Hyperbinary Numbers and the Calkin-Wilf Tree

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joint work with Samuel Coskey (Boise State), Japheth Wood (Bard)

Hyperinary Numbers - Definition

Every positive integer can be written in binary in precisely one way. For example, $27_{10} = 11011_2$, meaning that

$$27 = 16 + 8 + 2 + 1 = 1 \times 2^4 + 1 \times 2^3 + 0 \times 2^2 + 1 \times 2^1 + 1 \times 2^1 = 11011_2$$

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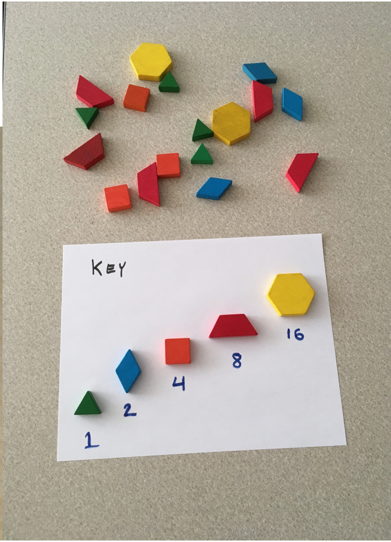
In the hyperbinary number system, each power of two can be used at most *twice* to represent a positive integer. This gives additional ways to represent numbers. For example,

$$27 = 16 + 4 + 4 + 2 + 1 = 10211_2$$

$$27 = 8 + 8 + 4 + 4 + 2 + 1 = 2211_2$$

Hyperbinary Numbers - Discovery



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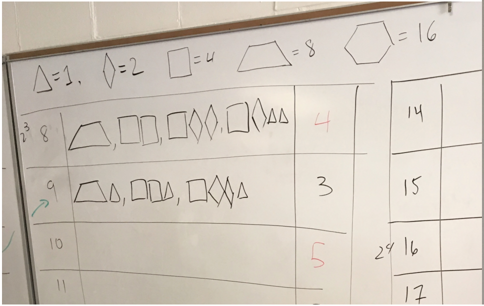

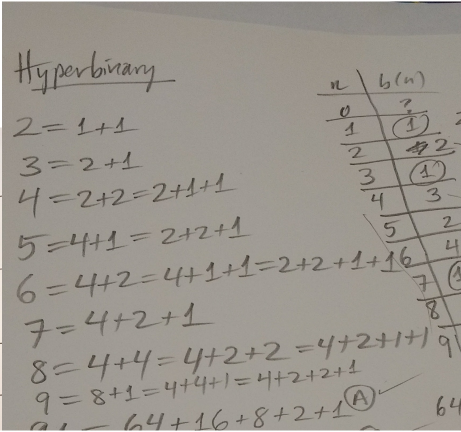

Hyperbinary Numbers - Discovery

n	Hyperbinary	b(n)
0	0	1
1	1	1
2	10, 2	2
3	11	1
4	100, 20, 12	3
5	101, 21	2
6	110, 102, 22	3
7	111	1
8	1000, 200, 120, 112	4
9	1001, 201, 121	3
10	1010, 210, 1002, 202, 122	5
11	1011, 211	2

Hyperbinary Numbers - Analysis

n	representations	n	representations
1	1	3	11
2	$10, 2$	5	$101, 21$
3	11	7	111
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10	5	4
1010	101	
210	21	
1002		100
202		20
122		12

$$b(2n) = b(n) + b(n-1)$$

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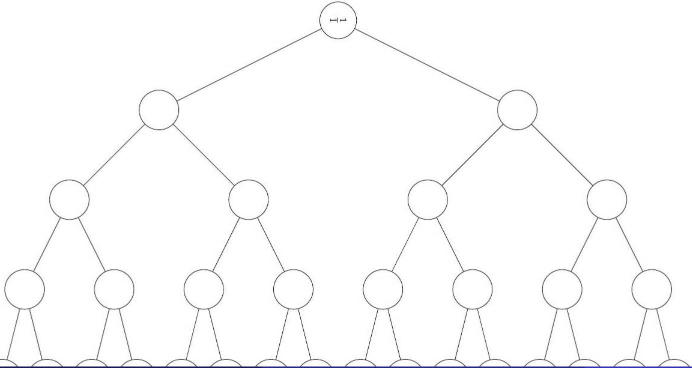
$$b(2n) = b(n) + b(n-1)$$

