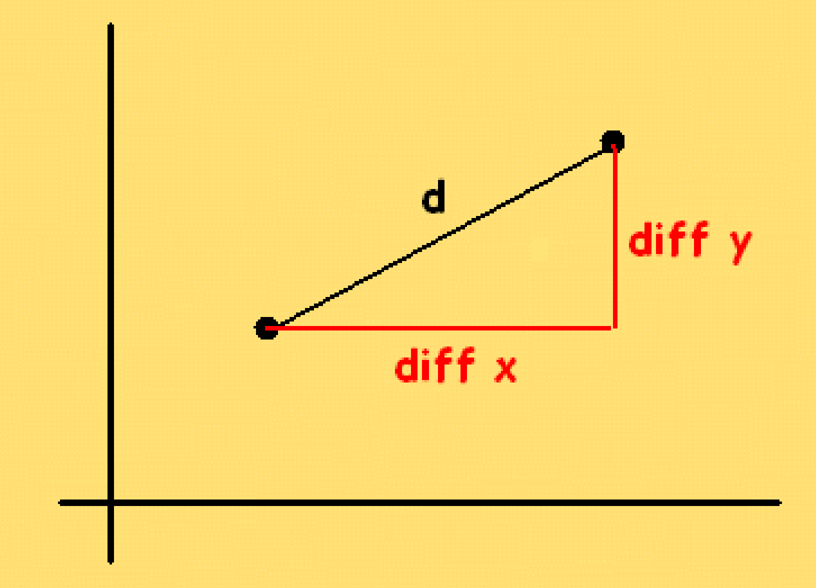
We like to believe that lines have the property “rise over run” always the same.

$$\frac{y-3}{x-2}=7$$

And feel free to manipulate this equation for the context at hand. (Maybe multiply through by x-2, at the very least.)

IV: UNDERSTANDING TRUMPS MEMORISING

DISTANCE FORMULA **= PYTHAGORAS'S THEOREM!**



$$d = \sqrt{(diff\ x)^2 + (diff\ y)^2}$$

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

HARD!

IV: UNDERSTANDING TRUMPS MEMORISING

Solve $x^2 + 4x + 1 = 86$

Well … This is about squares again. We have x^2.

	x	2
x	x^2	$2x$
2	$2x$	4

But there is more. Let's keep it a square.

$$x^2 + 4x + \overset{+3}{1} = \overset{+3}{86}$$

$$x^2 + 4x + 4 = 89$$

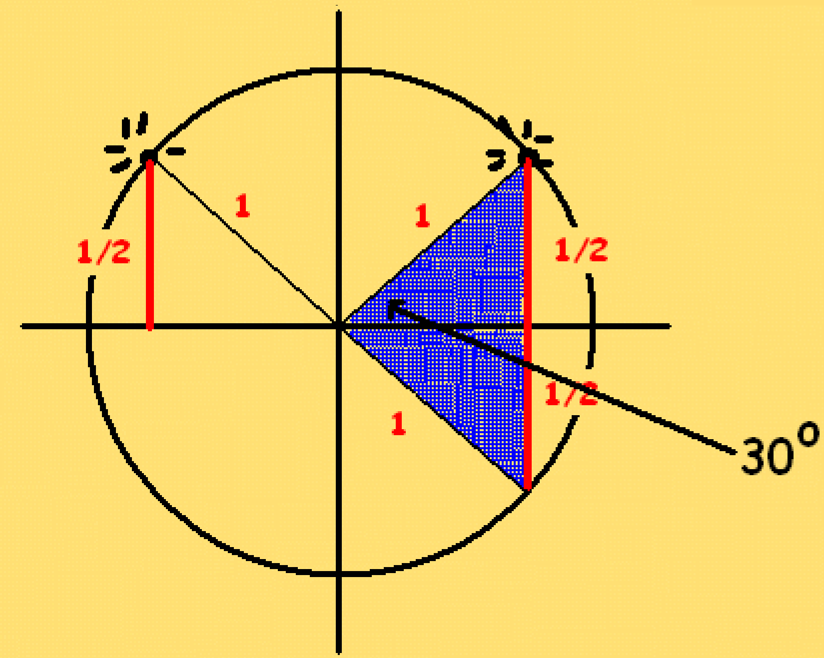
$$(x+2)^2 = 89$$

$$\vdots$$

IV: UNDERSTANDING TRUMPS MEMORISING

Find $\sin^{-1}\left(\frac{1}{2}\right)$

Where does a star have height one-half?



$$\sin^{-1}\left(\frac{1}{2}\right) = 30^{\circ} \;\; or \;\; 150^{\circ}$$

COMMENT: The numbers $\frac{1}{2}$ $\frac{\sqrt{3}}{2}$ $\frac{1}{\sqrt{2}}$ stick in your mind after a while.

