2013 Math Wrangle Solutions

MAA American Mathematics Competitions
SIGMAA Math Circles for Students and Teachers

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1. Let T and B be the points where the circle meets $\overline{PM}$ and $\overline{AP}$, respectively, with $\overline{ABP}$. Triangles POT and PAM are right triangles that share angle MPA, so they are similar. Let p_1 and p_2 be their respective perimeters. Then $OT/AM = p_1/p_2$. Because $AM = TM$, it follows that $p_1 = p_2 - (AM + TM) = 152 - 2AM$. Thus $19/AM = (152 - 2AM)/152$, so that $AM = 38$ and $p_1 = 76$. It is also true that $OP/PM = p_1/p_2$, so

$$\frac{1}{2} = \frac{OP}{PM} = \frac{OP}{152 - (38 + 19 + OP)}$$

It follows that $OP = 95/3$.

2. Let r_1 and r_2 be the radii and A_1 and A_2 be the centers of $\mathcal{C}_1$ and $\mathcal{C}_2$, respectively, and let $P = (u, v)$ belong to both circles. Because the circles have common external tangents that meet at the origin O, it follows that the first-quadrant angle formed by the lines $y = 0$ and $y = mx$ is bisected by the ray through O, A_1, and A_2. Therefore, $A_1 = (x_1, kx_1)$ and $A_2 = (x_2, kx_2)$, where k is the tangent of the angle formed by the positive x-axis and the ray OA_1. Notice that $r_1 = kx_1$ and $r_2 = kx_2$. It follows from the identity $\tan 2\alpha = \dfrac{2\tan\alpha}{1-\tan^2\alpha}$ that $m = \dfrac{2k}{1-k^2}$. Now $(PA_1)^2 = (kx_1)^2$, or

$$(u - x_1)^2 + (v - kx_1)^2 = k^2x_1^2, \text{ so}$$
$$(x_1)^2 - 2(u + kv)x_1 + u^2 + v^2 = 0.$$

In similar fashion, it follows that

$$(x_2)^2 - 2(u + kv)x_2 + u^2 + v^2 = 0.$$

Thus x_1 and x_2 are the roots of the equation

$$x^2 - 2(u + kv)x + u^2 + v^2 = 0,$$

which implies that $x_1x_2 = u^2+v^2$, and that $r_1r_2 = k^2x_1x_2 = k^2(u^2+v^2)$. Thus

$$k = \sqrt{\frac{r_1r_2}{u^2 + v^2}}$$

and

$$m = \frac{2k}{1 - k^2} = \frac{2\sqrt{r_1r_2(u^2 + v^2)}}{u^2 + v^2 - r_1r_2}.$$

When $u = 9$, $v = 6$, and $r_1r_2 = 68$, this gives $m = \dfrac{12\sqrt{221}}{49}$.

3. Let t be the number of members of the committee, n_k be the number of votes for candidate k, and let p_k be the percentage of votes for candidate k for $k = 1, 2, \dots, 27$. We have

$$n_k \geq p_k + 1 = \frac{100n_k}{t} + 1.$$

Adding these 27 inequalities yields $t \geq 127$. Solving for n_k gives $n_k \geq \dfrac{t}{t - 100}$, and, since n_k is an integer, we obtain

$$n_k \geq \left\lceil \frac{t}{t - 100} \right\rceil,$$

where the notation $\lceil x \rceil$ denotes the least integer that is greater than or equal to x. The last inequality is satisfied for all $k = 1, 2, \dots, 27$ if and only if it is satisfied by the smallest n_k, say n_1. Since $t \geq 27n_1$, we obtain

$$t \geq 27\left\lceil \frac{t}{t - 100} \right\rceil \qquad (1)$$

and our problem reduces to finding the smallest possible integer $t \geq 127$ that satisfies the inequality (1). If $\frac{t}{t-100} > 4$, that is, $t \leq 133$, then $27\left\lceil\frac{t}{t-100}\right\rceil \geq 27 \cdot 5 = 135$ so that the inequality (1) is not satisfied. Thus 134 is the least possible number of members in the committee. Note that when $t = 134$, an election in which 1 candidate receives 30 votes and the remaining 26 candidates receive 4 votes each satisfies the conditions of the problem.