Also $b(0) = 1$ and $b(1) = 1$

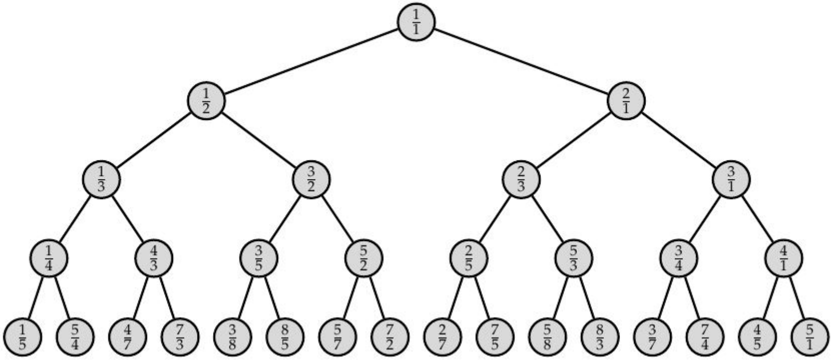
New Topic: Calkin-Wilf Tree

The Calkin-Wilf binary branching tree of fractions, or Calkin-Wilf tree for short, is constructed with the following rules.

- The root node is at the top and is labelled $\frac{1}{1}$
- Every node has a left and right "child" node below it. If the node is labelled $\frac{i}{j}$, then its left child is labelled $\frac{i}{i+j}$ and its right child is labelled $\frac{i+j}{j}$.

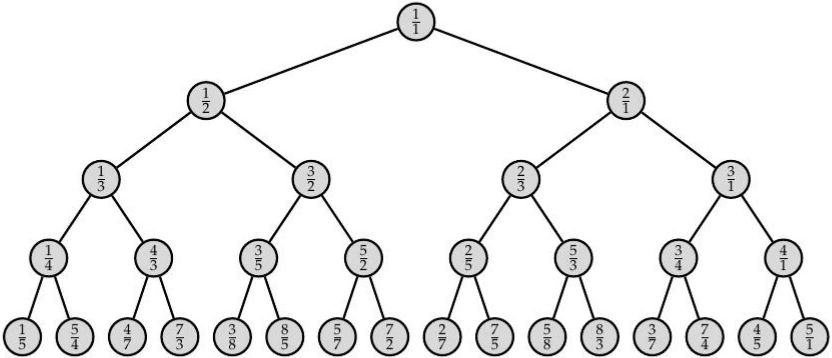


Calkin-Wilf Tree



Patterns?

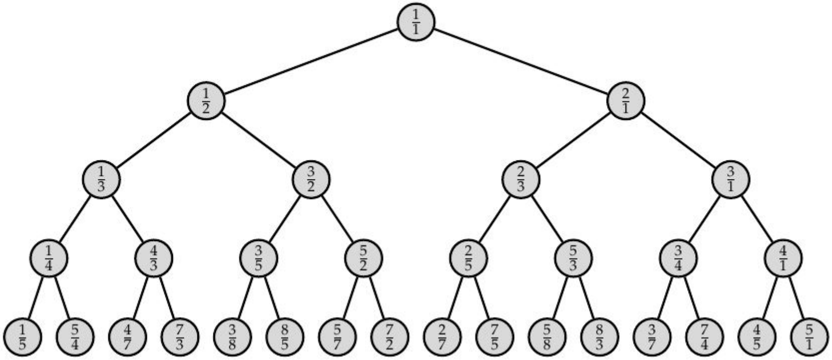
Calkin-Wilf Tree



Patterns?

Some just follow from the definition: Left children are always less than 1, and right children are always greater than 1

Calkin-Wilf Tree

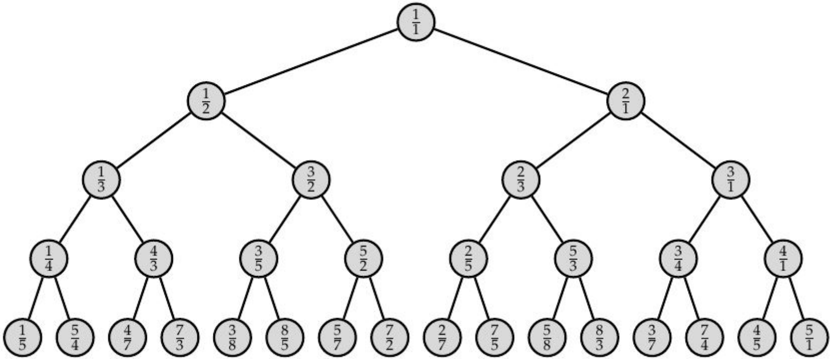


Patterns?

