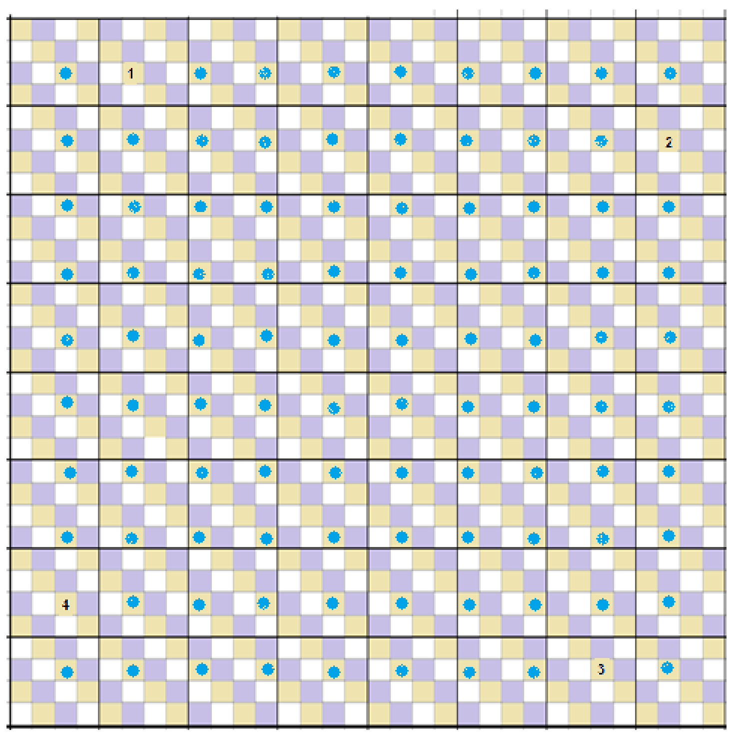
JMM 2019 Math Wrangle Solutions

1 *"I"-Trominos. One square from a* 32×32 *chessboard is removed at random. What it the probability that this 1023-square board can be tiled by* 3×1 *"I"-trominos?*
The answer is $\frac{100}{1024} = \frac{25}{256}$. We have colored the 32 by 32 grid below using 3 colors so that any horizontal or vertical triomino must cover one square of each color. Going from left to right and from top to bottom the colors yellow, purple, white appear in this order repeatedly. Since the grid starts and stops with yellow there is one extra yellow square. Thus we must choose a yellow square (leaving 341 of each of the three colors) for the tiling to be possible. However, not all yellow will work! Each yellow which works will give a tiling that is still a tiling even when we rotate the board by 90 degrees. Thus each working yellow must go to another yellow (and thus form an orbit of 4 yellow) under such a rotation. We have labeled a sample orbit of 4 of these. The other working yellow each have a blue dot and there are 100 of these. It is not too hard to see that a tiling is possible in each case.



2 *Missing digit. The following is the product of two twin primes, where "X" indicates a digit. What are the possible value(s) for X?*

$$85070591X30234644163149060928396584963$$

For a twin prime pair $p, p+2$ with $p > 3$ neither of these is divisible by 3 and thus we must have $p = 3n+2$ and $p+2 = 3n+4$. Hence $p(p+2) = 9n^2 + 18n + 8$ which is 8 modulo

9. Since a number is divisible by 9 iff the sum of the digits is also divisible by 9, we can analyze the above number by first crossing out the nines and zeroes (5 of each). Then we will cross out pairs of digits adding to 9 (five $6+3=9$ pairs, three $5+4=9$ pairs, three $8+1=9$ pairs, and one $7+2=9$ pair) thus leaving $X244$. So $X+10 \equiv X+1$ is congruent to 8 modulo 9 and thus $X=7$.

3 *Analog Clock. Between the start and end of most hours on a regular clock there is a time when the hour and minute hands coincide. When this happens we are interested in the small angle formed by the second hand with these other two. (Since on a circle we can go two directions from one place to another every angle between hands has a small version and a large version.) At which time(s) (to the nearest second) is this angle a maximum?*

The hour and minute hands coincide at noon and again at midnight, and the time between each coincidence must be the same (why?), so the next 11 coincidences happen every $12/11$ hours starting at noon. Converting to angles, note that one hour is $\pi/6$ radians. So when the minute and hour hands coincide, they will be located $2k\pi/11$ radians clockwise from 12 o'clock, for $k=1,2,\dots,10$ (we only go up to 10 since when $k=11$ we are back at 12 o'clock).

The second hand travels 720 times faster than the hour hand; in other words, if the hour hand has moved α radians clockwise from 12 o'clock, the second hand moves 720α radians, and the difference is 719α radians clockwise from 12 o'clock.

