Modeling with Mathematics:
MTC Sessions to Encourage and Illuminate CCSS MP4

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On the following pages, you can find session descriptions and resources for the sessions from my talk. Feel free to write and ask for more details!

The SUPER-M project site (lots of math activities of various levels available here):
http://superm.math.hawaii.edu/

The Math Teachers' Circle of Hawaii site:
http://math.crdg.hawaii.edu/match/

The Common Core State Standards Math Practices:
http://www.corestandards.org/Math/Practice/

MTC on Bioacoustics

Ideally, have a visitor from marine biology or other department visit to give the background on marine mammals. Alternately, here are a couple of blog posts that provide the background:

Alexis Rudd's blog post that led me to invite her:
http://bioacoustics.blogspot.com/2012/05/we-should-have-studied-dolphins-in-high.html

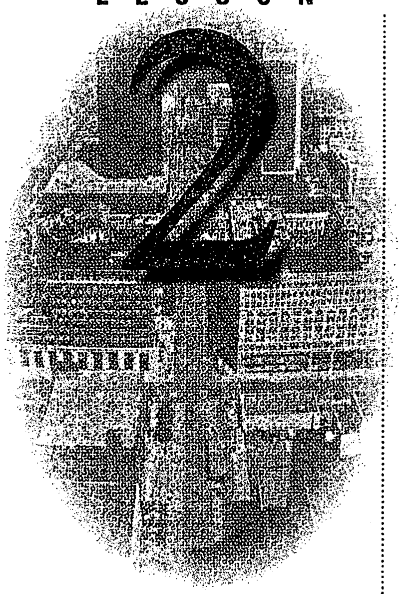
Alexis's reflections on her visit to our session:
http://bioacoustics.blogspot.com/2012/06/high-school-math-teachers-learn-about.html

A friend of Alexis blogs about a 2-mic hydrophone array (using trig to find the whale and turning the boat to decide on one of two possible locations):
http://www.michw.com/2014/10/hydrophone-arrays-ftw/

Flow of the session:

1) Group work on discovering the distance formula. I used a problem set from an old, out of print curriculum (scans attached on the next several pages). We asked teachers to work through the problems about distance between points in groups, and then a few groups shared out. We got the distance formula and put it on the board.
2) Alexis presented basics about marine mammals, including the biology behind the sounds they make. She answered lots of questions. (This is why it's good to invite an expert!) She also brought in a hydrophone array that the teachers could see & touch.
3) Alexis explained the basic setup of the problem: 15m between each hydrophone (not true, but works nicely with the speed of sound). Sound travels through salt water at a speed of 1500 m/sec (really depends on temperature and salinity, but this is a good estimate). If $\Delta t_1 = 0.0052$ sec and $\Delta t_2 = 0.0078$ sec, where is the whale in relation to the boat?
4) Teachers worked in groups for about 15 minutes; then we did some sharing out from one of the groups that had some ideas but hadn't solved it completely. Another 15 minutes of work and more sharing out. Repeat once more before getting to the answer.
5) Debrief. (Teacher comments, "That was hard!")

LESSON



Lines, Midpoints, and Distance

In this lesson, you will use coordinates to describe properties of li and find midpoints and lengths of segments.

Since coordinates are used to locate points, it should come as no surprise that are helpful for locating things, such as the midpoint of a line segment. And sir coordinates involve numbers, it makes sense that they might help you calculat lengths and distances better than measurement.

Notation: An easier way to write "A is at (5, 7)" is "A = (5, 7)."

Explore and Discuss

Suppose A is at (9, 5), and B is at (7, 5).

- **a** What is the distance between A and B?
- **b** Find the coordinates of the midpoint of $\overline{AB}$.

Suppose $C = (2, 5)$ and $D = (2, 396)$.

- **c** Find the distance between C and D.
- **d** Find the coordinates of the midpoint of $\overline{CD}$.

Suppose $E = (-5, -7)$ *and* $F = (12, -7)$.

- **e** What is the distance between E and F?
- **f** Find the coordinates of the midpoint of $\overline{EF}$.
- **g** State a conjecture about the coordinates of midpoints for horizontal and vertical segments.

ACTIVITY 1 Midpoints and Distance between Points

You have found some distances and midpoints for horizontal and vertical segn But what about segments that are not horizontal or vertical? The following prc will help you extend your methods to "slanted" segments.

1. Find the distances between the given pair of points.
   a. $I = (-110, -7)$ and $J = (-80, -7)$
   b. $K = (1, 5)$ and $L = (1, -15)$
   c. $M = (-93, 4)$ and $N = (90, 4)$
2. Here are the coordinates of four points.

   $A = (4, 2)$ $\quad B = (8, 5)$ $\quad C = (-4, 3)$ $\quad D = (-7, 7)$

   Which of the following statements can you justify? How?

   a. Segments AB and CD are guaranteed congruent.

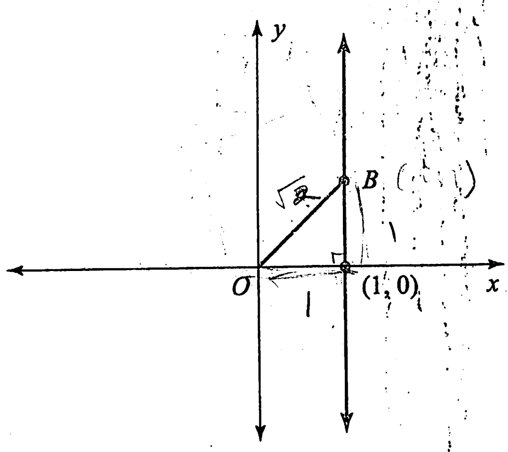
b. Segments AB and CD are approximately the same length, but are not guaranteed congruent.

3. Find the distances between the given pairs of points.

a. $P = (a, b)$ and $Q = (a, c)$

b. $P = (a, c)$ and $Q = (b, c)$

4. What about the distance from O to B, where $O = (0, 0)$ and $B = (1, 1)$? Solve the problem any way you can.



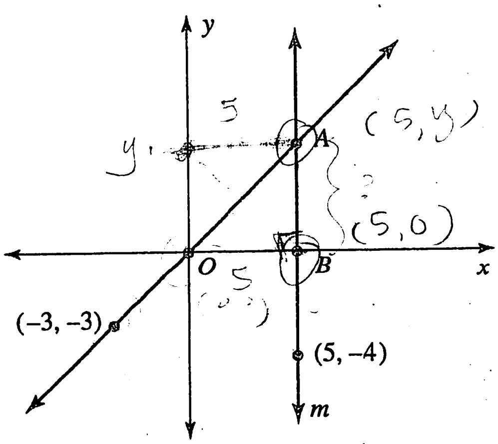
$1^2 + 1^2 = 2$

5. The following are the coordinates for the three vertices of a triangle: $E = (-1, -3)$, $F = (2, -3)$, and $G = (2, 1)$.

a. How long are $\overline{EF}$ and $\overline{FG}$?

b. Find the distance EG.

