# It's Circular:

## Conjecture, Compute, Iterate

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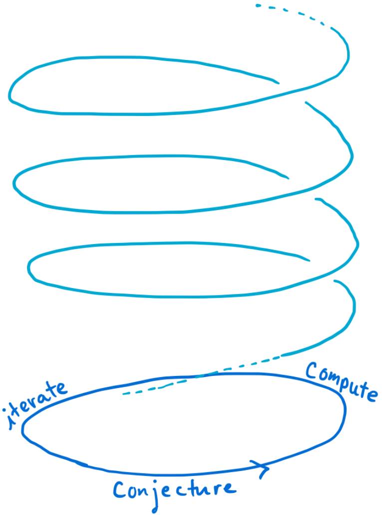
iterate
Compute
Conjecture

# Introduction

> "Not having achieved what they desired, they pretended to desire what they had achieved."
>
> *M. Montel*

In this series of lectures I will talk about several new directions in mathematical research. All of these are based on the idea of numerical experimentation. After looking at examples such as $5 \cdot 5 = 25$ and $6 \cdot 6 = 36$, we advance an hypothesis, such as $7 \cdot 7 = 47$. Further experimentation either supports or disproves it.

For example, Fermat's hypothesis (that the equation $x^n + y^n = z^n$ cannot be solved in natural numbers with $n > 2$) was advanced as a result of his attempts at a solution. This hypothesis led to the creation of a whole field of knowledge, but it was proved only after a few hundred years had passed.

The majority of hypotheses that we make are so far not proven (nor refuted). I decided to give these lectures exactly because of my hope that the listeners will help in the investigation of these problems, if only by conducting numerical experiments (which I have also conducted, without a computer, in the bounded region of the first million numbers).

Arnold - Experimental Mathematics (MSRI Mathematical Circles Library)

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Chapter 1

# Algebraic Structures

When you are introduced to people, at first you only learn their names and faces. Meeting them later, you begin to know them better, maybe even become friends with them.

In the first chapter, you will be only introduced to most of the algebraic structures considered in this book. A deeper understanding of them should come later, through reading and problem-solving.

Vinberg - A Course in Algebra

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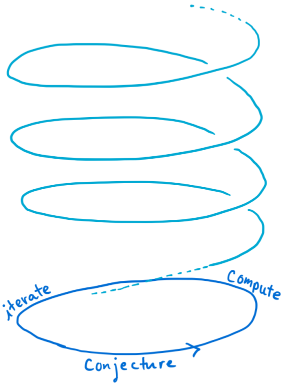
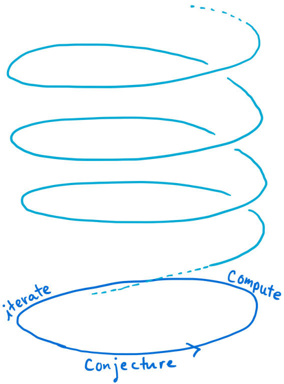
My own math circle experience:

1. Dr. Gabriella Pinter's math circle helped me learn to do research.
2. I didn't find or discover math at a young age.
3. A specific two days of English class helped me more than 12 years of math classes.
4. What do I hope to achieve in future math circles?



The basics on the bottom circle:

1. Experimentation!
2. Solve a simpler problem first.
3. Compute things, full details!
4. Draw pictures, lose details!
5. Math is the part of physics
   where the experiments are cheap.

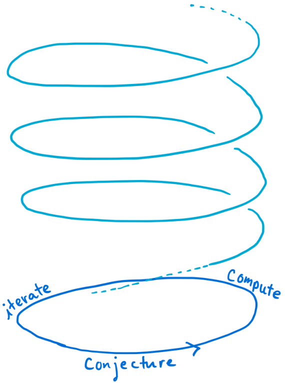


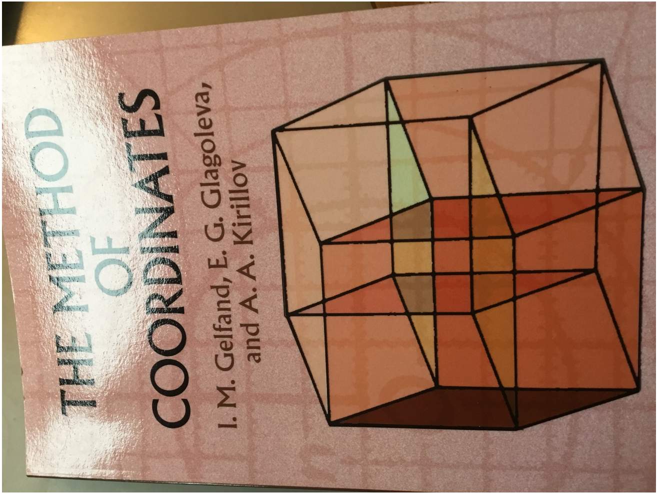
How to transcend and spiral upwards?

1. Simple joy, yes. But make room for profound, mysterious, deep joy too.
2. Black
   holes, dark matter, time travel

How to start?

1. Dimensions.
2. A good
   example is your friend. You will remember and cherish it forever.






Gelfand, Glagoleva, Kirillov - The Method of Coordinates

# Spaces of low dimensions

Yes, students should know the fundamental examples:

1. 1 dim: line, circle . . .
2. 2 dim: plane, the surface of a sphere . . .
3. 3 dim: the room we stand in . . .
4. 4 dim: the room we stand in, plus time

# Spaces of low dimensions

1. 1 dim: the edge of a piece of paper, **the (one!) edge of a Möbius strip**, rewinding/fast-forwarding a video, a robotic arm with one joint ...
2. 2 dim: plane, disc, **a musical melody**, the surface of a basketball, the space of all lines through a point, the space of **all possible recipes using 3 ingredients**, Newtonian motion of a particle on the line, the space of polynomials $x^3 + bx + c$ ...
3. 3 dim: **a sphere living inside 4-dimensional space**, the space of all rotations of a basketball, the space of all recipes using 4 ingredients, a robotic arm with 3 joints ...
4. 4 dim: the Pythagorean theorem in 4-dimensional space, **the Pythagorean theorem with one minus sign in 4-dimensional space**, the space of recipes using 5 ingredients, Newtonian motion of a particle in the plane ...