Some just follow from the definition: Left children are always less than 1, and right children are always greater than 1

Others are neat, but not the direction we want to go: Sum of numerators on level n is 3^n

Calkin-Wilf Tree



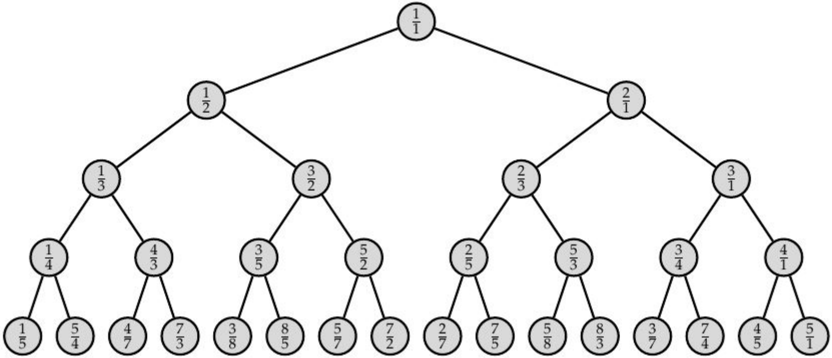
Initial observations:

All numberators down the left side are 1.

All denominators down the right side are 1.

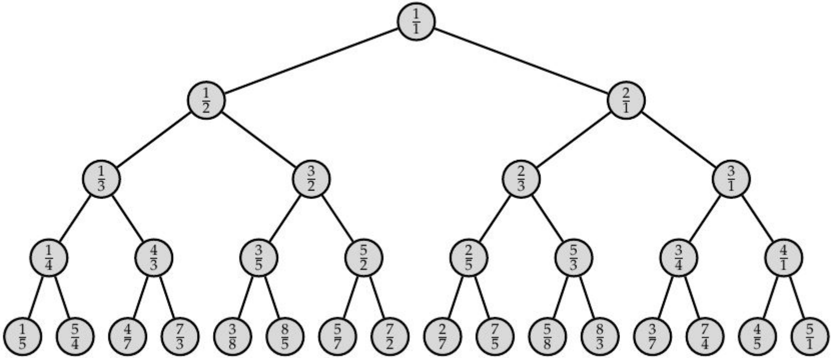
The denominator of any node is the same as the numerator of the next node to the right.

Calkin-Wilf Tree



Every reduced fraction appears exactly once in the tree!

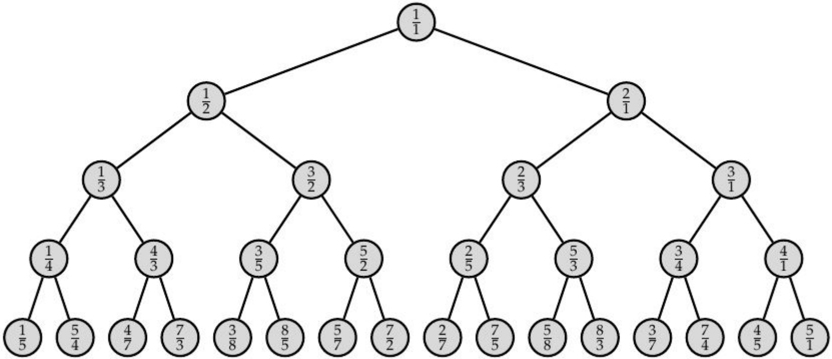
Calkin-Wilf Tree



Every reduced fraction appears exactly once in the tree!

- Every number in the tree is a reduced fraction;
- Every positive rational number appears in the tree; and
- No number appears more than once in the tree