6. In the picture below, m is a vertical line.



If you are not familiar with the dimensions of the particular triangles in Problems 5 and 6, use the Pythagorean Theorem.

a. Find the coordinates of A and B.

b. How long is $\overline{AB}$?

c. What is the area of $\triangle ABO$?

d. How long is $\overline{AO}$?

7. Use one set of axes for the following.

   a. Find eight points that are 5 units away from the origin.

   b. Draw the picture of all the points that are 5 units away from the origin. What shape is it? Why?

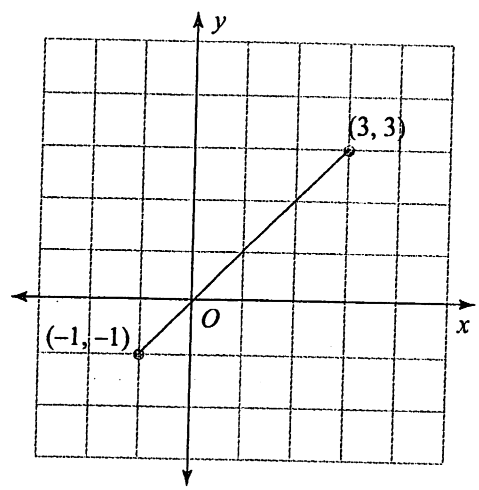
   c. Find eight more points on the figure you found in part **b.**

8. **Write and Reflect** Consider these problems.

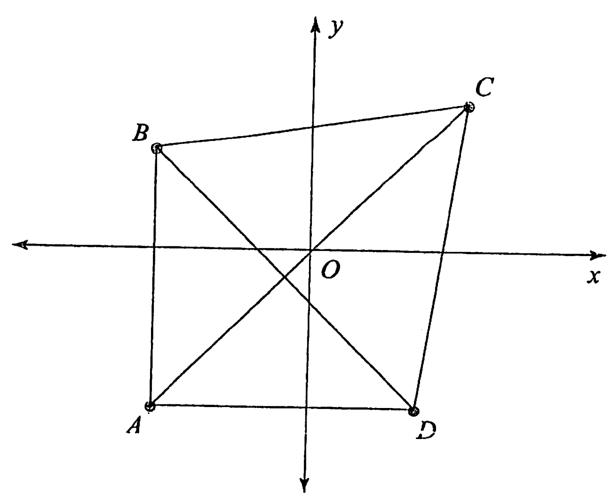
   a. Write a set of instructions that explains how to find the distance between *any* two points if you know their coordinates.

   b. Exchange your instructions with a partner. See if you can follow your partner's instructions to find the distance between (1, 3) and (6, 15).

9. Devise a method to find the coordinates of the midpoint of the segment below.



10. In the picture below, $A = (-3, -3)$, $B = (-3, 2)$, $C = (3, 3)$, and $D = (2, -3)$. Use your method from Problem **9** to find the coordinates of the midpoints of $\overline{AB}$, $\overline{CD}$, $\overline{AD}$, $\overline{BC}$, $\overline{AC}$, and $\overline{BD}$. If the method you used for Problem **9** does not work here, look at Problem **9** again. Try to develop a method that will work.



11. **Write and Reflect** Consider these problems.

a. There are many correct ways to find the coordinates of the midpoint of a line segment when you know the coordinates of the endpoints. Describe your method precisely. Explain why it works.

b. Exchange the methods you produced in Problems **9** and **10** with a partner. See if you can successfully follow your partner's method in finding the coordinates of the midpoint of $\overline{JK}$, where $J = (-2, 1)$ and $K = (1, 6)$.

12. Checkpoint Use your methods to find the midpoints and lengths of the segments whose endpoints are listed below.

a. (0, 7) and (5, 7)

b. (−3, −3) and (−1, 2)

ACTIVITY 2

Midpoint and Distance Formulas

In this activity, you will formalize the instructions you wrote for finding the midpoint and the distance between points.

13. Assume $G = (x, y)$ and $H = (w, z)$.

a. Use your method from Activity 1 to find the distance GH.

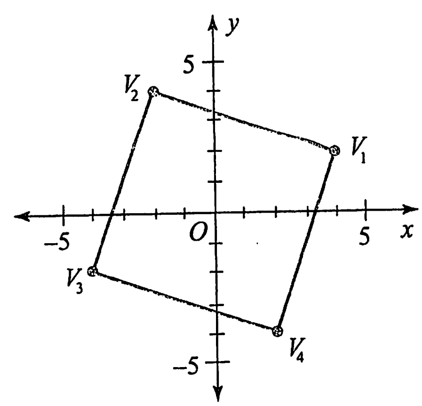
b. Use your method from Activity 1 to find the coordinates of the midpoint of $\overline{GH}$.

Subscript notation is used in the standard formulas for midpoints and distance, so it is introduced here.

When only a few points need names, it is convenient enough to call them A, B, C, and so on, with coordinates (a, b), (c, d), (e, f), and so on. But it is often important to have names for many points, and then one quickly runs out of letters. Numbers *never* run out, and so the convention is to name, say, the vertices of a decagon with names like these $A_1, A_2, A_3, \ldots, A_{10}$, and the vertices of an n-gon with names like these: $B_1, B_2, B_3, \ldots, B_n$. The notation is supposed to make sense. See if you can figure out the logic of the notation in the problems below.

14. Look at the drawing of the square. Copy and complete the table below. Assume that point V_1 has coordinates (x_1, y_1) and point V_2 has coordinates (x_2, y_2), and so on.

i	Coordinates of V_i	x_i	y_i
1	(,)		
2	(,)	−2	
3	(−4, −2)		
4	(,)	2	



Square $V_1V_2V_3V_4$

15. Here is a claim about the coordinates of the vertices of square $V_1V_2V_3V_4$: $x_i = y_{i+1}$. Is that claim true when $i = 1$? That is, is it true that $x_1 = y_2$? Is that claim true when $i = 2$? When $i = 3$? When $i = 4$?

16. Here is another claim about the vertices of square $V_1V_2V_3V_4$: If V_i has coordinates (x_i, y_i), then V_{i+1} has coordinates $(-y_i, x_i)$.

   a. When $i = 2$, the claim says: If V_2 has coordinates (x_2, y_2), then V_3 has coordinates $(-y_2, x_2)$. Look at the table and decide whether or not this is a true statement.

   b. Pick a value of i for which the statement does not make sense.

17. Name the vertices of the square for which it is true that $y_i = \frac{1}{2}x_i$.

$P_3 = (y_2, -x_2)$

18. Start with point $P_1 = (3, 4)$ and find the coordinates of P_2, P_3, and P_4, if the points follow the rule: If P_i has coordinates (x_i, y_i), then P_{i+1} has coordinates $(y_i, -x_i)$. Plot and label the points.