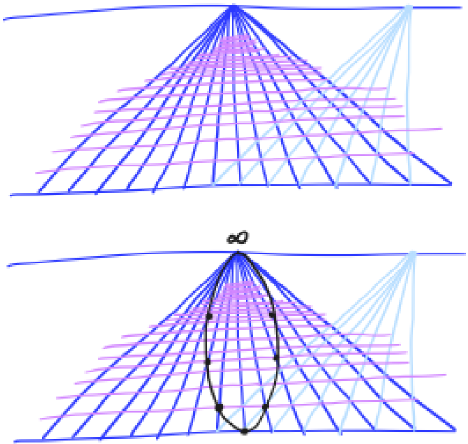
# One amazing example

Consider this wonderful example: the double cover $SU_2 \to SO_3$.

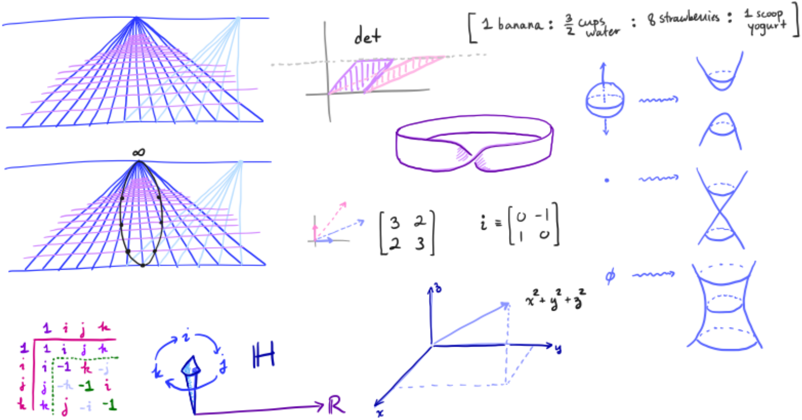
1. The unit sphere $S^3$ in $\mathbb{R}^4$
2. $\mathbb{P}(\mathbb{R}^4) = \mathbb{RP}^3$ projective space
3. The spin of electrons
4. Transformations of spacetime (special relativity)
5. The quaternions and $SO_4$

$$\begin{bmatrix} z & w \\ -\bar{w} & \bar{z} \end{bmatrix}$$

A challenge: devise a sequence of explorations over several/many months leading up to understanding this example.



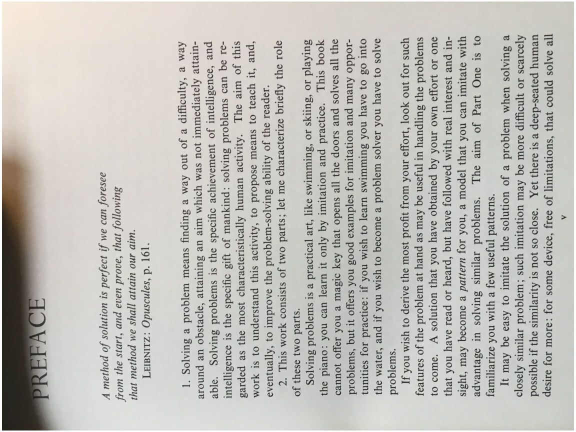

A parabola viewed projectively




A lot. But as part of this one example, this one story, it's worth it!

# For Further Reading I


- I.M. Gelfand, E.G. Glagoleva, and A.A. Kirillov
  The Method of Coordinates
- Stephanie F. Singer
  Linearity, Symmetry, and Prediction in the Hydrogen Atom
- Shlomo Sternberg
  Group Theory and Physics: Chapter 1
- Ernest B. Vinberg
  Linear Representations of Groups: Chapter 3



PREFACE

*A method of solution is perfect if we can foresee from the start, and even prove, that following that method we shall attain our aim.*
LEIBNITZ: *Opuscules*, p. 161.

1. Solving a problem means finding a way out of a difficulty, a way around an obstacle, attaining an aim which was not immediately attainable. Solving problems is the specific achievement of intelligence, and intelligence is the specific gift of mankind: solving problems can be regarded as the most characteristically human activity. The aim of this work is to understand this activity, to propose means to teach it, and, eventually, to improve the problem-solving ability of the reader.

2. This work consists of two parts; let me characterize briefly the role of these two parts.

Solving problems is a practical art, like swimming, or skiing, or playing the piano: you can learn it only by imitation and practice. This book cannot offer you a magic key that opens all the doors and solves all the problems, but it offers you good examples for imitation and many opportunities for practice: if you wish to learn swimming you have to go into the water, and if you wish to become a problem solver you have to solve problems.

If you wish to derive the most profit from your effort, look out for such features of the problem at hand as may be useful in handling the problems to come. A solution that you have obtained by your own effort or one that you have read or heard, but have followed with real interest and insight, may become a *pattern* for you, a model that you can imitate with advantage in solving similar problems. The aim of Part One is to familiarize you with a few useful patterns.

It may be easy to imitate the solution of a problem when solving a closely similar problem; such imitation may be more difficult or scarcely possible if the similarity is not so close. Yet there is a deep-seated human desire for more: for some device, free of limitations, that could solve all

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Pólya - Mathematical Discovery

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