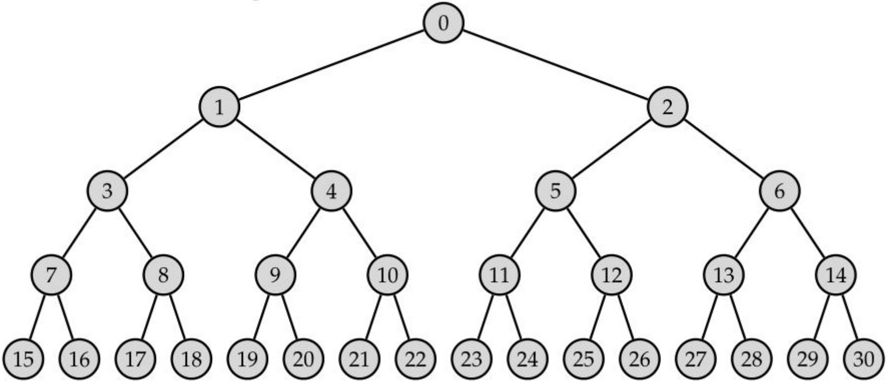
Putting the two investigations together



The sequence of numerators of each node, read left to right and then top to bottom, is exactly the hyperbinary sequence $b(n)$!

- The denominator of node n is the numerator of node $n+1$.
- Thus, the fraction label of node n has the form $f(n)/f(n+1)$ for some sequence $f(n)$.
- The sequence $f(n)$ is exactly the sequence $b(n)$ explored earlier.

Why are the two sequences the same?



The left child of node n is always node $2n+1$.

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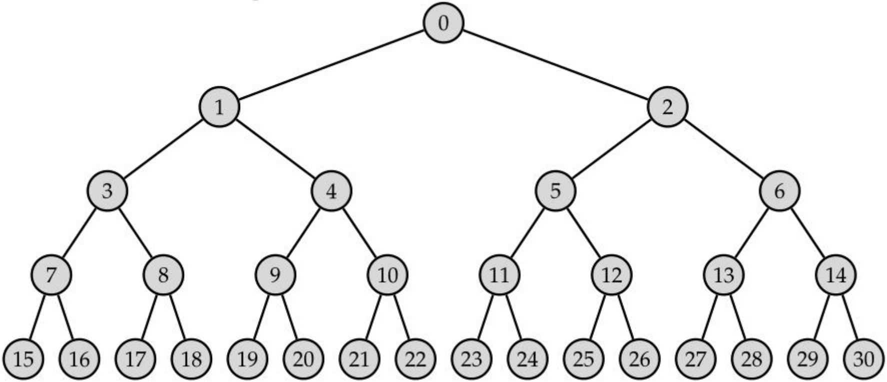

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The left child of the fraction labelled $\frac{f(n)}{f(n+1)}$ is the fraction labelled $\frac{f(2n+1)}{f(2n+2)}$.

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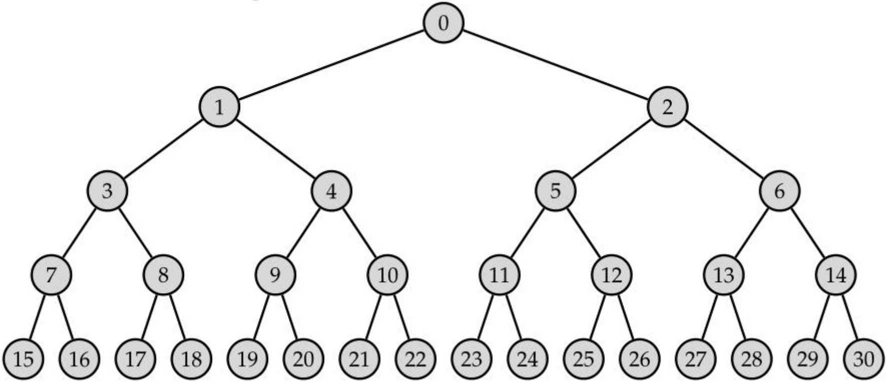

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The left child of the fraction labelled $\frac{f(n)}{f(n+1)}$ is the fraction labelled $\frac{f(2n+1)}{f(2n+2)}$.

Def of 'left child': $f(2n+1) = f(n)$ and $f(2n+2) = f(n) + f(n+1)$.

The Grand Conclusion

If one writes out the sequence $b(n)$, which counts the number of hyperbinary represenations of n, and then makes fractions out of the successive terms in the sequence, then one obtains an enumeration of all positive rational numbers, each in lowest terms, with no repeats!

References

Neil Calkin, Herbert S. Wilf
Recounting the Rationals
The American Mathematical Monthly Vol. 107, No. 4 (2000), pp. 360-363

Austin Purves
https://www.thingiverse.com/thing:2320981
(Those binary coins)

Our Upcoming Book:

Samuel Coskey, Paul Ellis, Japheth Wood
Five Fabulous Math Circle Activities
To be published with Natural Math in 2019

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