19. Here is a rule for deriving a new set of points Q_i from the points V_1, V_2, V_3, and V_4: If $V_i = (x_i, y_i)$, then $Q_i = (-3 + x_i, 4 + y_i)$.

   a. The rule is written in algebraic symbols. Explain it in words.

   b. Find the new points Q_1, Q_2, Q_3, and Q_4. Plot the new points.

20. If $P_1 = (x_1, y_1)$ and you know that P_2 is a second point on the same horizontal line, how could you write its coordinates?

21. $P_i = (x_i, y_i)$, $x_i = i + 3$, and $y_i = x_i - 4$. Plot P_i as i goes from 1 to 8.

Which of these is most like your method?

There are at least two ways to write a formula for the midpoint of a segment with endpoints at (x_1, y_1) and (x_2, y_2):

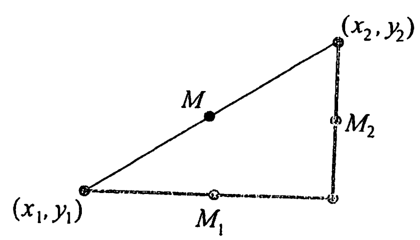
$$(1) \quad M = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

$$(2) \quad M = \left(x_1 - \frac{x_2 - x_1}{2}, y_1 - \frac{y_2 - y_1}{2}\right)$$

22. Pick one of the formulas and translate it into a sentence in English.

Both of these formulas work; you will work through a proof of the first one, filling in the details.

The goal is to find the coordinates of M, the midpoint of the segment connecting (x_1, y_1) and (x_2, y_2). You will use what you know about horizontal lines, vertical lines, and similar triangles.



23. In the picture above, a horizontal segment is drawn from (x_1, y_1) and a vertical segment has been drawn from (x_2, y_2). What are the coordinates of the point where these segments meet?

24. Find the coordinates of M_1.

25. a. Whatever expression you used for the first coordinate of M_1, show algebraically that

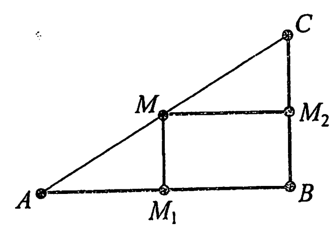
$$\text{your expression} - x_1 = x_2 - \text{your expression}.$$

b. Explain what the algebra was intended to prove about the midpoint.

26. a. What are the coordinates of M_2?

b. Explain how you can be sure that the coordinates you found describe a point that is not only equidistant from endpoints (x_2, y_1) and (x_2, y_2), but also *on the line* that connects them.

Below is a new drawing. Some of the points have new labels to make it easier to talk about them.



27. If a line is drawn through M_1, parallel to $\overline{BC}$, where does it intersect $\overline{AC}$? How do you know that?

28. If a line is drawn through M_2, parallel to $\overline{AB}$, where does it intersect $\overline{AC}$? How do you know that?

29. a. If you draw a vertical line through M_1, all of the points will have the same x-coordinate. What will it be?

b. If you draw a horizontal line through M_2, all of the points will have the same y-coordinate. What will it be?

c. So, what are the coordinates of M?

You have actually proved a new theorem.

Theorem 5.1

Each coordinate of the midpoint of a line segment is equal to the average of the corresponding coordinates of the endpoints of the line segment.

30. Use the midpoint theorem to find the midpoint between (1327, 94) and (–668, 17).

How many states actually comprised the U.S. in 1888?

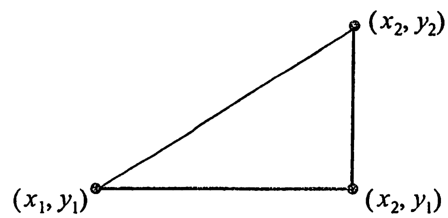
31. What is the midpoint between (1776, 13) and (2000, 50)?

32. Points A and B are endpoints of the diameter of a circle. Point C is the center of the circle. Find the coordinates of C given the following coordinates for A and B.

a. $A = (-79, 687), B = (13, 435)$

b. $A = (x, 0), B = (5x, y)$

To find a formula for the distance between two points, you can use the same right triangle that you used for midpoints.



33. What is the length of the horizontal segment?

34. What is the length of the vertical segment?

35. Let d be the distance between (x_1, y_1) and (x_2, y_2). Use the Pythagorean Theorem to find d.

It's nice to know that even if you forget the distance formula, you can figure it out just by remembering $a^2 + b^2 = c^2$.

Theorem 5.2

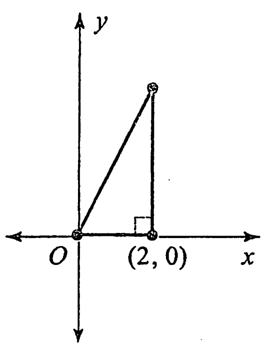
The distance d between two points (x_1, y_1) and (x_2, y_2) can be found using the Pythagorean Theorem. It is the square root of the sum of the difference in the x-coordinates squared and the difference in the y-coordinates squared.

36. Find the distance between the following pairs of points.

a. (1, 1) and (−1, −1)

b. (1, 1) and (4, 5)

c. (2, 4) and (−4, −2)



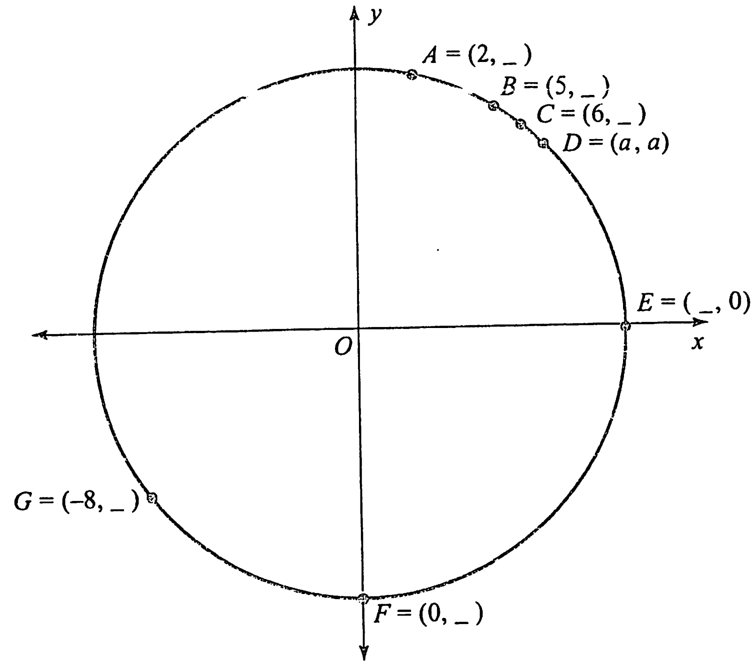
37. a. Imagine a right triangle with its right angle sitting on the x-axis but not at the origin. Its first two vertices are at (0, 0) and (2, 0). Find the coordinates of the third vertex when the hypotenuse is

i. 5 units long.

ii. 6 units long.

b. A different triangle has a hypotenuse 5 units long, one vertex at (0, 0), and legs of equal length. Again, the right angle is on the x-axis but not at the origin. Find the coordinates of the second and third vertices of the right triangle.