Doing $\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$, for example, becomes just as easy.

IV: UNDERSTANDING TRUMPS MEMORISING

Please memorize the following division rule for 9:

Read the number from left to right
and write down the partial sums of its digits.

e.g.

$$213011 \div 9 = 23667\ R\ 8$$

Aren't you just burning to know why this works?

IV: UNDERSTANDING TRUMPS MEMORISING

Who memorizes … ?

$$\left(\frac{f}{g}\right)' = \frac{f'g - g'f}{g^2}$$

Differentiate $f \cdot g^{-1}$

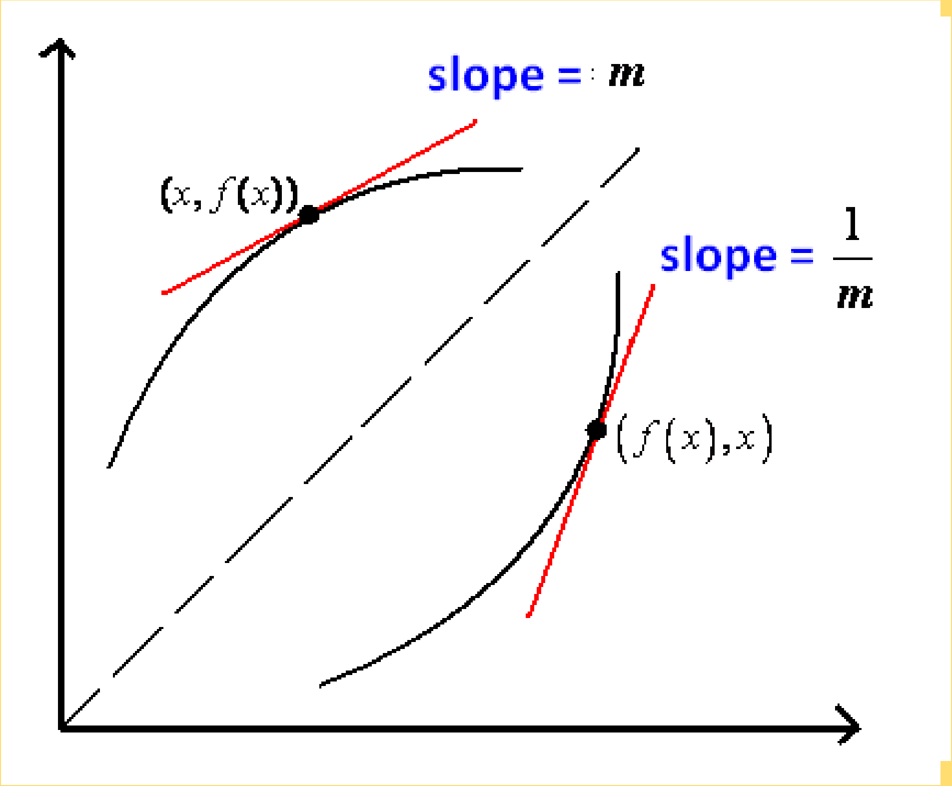
$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$

Differentiate

$$f\left(f^{-1}(x)\right) = x$$

IV: UNDERSTANDING TRUMPS MEMORISING

$$\left(f^{-1}\right)'(x) = \frac{1}{f'\left(f^{-1}(x)\right)}$$



V: MATH PEOPLE ARE CLEAR ON WHAT THEY DON'T KNOW
- and are comfortable admitting that they don't know!

Absolutely no shame in not knowing!

One needs to struggle and flail with ideas in order to truly understand them, to truly internalize them.

Often (usually!) one needs to see why something doesn't work in order to properly understand why what works does.

V: MATH PEOPLE ARE CLEAR ON WHAT THEY DON'T KNOW
- and are comfortable admitting that they don't know!

Example: **ZERO IS HARD!**

Is zero even or odd?

Is -0 the same as +0?

Is zero even a counting number? What does it count?

Why can't we divide by zero?

$\frac{20}{5} = 4$ CHECK: $4 \times 5 = 20$ ✓

$\frac{5}{0} = 2$ CHECK: $2 \times 0 = 5$ ✗

$\frac{0}{0} = 17$ CHECK: $17 \times 0 = 0$ ✓

V: MATH PEOPLE ARE CLEAR ON WHAT THEY DON'T KNOW
- and are comfortable admitting that they don't know!

I don't really understand exponents!