OR

Let t be the number of members of the committee, and let m be the least number of votes that any candidate received. It is clear that $m \neq 0$ and $m \neq 1$. If $m = 2$, then $2 \geq 1 + 100(2/t)$, so $t \geq 200$. Similarly, if $m = 3$, then $3 \geq 1 + 100(3/t)$, and $t \geq 150$; and if $m = 4$, then $4 \geq 1 + 100(4/t)$, so $t \geq 134$. When $m \geq 5$, $t \geq 27 \cdot 5 = 135$. Thus $t \geq 134$. Verify that t can be 134 by noting that the votes may be distributed so that 1 candidate receives 30 votes and the remaining 26 candidates receive 4 votes each.

4. If the bug is at the starting vertex after move n, the probability is 1 that it will move to a non-starting vertex on move $n+1$. If the bug is not at the starting vertex after move n, the probability is $1/2$ that it will move back to its starting vertex on move $n+1$, and the probability is $1/2$ that it will move to another non-starting starting vertex on move $n+1$. Let p_n be the probability that the bug is at the starting vertex after move n. Then $p_{n+1} = 0 \cdot p_n + \frac{1}{2}(1 - p_n) = -\frac{1}{2}p_n + \frac{1}{2}$. This implies that $p_{n+1} - \frac{1}{3} = -\frac{1}{2}(p_n - \frac{1}{3})$. Since $p_0 - \frac{1}{3} = 1 - \frac{1}{3} = 2/3$, conclude that $p_n - \frac{1}{3} = \frac{2}{3} \cdot (-\frac{1}{2})^n$. Therefore

$$p_n = \frac{2}{3} \cdot \left(-\frac{1}{2}\right)^n + \frac{1}{3} = \frac{1 + (-1)^n \frac{1}{2^{n-1}}}{3} = \frac{2^{n-1} + (-1)^n}{3 \cdot 2^{n-1}}.$$

Substitute 10 for n to find that $p_{10} = 171/512$.

OR

A 10-step path can be represented by a 10-letter sequence consisting of only A's and B's, where A represents a move in the clockwise direction and B represents a move in the counterclockwise direction. Where the path ends depends on the number of A's and B's, not on their arrangement. Let x be the number of A's, and let y be the number of B's. Note that the bug will be home if and only if $x - y$ is a multiple of 3. After 10 moves, $x + y = 10$. Then $2x = 10 + 3k$ for some integer k, and so $x = 5 + 3j$ for some integer j. Thus the number of A's must be 2, 5, or 8, and the desired probability is

$$\frac{\binom{10}{2} + \binom{10}{5} + \binom{10}{8}}{2^{10}} = \frac{171}{512}.$$

OR

Let X be the bug's starting vertex, and let Y and Z be the other two vertices. Let x_k, y_k, and z_k be the probabilities that the bug is at vertex X, Y, and Z, respectively, at move k, for $k \geq 0$. Then $x_{k+1} = .5y_k + .5z_k$, $y_{k+1} = .5x_k + .5z_k$, and $z_{k+1} = .5x_k + .5y_k$. This can be written as

$$\begin{bmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{bmatrix} = \begin{bmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{bmatrix} \begin{bmatrix} x_k \\ y_k \\ z_k \end{bmatrix}.$$

Thus

$$\begin{bmatrix} x_{10} \\ y_{10} \\ z_{10} \end{bmatrix} = \begin{bmatrix} 0 & .5 & .5 \\ .5 & 0 & .5 \\ .5 & .5 & 0 \end{bmatrix}^{10} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix},$$

and $x_{10} = 171/512$.

5. The lateral surface area of a cone with radius R and slant height S can be found by cutting the cone along a slant height and then unrolling it to form a sector of a circle. The sector's arc has length $2\pi R$ and its radius is S, so its area, and the cone's lateral surface area, is $\pi S^2 \cdot \frac{2\pi R}{2\pi S} =$

πRS. Let r, h, and s represent the radius, height, and slant height of the smaller cone formed by the cut. Then

$$k = \frac{r^2h}{36 - r^2h} = \frac{rs}{9+15-rs}, \quad \text{so}$$
$$\frac{1}{k} = \frac{36}{r^2h} - 1 = \frac{24}{rs} - 1.$$

Thus $3s = 2rh$. Because $r : h : s = 3 : 4 : 5$, let $(r, h, s) = (3x, 4x, 5x)$, and substitute to find that $x = 5/8$, and then that $(r, h, s) = (15/8, 20/8, 25/8)$. The ratio of the volume of $\mathcal{C}$ to that of the large cone is therefore $\left(\frac{15/8}{3}\right)^3 = \frac{125}{512}$, so the ratio of the volumes of $\mathcal{C}$ and $\mathcal{F}$ is $125/(512 - 125) = 125/387$.