38. Find the missing coordinates for points A through G as shown in the circle with radius 10.



39. The vertices of $\triangle ABC$ are $A = (2, 1)$, $B = (4, 8)$, and $C = (6, -2)$.

a. Find the lengths of all three sides of the triangle.

b. Find the length of all three medians of the triangle.

40. Consider the six points $A = (5, 1)$, $B = (10, -2)$, $C = (8, 3)$, $A' = (2, 3)$, $B' = (7, 0)$, and $C' = (5, 5)$. Show that $\triangle ABC \cong \triangle A'B'C'$.

41. Consider the six points $A = (-800, -500)$, $B = (160, 12)$, $C = (-737, -484)$, $A' = (0, 0)$, $B' = (3840, 2048)$, and $C' = (252, 64)$. Show that $\triangle ABC \sim \triangle A'B'C'$.

42. **Write and Reflect** Consider these problems.

a. Explain how to tell if two triangles are congruent by calculations on the coordinates of their vertices.

b. Explain how to tell if two triangles are similar by looking at the coordinates of their vertices.

One way to show that a quadrilateral is a parallelogram is to show that the opposite sides are parallel. What is another way?

43. **Checkpoint** Pick four points that form a quadrilateral in a Cartesian plane. Find the midpoints of all four sides. Show that if you connect the midpoints, you get a parallelogram.

ACTIVITY 3

Lines

Remember Problem **15** from Lesson 1, where you drew the picture of all the points for which the x-coordinate was the same as the y-coordinate? You got a figure that looked like a line. Now you will look at some other lines in the plane, and devise a way to tell if three points are all on the same line.

44. Plot several points whose y-coordinates are three more than their x-coordinates Is there any regularity to the points? Explain.

45. Plot several points whose y-coordinates are two more than their x-coordinates. Draw the picture of *all* the points with this property.

46. Plot several points whose y-coordinates are one more than their x-coordinates. Draw the picture of *all* the points with this property.

47. Plot several points whose y-coordinates are

   a. twice their x-coordinates.

   b. three times their x-coordinates.

   c. four times their x-coordinates.

48. **Write and Reflect** Consider these problems.

   a. Some of the lines you sketched had points whose coordinates were of the form $(x, x + \textit{something})$. What did those lines have in common?

   b. Some other lines you sketched had points whose coordinates were $(x, x \times \textit{something})$. What did *those* lines have in common?

49. Find a point that is collinear with $A = (5, 1)$ and $B = (8, -3)$. Explain how yo did it.

50. Give a set of instructions for finding points that are collinear with $A = (5, 1)$ and $B = (8, -3)$. Explain why your method works.

51. **Write and Reflect** Consider these problems.

   a. Give a set of instructions for testing points that are collinear with $R = (-40, -30)$ and $S = (80, 20)$. Pick some points that are collinear with R and S and some that are not. Explain why your method works.

   b. Generalize your method. Suppose R and S are two points. Give a set of instructions for testing a third point P to see if it is collinear with R and S. Explain why your method works.

52. **Checkpoint** Is $P = (-4, -14)$ collinear with $R = (-40, -30)$ and $S = (80, 20$ Explain.

On Your Own

1. Three vertices of a square are $(-114, 214)$, $(186, 114)$, and $(-214, -86)$.

   a. Find the center of the square.

   b. Find the fourth vertex.

2. Three vertices of a square are $(-1, 5)$, $(5, 3)$, and $(3, -3)$.

   a. Find the center of the square.

   b. Find the fourth vertex.

3. Points D and E are the endpoints of one of the sides of a square. If the coordinates of the midpoint of the side, F, are $(4.5, 17)$, and the coordinates of D are $(2, 16)$, what are the coordinates of the other endpoint, E?

Two segments bisect each other if they intersect at each other's midpoint.

If you call the length of the third side of the triangle d_1, then you must show that d_2, the segment connecting the midpoints, is $\frac{1}{2}d_1$.

4. Segments AB and CD bisect each other. Find E, the point of bisection, and D, if $A = (110, 15)$, $B = (116, 23)$, and $C = (110, 23)$.

5. A segment has a length of 25 units. Give possible coordinates for the endpoints of this segment if it is
   a. a horizontal segment.
   b. a vertical segment.
   c. neither a horizontal nor a vertical segment.

6. A segment has its midpoint at (8, 10). List four possibilities for the coordinates of its endpoints.

7. A segment has one endpoint at $(-7, -2)$. Its midpoint is at $(-2, 1.5)$. What are the coordinates of the other endpoint?

8. Consider point P at (5, 0) and Q at (15, 0).
   a. Find six points that are just as far from P as they are from Q.
   b. Find six points that are closer to P than they are to Q.
   c. How can you tell if a point is equidistant from P and Q just by looking at its coordinates?

9. Show that the length of the line segment whose endpoints are the midpoints of two sides of a triangle is one half the length of the third side of that triangle. Do this for *any* triangle. Use subscript notation.

10. Use subscript notation to prove a general version of Problem **43** on page 373.

11. Suppose $A = (4, 1)$, $B = (8, -2)$, $C = (7, 1)$, and $D = (3, 4)$.
    a. Show that the diagonals of $ABCD$ bisect each other.
    b. What does this say about the kind of quadrilateral $ABCD$ is? Why?

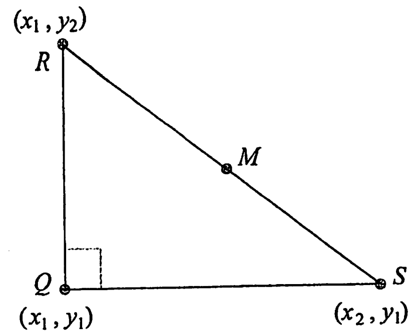
12. If two vertices of an equilateral triangle lie in the Cartesian plane at (1, 0) and (9, 0), find coordinates for the third vertex.

13. Suppose $P = (12, -5)$. How far is P from the origin? How far is P from $(15, -9)$?

14. The vertices of $\triangle DEF$ are $D = (11, -1)$, $E = (13, 10)$, and $F = (3, 5)$.
    a. Show that $\triangle DEF$ is isosceles.
    b. Call the midpoint of $\overline{FD}$ L. Find the length of the median $\overline{EL}$.
    c. Show that $\triangle ELD$ is a right triangle.
    d. Show that M, the midpoint of $\overline{DE}$, is equidistant to D, L, and E.