Therefore we must examine the angle

$$\frac{719\cdot 2k\pi}{11} = \frac{1438k\pi}{11} = \left(130+\frac{8}{11}\right)k\pi,$$

which reduces to $8k\pi/11$ (modulo 2π). We want this angle (modulo 2π) to be as close to π as possible, so we write the angle as $\frac{4k}{11}(2\pi)$, and we want the fractional part of $4k/11$ to be as close to $1/2$ as possible, which happens when $4k$ is congruent to 5 or 6 (modulo 11). This happens when $k=4$ or $k=7$. The times when this happens are $48/11$ and $84/11$ hours past 12, or **4:21:49** and **7:38:11**, to the nearest second.

4 *A Slightly Weird Function. Specify a function $f(n)$ from the positive integers to the positive integers satisfying the following two conditions for all positive integers n:*

(a) $f(n+1) > f(n)$,

(b) $f(f(n)) = 3n$.

Find $f(2019)$.

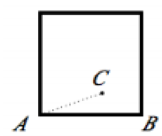
The function meeting these specifications is unique and we encourage the reader to prove this. To define f we will work in base 3. Let $n=a_1a_2\cdots a_k$ in base 3 where without loss of generality we can assume that $a_1 \neq 0$. Then

$$f(n) = \begin{cases} 2a_2\cdots a_k & \text{if} \quad a_1 = 1 \\ 1a_2\cdots a_k0 & \text{if} \quad a_2 = 2 \end{cases}$$

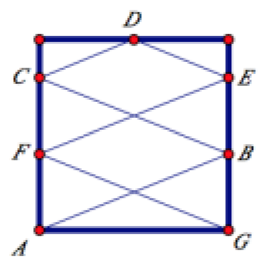
Since f either increases the leading digit or increases the number of digits, it is clear that f increases the value of it's input. Further, regardless of the a_1, $f(f(n)) = a_1\cdots a_k0 = 3n$.

Now in base 3, $2019 = 2202210$ and thus in base 3, $f(2019) = 12022100$ which is 3870 in base 10.

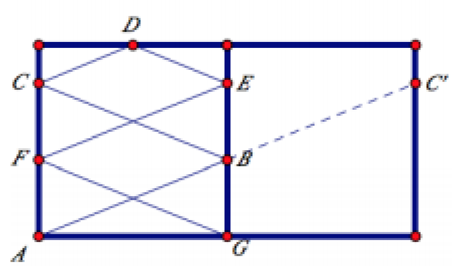
5 *Billiard ball. A mathematical billiard ball (i.e., a point with zero radius) is shot from corner A of the square below at an angle* $\theta = \angle BAC$ *where* $\tan\theta = 1000/2019$. *How many times will it bounce off a wall of the square before it returns to a corner, and which corner will it return to?*



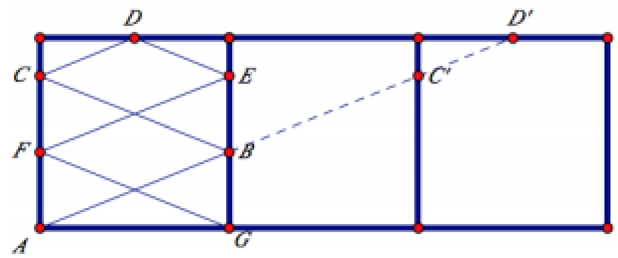
Here is a simpler example, where the slope of the line is $2/5$. Starting at A, the ball bounces 5 times, at B, C, D, E, F, and ends up at G.



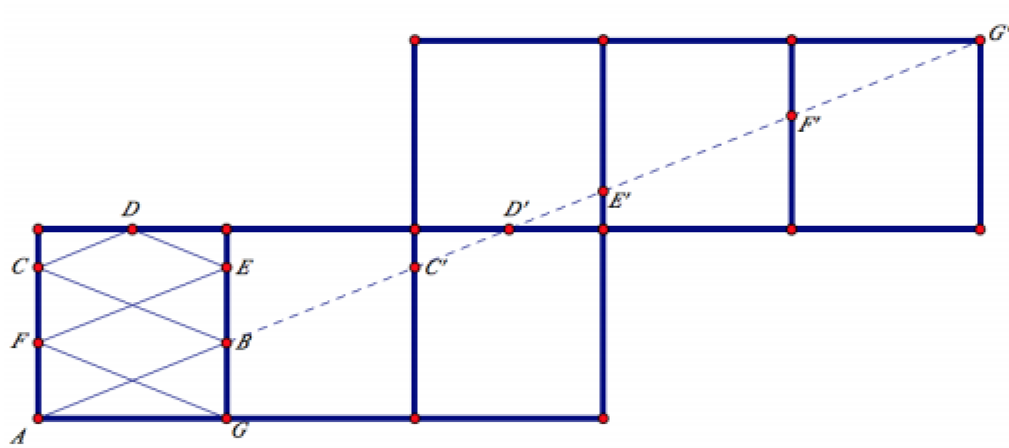
To see why this is correct, put a copy of the square next to the original one, and continue drawing the original line from A until it hits the border of this new square, at C'. Notice that the dashed line BC' is a reflection of the actual billiard path BC.