6. Number the squares from left to right, starting with 0 for the leftmost square, and ending with 1023 for the rightmost square. The 942nd square is thus initially numbered 941. Represent the position of a square after f folds as an ordered triple (p, h, f), where p is the position of the square starting from the left, starting with 0 as the leftmost position, h is the number of paper levels below the square, and f is the number of folds that have been made. For example, the ordered triple that initially describes square number 941 is $(941, 0, 0)$. The first 0 indicates that at the start there are no squares under this one, and the second 0 indicates that no folds have been made.

   Note that the function F, defined below, describes the position of a square after $(f + 1)$ folds:

$$F(p,h,f) = \begin{cases} (p, h, f+1) & \text{for } 0 \le p \le 2^{10-f-1} - 1 \\ (2^{10-f} - 1 - p, 2^f + (2^f - 1 - h), f+1) & \\ & \text{for } 2^{10-f-1} \le p \le 2^{10-f} - 1 \end{cases}$$

   The top line in the definition indicates that squares on the left half of the strip do not change their position or height as a result of a fold. The second line indicates that, as a result of a fold, the position of a square on the right half of the strip is reflected about the center line of the strip, and that the stack of squares in that position is inverted and placed on the top of the stack that was already in that position's reflection.

Because of the powers of 2 in the definition of F, evaluating F can be made easier if the position and height are expressed in base two. In particular, after f folds, the strip has length 2^{10-f}, so the positions 0 through $2^{10-f} - 1$ are represented by all possible binary strings of $10 - f$ digits. In this representation, $0 \leq p \leq 2^{10-f-1} - 1$ if and only if the leading digit is 0, and $2^{10-f-1} \leq p \leq 2^{10-f} - 1$ if and only if the leading digit is 1. In the former case, the new position, now represented by the string of length $10 - f - 1$, is obtained by deleting the leading 0. In the latter case, the new position $2^{10-f} - 1 - p$ is obtained by truncating the leading 1 and for the remaining digits, changing each 0 to a 1 and each 1 to a 0. Likewise, in this latter case, the new height is $2^f + (2^f - 1 - h)$. When $f \geq 1$, the new height is obtained in the case $2^{10-f-1} \leq p \leq 2^{10-f} - 1$ by taking the f-digit binary string representing the height, changing each 1 to a 0 and each 0 to a 1, and then appending a 1 on the left. In the case $0 \leq p \leq 2^{10-f-1} - 1$, the new $(f+1)$-digit string representing the new height is obtained by appending a 0 to the left of the string.

With these conditions, square number 941 is initially described by $(1110101101, 0, 0)$. In the display below, an arrow is used to denote an application of F. For the first fold

$$(1110101101, 0, 0) \rightarrow (001010010, 1, 1),$$

indicating that after the first fold, square 941 is in position $001010010_2 = 82$, and there is one layer under this square. Continue to obtain $(001010010, 1, 1) \rightarrow (01010010, 01, 2) \rightarrow (1010010, 001, 3) \rightarrow (101101, 1110, 4) \rightarrow (10010, 10001, 5) \rightarrow (1101, 101110, 6) \rightarrow (010, 1010001, 7) \rightarrow (10, 01010001, 8) \rightarrow (1, 110101110, 9) \rightarrow (0, 1001010001, 10)$. After 10 folds, the number of layers under square 941 is $1001010001_2 = 593$.

OR

If a square is to the left of the center after n folds, its positions counting from the left and the bottom do not change after $(n+1)$ folds. Otherwise, its positions counting from the right and bottom after n folds become its positions counting from the left and top after $(n+1)$ folds.

Also, after n folds the sum of the positions of each square counting from the left and right is $2^{10-n}+1$, and the sum of the positions counting from the bottom and top is 2^n+1. The position of the 942nd square can be described in the table below.

Folds	Position Counting From Left	Position Counting From Right	Position Counting From Bottom	Position Counting From Top
0	942	83	1	1
1	83	430	2	1
2	83	174	2	3
3	83	46	2	7
4	46	19	15	2
5	19	14	18	15
6	14	3	47	18
7	3	6	82	47
8	3	2	82	175
9	2	1	431	82
10	1	1	594	431

Thus there are 593 squares below the final position of the 942nd square.