You have done this already for a specific right triangle in Problem 14.

15. Let the coordinates of the vertices of right triangle QRS be (x_1, y_1), (x_1, y_2), and (x_2, y_1). Show that M, the midpoint of the hypotenuse, is equidistant from the three vertices.



16. Group the following lines as "parallel" or "meet at the origin."

  a. The y-coordinate is one half the x-coordinate.
  b. The y-coordinate is three more than the x-coordinate.
  c. The y-coordinate is two less than the x-coordinate.
  d. The y-coordinate is negative two times the x-coordinate.
  e. The y-coordinate is five times the x-coordinate.
  f. The y-coordinate is one third the x-coordinate.

17. Is (110, 9) collinear with (60, 10) and (10, 11)? Why or why not?

Take It Further

For the first few problems, if you are going to add the coordinates and try to divide by 3, then you are taking the average of three points. In this case it's like a weighted average. If you are going to divide by 3, you have got to *add* three points. The question is, *which* three points?

18. Suppose $A = (2, 1)$ and $B = (32, 1)$. Find point P that is one third of the way from A to B. Explain how you did it.

19. Suppose $C = (2, 2)$ and $D = (30, 2)$. Find point Q that is one fourth of the way from C to D. Explain how you did it.

20. Consider the points $E = (5, 2)$ and $F = (11, -1)$. Find point S that is one third of the way from E to F. Explain how you did it.

21. Let $A = (-3, 5)$, $B = (5, 1)$, and $C = (7, -9)$.

  a. Calculate midpoint D of $\overline{AB}$.
  b. Calculate midpoint E of $\overline{BC}$.
  c. Calculate F to be two thirds of the way from A to E.
  d. Calculate G to be two thirds of the way from C to D.
  e. Calculate midpoint H of $\overline{AC}$.

f. Calculate J to be two thirds of the way from B to H.

g. If you have not already done so, draw the picture that goes with these calculations.

h. What theorem does this remind you of?

22. Line a passes through (0, 1) and (1, 0). Line b passes through the origin and makes a 45° angle with the axes as it enters quadrant I. Find the coordinates where these two lines intersect.

23. Let C be the circle whose center is at the origin of the plane and whose radius is 5.

a. Find any points on C that are also on the vertical line that contains (4, –9).

b. Find the intersection of C with the vertical line that contains (3, 2).

c. Find any points on C that are also on the horizontal line that contains (8, 0).

d. Find the intersection of C with the horizontal line that contains (8, 3).

e. Find any points on C that are 13 units from the origin.

f. **Challenge** Find any points on C that are 13 units from (8, 16).

MTC on Sue Fuller's "String Composition #337"

Show picture of the artwork and describe it:

Two pieces of plexiglass with 23 nails equally spaced in a circle. A thin thread is strung between every possible pair of nails.

1) Ask participants to brainstorm: What mathematical questions can you ask about this piece of art? That was how we came up with the question... you might find a different one!
2) Question we decided on: How many *regions* are inside this circle? Participants had to make some definitions: what do we mean by a *region*?
3) Build the model: What are the essential / inessential features? Participants quickly settled on a 2D model: a circle with 23 equally space points, every point connected to every other point.
4) The obvious thing to do (which most groups did) was tackle the problem with a smaller number of dots. This problem has a very nice feature that there is an "obvious" pattern that breaks down after a few steps.

# of dots	1	2	3	4	5	6	7
# of regions	1	2	4	8	16	31	57

This is a pretty well-known example in Math Circles. See:
https://www.mathcircles.org/content/regions-circle-and-difference-equations

Note: We tackled this problem over several days in a summer retreat. We spent about 1.5 hours per day on it, and asked teachers to think about it in-between and to come up with new ideas to explore the next day. Most of them did. Also, I had no idea how to solve this problem when we started, but I figured it out the night of Day 2.

Some mathematical notes: It's true (but not so elementary to prove) that if the number of dots around the circle is odd and they are equally spaced, then the diagonals of the polygon formed by the dots only intersect in pairs. This is actually a key feature, so we are lucky that Sue Fuller used 23 nails! Teachers were willing to take this as an unproven lemma.

If you accept that, then there are 23-choose-2 lines (handshake problem) and 23-choose-4 intersection points (four points determine a unique pair of intersecting diagonals). This plus Euler's formula V-E+F=2 lets you count the number of regions. (Don't forget the 23 regions outside of the polygon and bounded by the circle!)

MTC on Ulu Maika

We essentially followed the lesson described here, with only minor alterations:
http://superm.math.hawaii.edu/_lessons/ninth_twelveth/ulu_maika.pdf

(The lesson appears on the next few pages for your convenience, but you might want to check out other activities at http://superm.math.hawaii.edu/activities.html.)

‘Ulu Maika

Introduction

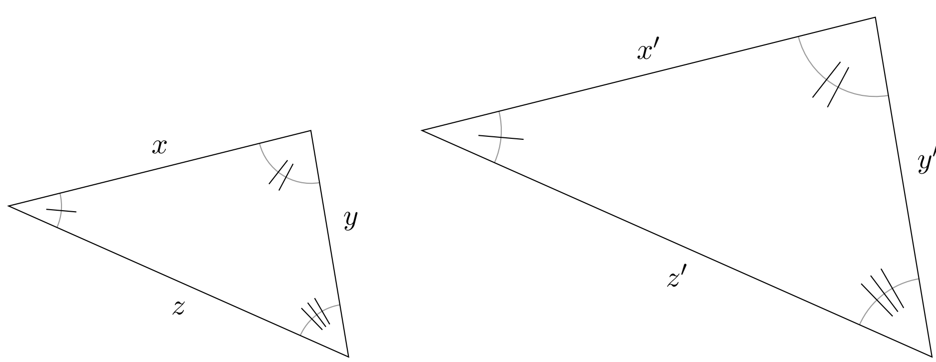
Early Hawaiians devoted large amounts of time to games, amusements, and relaxing pastimes. Games were played to develop strength, endurance, and skills. ‘Ulu maika (or ‘olohū), one of the most popular sports in early Hawai‘i, is an example a skill game where competitors roll stones towards two stakes, the victor decided upon by a variety of criteria (proximity to stakes, furthest thrown, etc.) In early Hawai‘i only men were allowed to roll the stone disks, ‘ulu, between stakes to test their skill or down long courses, free of stakes, to show their strength. Even to this day the sport is played, as hundreds of the skillfully fashioned stones of the era existing in museums and private collections. Unfortunately, many kahua maika (specially prepared courses on which the stones were rolled) used in the days of the early Hawaiians have been destroyed.

Grade Levels and Topics

This activity is intended for high school students as a geometry activity.