Continuing this process, we add another square and extend the line to D'. Note that the line segment $C'D'$ is *exactly* the same as the "real" path CD, because now we have a reflection of a reflection!



We keep going, until the path hits a corner, and we count, keeping track of what got reflected. In this case We will need a few more squares, as seen below.



Now look at the line starting at A, ending at G'. Each time this line hits a vertical or horizontal line, it indicates a bounce. It is easy to count a total of 5 bounces (counting the hit at B, 4 vertical and 1 horizontal before the corner), but how do we determine the final corner? When the line hits a vertical line, in the "real" world, the ball bounces horizontally, flipping from eastward to westward, or vice-versa. Likewise, hitting a horizontal line causes a flip from either north to south or south to north. Since we hit an even number of vertical lines, the final segment of the ball will be east (the same direction it started at), but because the number of horizontal hits was odd, the ball will be heading south (the opposite of its initial direction). Hence the ball is traveling southeast in its final segment, clearly ending up at the lower-right corner. Notice that by treating the table as a unit square, if the starting position at A is $(0,0)$, then in the extended diagram the ending one at G' is $(5,2)$.

Finally we can apply this method to the actual problem. Since 2019 and 1000 have no factors in common, we will need to draw a line from $(0,0)$ to $(2019,1000)$ because the line with slope $1000/2019$ starting at $(0,0)$ hits no lattice points until $(2019,1000)$. So the line will hit 2018 vertical lines and 999 horizontal lines, for a total of **3017** bounces before it travels along its final segment, heading for a corner. It begins its trip heading NE, and the 2018 vertical hits will keep E as E, but the 999 horizontal hits means that N becomes S. So the final direction is SE, which means that the corner we hit is the **lower-right corner**.

6 *Cosines and Squares. Let* $\theta = 2\pi/17$. *Compute*

$$\sum_{k=1}^{16} \cos\left(k^2\theta\right) = \cos\theta + \cos 4\theta + \cos 9\theta + \cos 16\theta + \cdots + \cos 16^2\theta.$$

Let $z = e^{i\theta}$ and consider the sum $S = \sum_{k=0}^{16} z^{k^2}$. Since the original sum did not involve $k = 0$, the answer to the problem will be the real part of S minus 1. If we now compute S^2, this unspools into the sum of all 17^2 possible terms of the form $z^{a^2+b^2}$, where a, b range through all 17 residues modulo 17. Now replace b with $u \equiv 4b \pmod{17}$. This gives summands of the form $z^{a^2+(4b)^2}$. Since 17 is prime, the residues for u will also range through all 17 possibilities. Since $4^2 = 16 \equiv -1 \pmod{17}$, we are now looking at terms of the form $z^{a^2+u^2} = z^{a^2-b^2} = z^{(a-b)(a+b)}$. We will analyze this by distinguishing between terms where $a = b$ and $a \neq b$.

- If $a \equiv b$, we get $z^0 = 1$, and there will be 17 such instances, adding up to 17.
- If $a - b \equiv c \neq 0$, we get $z^{c(a+b)} = z^{c(2b+c)} = z^{c^2} z^{2bc}$. As b ranges through all 17 residues, so too will $2bc$, and we thus get the sum $z^{c^2}(z^0 + z^1 + z^2 + \cdots + z^{16})$, and since the quantity in the parentheses equals zero, we see that all of these sums equal zero.

Thus $S = \sqrt{17}$. Since this is already real, the answer to the problem is $\sqrt{17} - 1$.

7 *A 9th-degree equation. Find all solutions, real and complex, to*

$$(2000x^3 - 17)^3 + (19x^3 - 12)^3 = (2019x^3 - 29)^3$$

If we let $a = 2000x^3 - 17$ and $b = 19x^3 - 12$ then $a + b = 2019x^3 - 29$ and the above is equivalent to $a^3 + b^3 = (a+b)^3$.

$$a^3 + b^3 - (a+b)^3 = -3a^2b - 3ab^2 = -3ab(a+b) = 0 \Longrightarrow a = 0, b = 0 \text{ or } a + b = 0$$

Thus we have the nine possibilities $x, x\omega, x\omega^2$, where $x = \sqrt[3]{\frac{17}{2000}}, \sqrt[3]{\frac{12}{19}}, \sqrt[3]{\frac{29}{2019}}$, and $\omega = (-1 + i\sqrt{3})/2$.

8 *A Boring Polynomial. Let* $P(x)$ *be a 2018th-degree polynomial with real coefficients, such that* $P(n) = n$, *for* $n = 1, 2, 3, \ldots, 2018$. *Find all possible values for* $P(2019)$.
Consider $Q(x) := P(x) - x$. We know its roots, so $Q(x) = A(x-1)(x-2)\cdots(x-2018)$, where A is a non-zero real number. Thus suppose $P(2019) = B$. Plugging in $x = 2019$, we see that $P(2019) = 2019 + Q(2019) = 2019 + (2018!)A = B$. Thus we conclude that $P(2019) = B$ can be *any* real number with $B \neq 2019$.