$2^3 = 2 \times 2 \times 2$

$2^5 = 2 \times 2 \times 2 \times 2 \times 2$

$2^1 = ??$

$2^0 = ??$

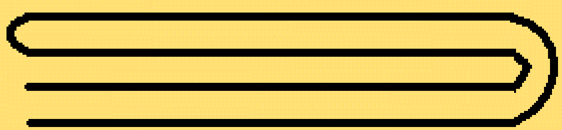
$2^{1/2} = ??$

$2^{-1} = ??$

V: MATH PEOPLE ARE CLEAR ON WHAT THEY DON'T KNOW
- and are comfortable admitting that they don't know!

Paper-folding maybe …

1 fold = two layers: $2^1 = 2$



2 folds = four layers: $2^2 = 4$

0 folds = one layer: $2^0 = 1$

-1 folds = peel the paper open = half a layer: $2^{-1} = \frac{1}{2}$

Half a fold? $2^{1/2} = ??$

V: MATH PEOPLE ARE CLEAR ON WHAT THEY DON'T KNOW - and are comfortable admitting that they don't know!

An age-old question:

What's 0.9999… ?

$$0.99999999\ldots = x$$

$$9.999999\ldots. = 10x$$

$$9 + x = 10x$$
$$9 = 9x$$
$$x = 1$$

So

$$0.999999\ldots. = 1$$

What's ….99999?

$$\ldots 99999 = y$$

$$\ldots 999990 = 10y$$

$$y - 9 = 10y$$
$$-9 = 9y$$
$$y = -1$$

So

$$\ldots 99999 = -1$$

What's …999.999…?

$$\ldots 999.999\ldots = w$$

$$\ldots 9999.99\ldots = 10w$$

$$w = 10w$$
$$w = 0$$

So

$$\ldots 999.999\ldots = 0$$

IS ANY OF THIS TRUE?

V: MATH PEOPLE ARE CLEAR ON WHAT THEY DON'T KNOW
- and are comfortable admitting that they don't know!

Lots of deep questions to be had. (Usually deemed "elementary"!)

Why is the value of pi the same for all circles?

Why circles all the time? Is there a value of pi for a square?

Why are quadratics called ***QUAD***?
What have they got to do with the number four?

Why do factor trees always give the same list of primes in the end, no matter the choices you make along the way?

Why are the powers of 11 the rows of Pascal's triangle?

Is 0.9999… equal to 1 or is it not?

V: MATH PEOPLE ARE CLEAR ON WHAT THEY DON'T KNOW
- and are comfortable admitting that they don't know!

Why is 0! equal to 1?

What is “tangent” in trigonometry called “tangent” from geometry? (And secant “secant”?)

Why is the graph of an inverse function just a reflection across the diagonal line?

Why does the shell method work? (After all, unrolling a shell does change its volume!!)

Why is log base “e” the <u>natural</u> logarithm to use?

It is okay not to know.

It is not okay not to want to find out.

Something fun: **Sketch the graph of** $y = x^{\frac{1}{\ln x}}$. **What do you notice?**

V: MATH PEOPLE ARE CLEAR ON WHAT THEY DON'T KNOW
- and are comfortable admitting that they don't know!

Admitting that you don’t know is the most powerful of principles.

Top-notch math people have the confidence …

To pin down what it is they don’t know

To muddle and try things

To make lots of mistakes

To use insights about what doesn’t work to understand what does.

To ask questions and consult resources
– and respond to answers with personal skepticism!
“Do I really believe that?”

To have fun with it all!

All sound like good life management to me!

THANKS!

JAMES TANTON
tanton.math@gmail.com

For great mathematical problem-solving in the classroom, see:

MAA's CURRICULUM INSPIRATIONS

www.maa.org/ci .

How to Think Brilliantly and Creatively in Mathematics: A Guide for K–12 Educators and Their Students.

Saturday, January 9, 2016
8:00 am–8:50 am.

This lecture is a guide for thinking brilliantly and creatively in mathematics for K–12 educators, their students, and all seeking joyful mathematics doing. How do we model and practice uncluttered thinking and joyous doing in the classroom? Pursue deep understanding over rote practice and memorization? Develop the art of successful flailing? Our complex society demands of its next generation not only mastery of quantitative skills, but also the confidence to ask new questions, explore, wonder, flail, persevere, innovate, and succeed. Let's not only send humans to Mars, let's teach our next generation to solve problems and get those humans back if something goes wrong! In this talk, James Tanton, MAA, will explore five natural principles of mathematical thinking. We will all have fun seeing how school mathematical content is the vehicle for ingenuity and joy. All are so welcome to attend!

The sponsor for this lecture is the MAA Council on Outreach.