Geometry:

- **Pythagorean Theorem:** $a^2 + b^2 = c^2$ where c is the hypotenuse of a right triangle with legs a, b.
- **Similar triangles:** Triangles are similar if they have the same shape, but can be different sizes. The property of two triangles that are similar is that you can multiple all the sides of one triangle by the same number to get the resulting triangles sides (as shown bellow).



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- **Law of sines and law of cosines:** If we have the following triangle (next page):

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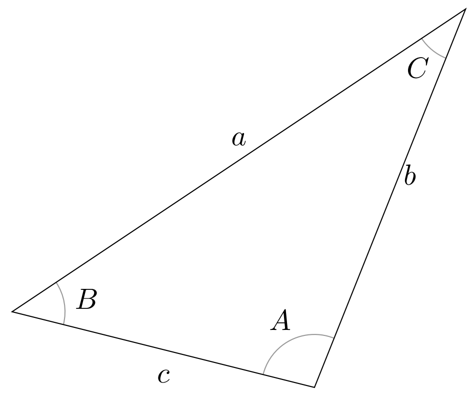

Then the following equalities hold true:

– Law of sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

– Law of cosines:

$$a^2 = b^2 + c^2 - 2bc\cos A$$

Objectives

- Students should gain an understanding of what mathematical models are.
- Students will learn how to apply and manipulate the Pythagorean theorem, law of sine, law of cosine.
- Students should gain more familiarity with similar triangles.

Note: the images are not to scale. For example, the pegs on an ʻulu maika field are supposed to be 10 in. apart, but on the images it does not look 10 in. apart. Mathematical models/diagrams do not need to be drawn to scale; they are only used as a visual aid. In the homework problems, they will use the models to solve certain measurements about ʻulu maika rolls by applying properties of triangles.

Activity

Materials and Resources

1) Handouts and worksheets (see below)
2) Stop Watch
3) Rolling Measure (tape measure would also work, but is limited and less efficient)
4) String (used to visualize the triangle on the field)

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5) Protractor (used for a more trig based lesson)

6) ʻUlu maika disk and goal sticks. Here are some examples:

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NOTE: If you do not have access to ʻulu maika materials, don't worry! For the ʻulu maika, all you need is a 3in diameter cylindrical shaped item or you can make one using either commercial quick drying cement or plaster of paris. For the goal, you need two sticks.

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Procedure

Geometry Part 1: Right Triangles

1. Set up the ʻulu make field prior to class
2. Organize your class into teams of 3-4 players.
3. Each team will have 3 positions:
   - 1 person to act as timer with the stop watch
   - 1 person to act as the distance recorder with the measuring tool
   - 1 person to throw the ʻulu
4. Have the students record their measurements in the tables (provided in the handouts below).
   - If there is an abundance of ʻulu resources available, each team can play their own ʻulu maika, alternating through the positions.
   - Otherwise, teams will alternate rolling and measuring the results.
   - Either way, students should record the data of the other teams/players, not just their own team or self.
5. Using this data, the students must determine a set of rules. In other words, should winner be determined by the distance of the roll, the distance from the goals, or the speed? Do the rules make sense? If the units of measure change, will the winning team change?

Example outcome table of the students' data after playing ʻulu maika.

Table for part 1: Right Triangles

Team/Player	Distance ʻulu traveled (ft)	Starting distance from goal (ft)	Time traveled (sec)	Speed of ʻulu (ft/sec)	Angle ʻulu diverged off straight path
Nā ʻŌʻō	15	2	2	7.5	10
Nā ʻAmaʻama	16	4	2	8	20
Nā Aliʻi	21	0.5	3	7	5
Nā Koa	25	8	2	12.5	25

Geometry Part 2: Law of Sines and Law of Cosines

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Discussion

- In the previous section we saw one way of determining which ʻulu roll is most accurate. In this section we see new ways of determining the most accurate throw. In this version of the game, students will measure the distance they are away from the goal when they rolled the ʻulu, the distance from

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the center of the goal to the ʻulu, and the distance from the starting position to the ʻulu. With this information, the students will create a model of their throw. The problems will show the students how you can determine the most accurate throw using these distances.

- The law of cosines are a special case of the Pythagorean theorem. You can show them why, or have them prove why that is, using a generalized right triangle.
- Discuss with the students what happens if we use the angle θ in number 1 to compare accuracies. In this case the winner will be the one with the smallest angle. Where as in number 3 if we use the angle θ to compare accuracies then the winner will be the one with the angle closest to $180°$.

Example outcome table of the students' data after playing ʻulu maika.

Table for part 2: Law of Cosines & Law of Sines

Team/Player	Distance ʻulu traveled (ft)	Starting distance from goal (ft)	Time traveled (sec)	Speed of ʻulu (ft/sec)	Angle opposite of the ʻulu path
Nā ʻŌʻō	15	2	2	7.5	30
Nā ʻAmaʻama	16	4	2	8	40
Nā Aliʻi	21	0.5	3	7	175
Nā Koa	25	8	2	12.5	141

The following pages contain tables and worksheets for the students. Please note the second to last page is the answer key.

‘Ulu Maika

1. Organize yourselves into teams of 3-4 players.
2. Each team will have 3 positions:
   - 1 person to act as timer with the stop watch
   - 1 person to act as the distance recorder with the measuring tool
   - 1 person to throw the ‘ulu
3. Record all the measurements in the tables (provided below) for each roll of the ‘ulu.
4. Using this data, how do we determine a winner? Strength, speed, or accuracy? If the units of measure change, will the winning team change?

Table for part 1: Right Triangles

Team/Player	Distance ‘ulu traveled (ft)	Starting distance from goal	Time traveled (sec)	Speed of ‘ulu (ft/sec)	Angle ‘ulu diverged off straight path

Table for part 2: Law of Cosines & Law of Sines

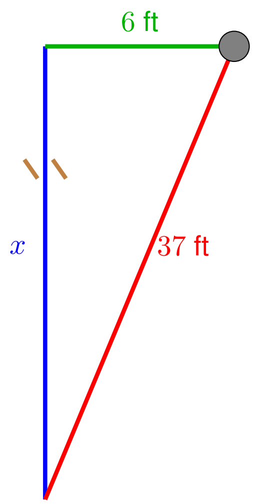
Team/Player	Distance ‘ulu traveled (ft)	Starting distance from goal (ft)	Time traveled (sec)	Speed of ‘ulu (ft/sec)	Angle opposite of the ‘ulu path

Geometry Part 1: Right Triangles

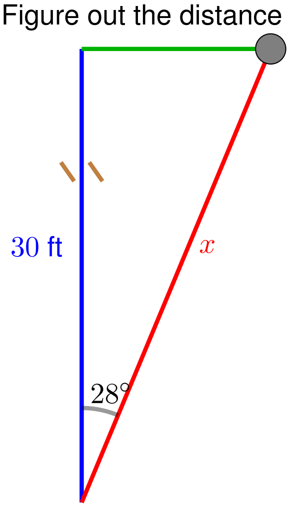
‘Ulu MaikaTriangles

1. Suppose the goal is 10 in. wide and 20 feet away. At what angle will you completely miss the goal? What is the largest angle that guarantees you will make it in the goal?

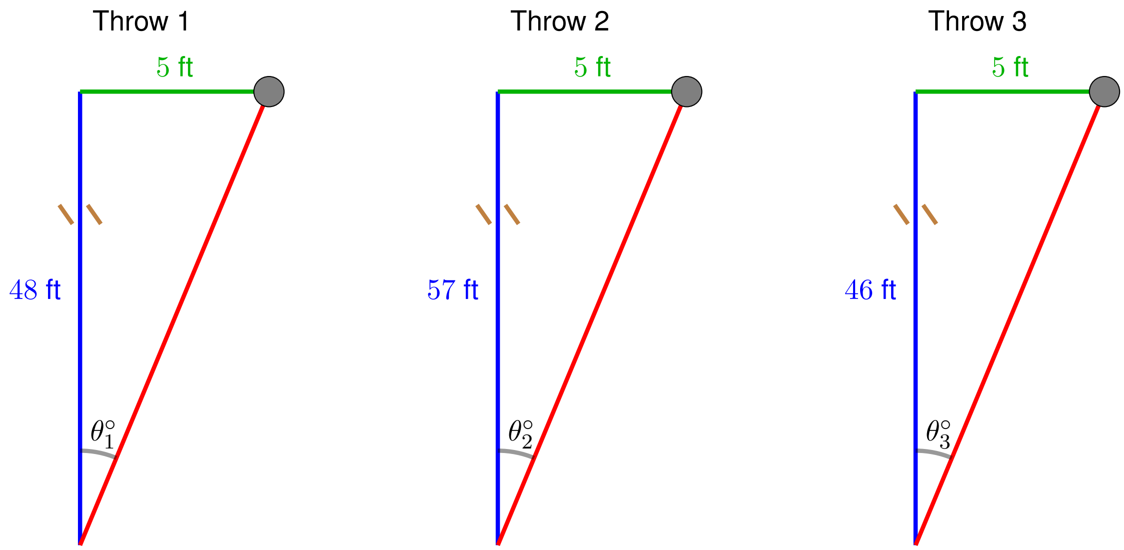
2. Figure out the vertical distance x the disk traveled.



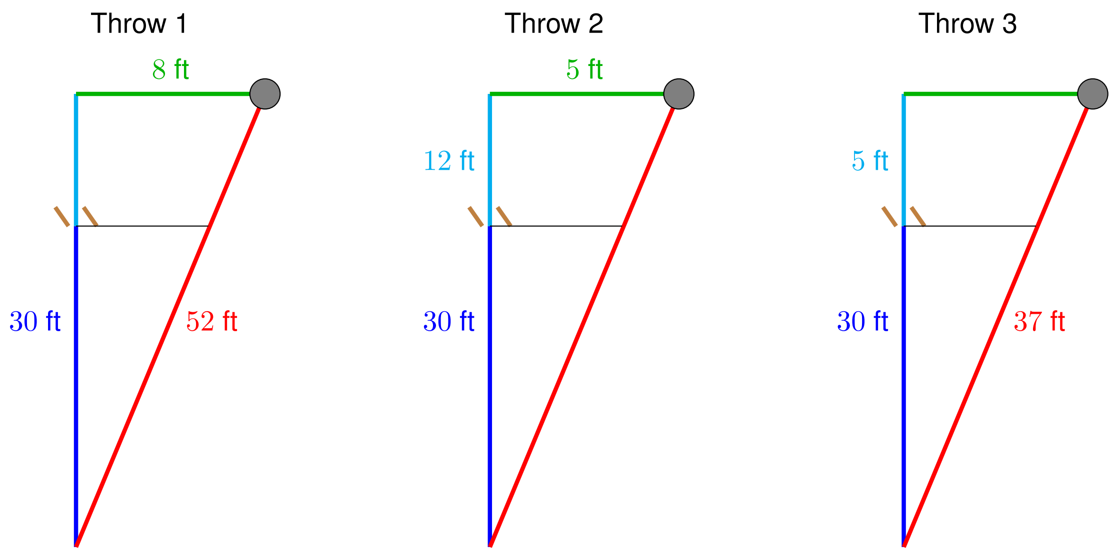
3. Figure out the distance the rock traveled x.



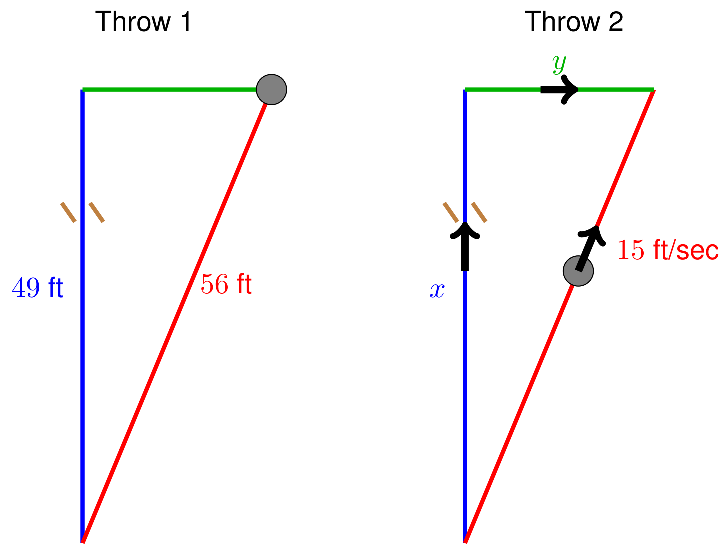
4. Among the previous problems, determine which ‘ulu roll was the most accurate by calculating and comparing the angles that the ‘ulu has diverged off of the straight path θ_1, θ_2, and θ_3. Determine which roll was the most accurate using your intuition. Now prove that your intuition is correct.



5. The person with their disk closest to the goal at the time it passes the goal is most accurate. Use this fact to determine which of the following ‘ulu rolls were the most accurate.



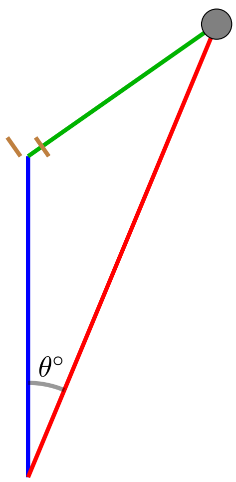
6. Use the two models (left: showing the distance the ‘ulu traveled; right: showing the velocity of that same ‘ulu) to determine the horizontal speed y and the vertical speed x of the disk roll.



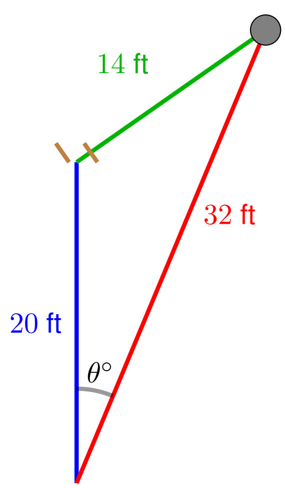
Geometry Part 2: Law of Sines and Law of Cosines

Questions

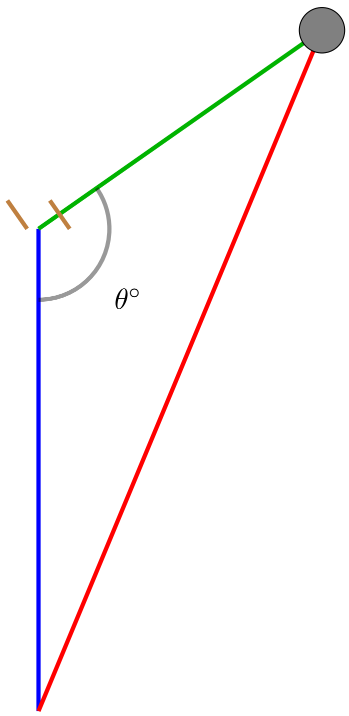
1. We are comparing multiple throws from people competing in an ʻulu maika tournament where the winner is the person who has the most accurate throw. If we only have the angle θ from everyones' throw, then which θ value would be the winner?



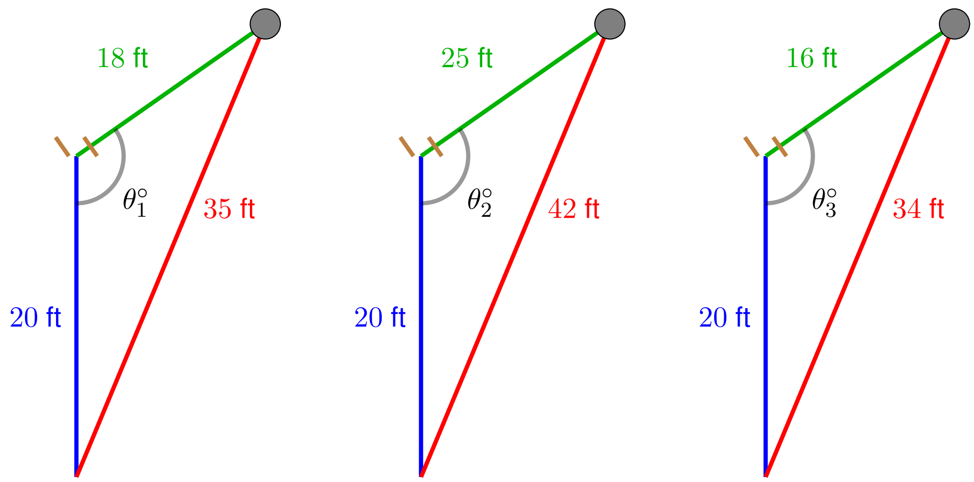
2. Determine the angle that the player's *ulu* was off of the straight path θ using law of cosines.



3. We are comparing multiple throws of different players and we are using the angle θ, as defined in the picture below, to determine which throw was the most accurate. Which θ value would determine the winner? Hint: Think about what value of θ would be a "perfect" throw.



4. Determine which throw was the most accurate and least accurate by comparing θ_1, θ_2, and θ_3.



Answer Key

Geometry Part 1: Right Triangles

1. Rolling the 'ulu at an angle larger then $2.386°$ (or $1.193°$ to either the left or right) will result in missing the goals.
2. $x = \sqrt{37^2 - 6^2} = \sqrt{1333} \approx 36.510272523$
3. $\frac{x}{\sin(90°)} = \frac{30}{\sin(62°)} \implies x \approx 33.97710152$
4. $\theta_1 = 5.9469°$
   $\theta_2 = 5.0131°$
   $\theta_3 = 6.2034°$
   Hence throw 2 was the most accurate.
5. Throw 1: $\frac{60}{\sqrt{165}} = 4\sqrt{\frac{15}{11}} \approx 4.67099$
   Throw 2: $\frac{25}{7} \approx 3.571429$
   Throw 3: $\frac{72}{7} \approx 10.2857$
   Hence the Second roll is most accurate, roll 1 is second to most accurate, and roll 3 is least accurate.
6. $x = \frac{105}{8} = 13.125 y = \frac{15}{8}\sqrt{15} \approx 7.26184$

Geometry Part 2: Law of Sines and Law of Cosines

1. The winner would be the person with the θ value closest to zero.
2. $\theta \approx 23.7689$
3. The player with a θ closest to $180°$ would be the person with the most accurate throw.
4. $\theta_1 \approx 134.094$, $\theta_2 \approx 137.646$, and $\theta_3 \approx 141.375$. Hence roll 3 is most accurate, roll 2 is second, and roll 1 is least accurate.

Common Core Standards

- CCSS.Math.Content.HSG-SRT.B.5 Use congruence and similarity criteria for triangles to solve problems and to prove relationships in geometric figures.

- CCSS.Math.Content.HSG-SRT.C.6 Understand that by similarity, side ratios in right triangles are properties of the angles in the triangle, leading to definitions of trigonometric ratios for acute angles.

- CCSS.Math.Content.HSG-SRT.C.8 Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.(Modeling)

- CCSS.Math.Content.HSG-SRT.D.11 (+) Understand and apply the Law of Sines and the Law of Cosines to find unknown measurements in right and non-right triangles (e.g., surveying problems, resultant forces).

MTC on Bee Population Dynamics

You can find lots of information and resources here:
http://math.hawaii.edu/~sarah/HoneybeeMath.html

This file has tons of information and activity about the life cycle of bees and how the hive works:
http://math.hawaii.edu/~sarah/beepop_combined.pdf

Here are some spreadsheets for modeling the dynamics:
http://math.hawaii.edu/~sarah/BeepopitaExcelMatchHonolulu.xlsx
http://math.hawaii.edu/~sarah/BeepopitaExcelMatchBoston.xlsx

Flow of the session:

1) Overview of why bees are important and how they contribute to the food chain by the guest presenters.
2) Population sampling activity as described in the beepop_combined.pdf packet above.
3) More information about bees from the presenters, specifically what the different bees do in the hive, how the activity changes with seasons, how hives split (and collapse), why swarming happens, etc.
4) Discussion of the variables that affect bee population dynamics, everyone downloads spreadsheet or TI program (in the packet). Play with variables so that you can have a hive in Honolulu that doesn't collapse. What variables are the most important? What doesn't matter so much?
5) Wrap-up and questions.